If $z = 0$, then solving Eqs. (3) and (4) simultaneously to find the corresponding points on the ellipse gives the two points (1, 0, 0) and (0, 1, 0). This makes sense when you look at Fig. 12.66.

If $x = y$, then Eqs. (3) and (4) give

$$\begin{aligned} x^2 + x^2 - 1 &= 0 & x + x + z - 1 &= 0 \\ 2x^2 &= 1 & z &= 1 - 2x \\ x &= \pm\frac{\sqrt{2}}{2} & z &= 1 \mp \sqrt{2}. \end{aligned}$$

The corresponding points on the ellipse are

$$P_1 = \left(\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}, 1 - \sqrt{2}\right) \quad \text{and} \quad P_2 = \left(-\frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2}, 1 + \sqrt{2}\right).$$

But here we need to be careful. While P_1 and P_2 both give local maxima of f on the ellipse, P_2 is farther from the origin than P_1.

The points on the ellipse closest to the origin are (1, 0, 0) and (0, 1, 0). The point on the ellipse farthest from the origin is P_2. □

Exercises 12.9

Two Independent Variables with One Constraint

1. Find the points on the ellipse $x^2 + 2y^2 = 1$ where $f(x, y) = xy$ has its extreme values.

2. Find the extreme values of $f(x, y) = xy$ subject to the constraint $g(x, y) = x^2 + y^2 - 10 = 0$.

3. Find the maximum value of $f(x, y) = 49 - x^2 - y^2$ on the line $x + 3y = 10$ (Fig. 12.58).

4. Find the local extreme values of $f(x, y) = x^2y$ on the line $x + y = 3$.

5. Find the points on the curve $xy^2 = 54$ nearest the origin.

6. Find the points on the curve $x^2y = 2$ nearest the origin.

7. Use the method of Lagrange multipliers to find
   a) the minimum value of $x + y$, subject to the constraints $xy = 16, x > 0, y > 0$;
   b) the maximum value of xy, subject to the constraint $x + y = 16$.

   Comment on the geometry of each solution.

8. Find the points on the curve $x^2 + xy + y^2 = 1$ in the xy-plane that are nearest to and farthest from the origin.

9. Find the dimensions of the closed right circular cylindrical can of smallest surface area whose volume is 16π cm^3.

10. Find the radius and height of the open right circular cylinder of largest surface area that can be inscribed in a sphere of radius a. What *is* the largest surface area?

11. Use the method of Lagrange multipliers to find the dimensions of the rectangle of greatest area that can be inscribed in the ellipse $x^2/16 + y^2/9 = 1$ with sides parallel to the coordinate axes.

12. Find the dimensions of the rectangle of largest perimeter that can be inscribed in the ellipse $x^2/a^2 + y^2/b^2 = 1$ with sides parallel to the coordinate axes. What *is* the largest perimeter?

13. Find the maximum and minimum values of $x^2 + y^2$ subject to the constraint $x^2 - 2x + y^2 - 4y = 0$.

14. Find the maximum and minimum values of $3x - y + 6$ subject to the constraint $x^2 + y^2 = 4$.

15. The temperature at a point (x, y) on a metal plate is $T(x, y) = 4x^2 - 4xy + y^2$. An ant on the plate walks around the circle of radius 5 centered at the origin. What are the highest and lowest temperatures encountered by the ant?

16. Your firm has been asked to design a storage tank for liquid petroleum gas. The customer's specifications call for a cylindrical tank with hemispherical ends, and the tank is to hold 8000 m^3 of gas. The customer also wants to use the smallest amount of material possible in building the tank. What radius and height do you recommend for the cylindrical portion of the tank?

Three Independent Variables with One Constraint

17. Find the point on the plane $x + 2y + 3z = 13$ closest to the point (1, 1, 1).

18. Find the point on the sphere $x^2 + y^2 + z^2 = 4$ which is farthest from the point (1, −1, 1).

19. Find the minimum distance from the surface $x^2 + y^2 - z^2 = 1$ to the origin.

20. Find the point on the surface $z = xy + 1$ nearest the origin.

21. Find the points on the surface $z^2 = xy + 4$ closest to the origin.

22. Find the point(s) on the surface $xyz = 1$ closest to the origin.

23. Find the maximum and minimum values of

$$f(x, y, z) = x - 2y + 5z$$

on the sphere $x^2 + y^2 + z^2 = 30$.

24. Find the points on the sphere $x^2 + y^2 + z^2 = 25$ where $f(x, y, z) = x + 2y + 3z$ has its maximum and minimum values.

25. Find three real numbers whose sum is 9 and the sum of whose squares is as small as possible.

26. Find the largest product the positive numbers x, y, and z can have if $x + y + z^2 = 16$.

27. Find the dimensions of the closed rectangular box with maximum volume that can be inscribed in the unit sphere.

28. Find the volume of the largest closed rectangular box in the first octant having three faces in the coordinate planes and a vertex on the plane $x/a + y/b + z/c = 1$, where $a > 0$, $b > 0$, and $c > 0$.

29. A space probe in the shape of the ellipsoid

$$4x^2 + y^2 + 4z^2 = 16$$

enters the earth's atmosphere and its surface begins to heat. After one hour, the temperature at the point (x, y, z) on the probe's surface is

$$T(x, y, z) = 8x^2 + 4yz - 16z + 600.$$

Find the hottest point on the probe's surface.

30. Suppose that the Celsius temperature at the point (x, y, z) on the sphere $x^2 + y^2 + z^2 = 1$ is $T = 400xyz^2$. Locate the highest and lowest temperatures on the sphere.

31. *An example from economics.* In economics, the usefulness or *utility* of amounts x and y of two capital goods G_1 and G_2 is sometimes measured by a function $U(x, y)$. For example, G_1 and G_2 might be two chemicals a pharmaceutical company needs to have on hand and $U(x, y)$ the gain from manufacturing a product whose synthesis requires different amounts of the chemicals depending on the process used. If G_1 costs a dollars per kilogram, G_2 costs b dollars per kilogram, and the total amount allocated for the purchase of G_1 and G_2 together is c dollars, then the company's managers want to maximize $U(x, y)$ given that $ax + by = c$. Thus, they need to solve a typical Lagrange multiplier problem.

Suppose that

$$U(x, y) = xy + 2x$$

and that the equation $ax + by = c$ simplifies to

$$2x + y = 30.$$

Find the maximum value of U and the corresponding values of x and y subject to this latter constraint.

32. You are in charge of erecting a radio telescope on a newly discovered planet. To minimize interference, you want to place it where the magnetic field of the planet is weakest. The planet is spherical, with a radius of 6 units. Based on a coordinate system whose origin is at the center of the planet, the strength of the magnetic field is given by $M(x, y, z) = 6x - y^2 + xz + 60$. Where should you locate the radio telescope?

Lagrange Multipliers with Two Constraints

33. Maximize the function $f(x, y, z) = x^2 + 2y - z^2$ subject to the constraints $2x - y = 0$ and $y + z = 0$.

34. Minimize the function $f(x, y, z) = x^2 + y^2 + z^2$ subject to the constraints $x + 2y + 3z = 6$ and $x + 3y + 9z = 9$.

35. Find the point closest to the origin on the line of intersection of the planes $y + 2z = 12$ and $x + y = 6$.

36. Find the maximum value that $f(x, y, z) = x^2 + 2y - z^2$ can have on the line of intersection of the planes $2x - y = 0$ and $y + z = 0$.

37. Find the extreme values of $f(x, y, z) = x^2yz + 1$ on the intersection of the plane $z = 1$ with the sphere $x^2 + y^2 + z^2 = 10$.

38. a) Find the maximum value of $w = xyz$ on the line of intersection of the two planes $x + y + z = 40$ and $x + y - z = 0$.
    b) Give a geometric argument to support your claim that you have found a maximum, and not a minimum, value of w.

39. Find the extreme values of the function $f(x, y, z) = xy + z^2$ on the circle in which the plane $y - x = 0$ intersects the sphere $x^2 + y^2 + z^2 = 4$.

40. Find the point closest to the origin on the curve of intersection of the plane $2y + 4z = 5$ and the cone $z^2 = 4x^2 + 4y^2$.

Theory and Examples

41. *The condition $\nabla f = \lambda \nabla g$ is not sufficient.* While $\nabla f = \lambda \nabla g$ is a necessary condition for the occurrence of an extreme value of $f(x, y)$ subject to the condition $g(x, y) = 0$, it does not in itself guarantee that one exists. As a case in point, try using the method of Lagrange multipliers to find a maximum value of $f(x, y) = x + y$ subject to the constraint that $xy = 16$. The method will identify the two points $(4, 4)$ and $(-4, -4)$ as candidates for the location of extreme values. Yet the sum $(x + y)$ has no maximum value on the hyperbola $xy = 16$. The farther you go from the origin on this hyperbola in the first quadrant, the larger the sum $f(x, y) = x + y$ becomes.

42. *A least squares plane.* The plane $z = Ax + By + C$ is to be "fitted" to the following points (x_k, y_k, z_k):

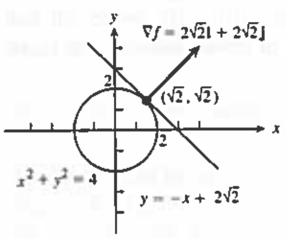
$$(0, 0, 0), \quad (0, 1, 1), \quad (1, 1, 1), \quad (1, 0, -1).$$

Find the values of A, B, and C that minimize the sum

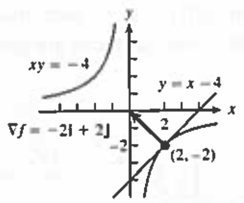
$$\sum_{k=1}^{4} (Ax_k + By_k + C - z_k)^2,$$

the sum of the squares of the deviations.

27. Tangent: $x + y + z = 3$, normal line: $x = 1 + 2t$, $y = 1 + 2t$, $z = 1 + 2t$
29. Tangent: $2x - z - 2 = 0$, normal line: $x = 2 - 4t$, $y = 0$, $z = 2 + 2t$
31. Tangent: $2x + 2y + z - 4 = 0$, normal line: $x = 2t$, $y = 1 + 2t$, $z = 2 + t$
33. Tangent: $x + y + z - 1 = 0$, normal line: $x = t$, $y = 1 + t$, $z = t$
35. $2x - z - 2 = 0$ 37. $x - y + 2z - 1 = 0$
39.



41.

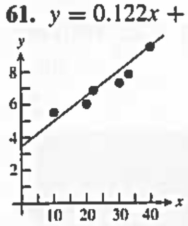


43. $x = 1$, $y = 1 + 2t$, $z = 1 - 2t$
45. $x = 1 - 2t$, $y = 1$, $z = \frac{1}{2} + 2t$
47. $x = 1 + 90t$, $y = 1 - 90t$, $z = 3$
49. $\mathbf{u} = \frac{7}{\sqrt{53}}\mathbf{i} - \frac{2}{\sqrt{53}}\mathbf{j}$, $-\mathbf{u} = -\frac{7}{\sqrt{53}}\mathbf{i} + \frac{2}{\sqrt{53}}\mathbf{j}$
51. No, the maximum rate of change is $\sqrt{185} < 14$. 53. $-\frac{7}{\sqrt{5}}$
55. a) $\frac{\sqrt{3}}{2}\sin\sqrt{3} - \frac{1}{2}\cos\sqrt{3} \approx 0.935°\text{C/ft}$
b) $\sqrt{3}\sin\sqrt{3} - \cos\sqrt{3} \approx 1.87°\text{C/sec}$
57. At $-\frac{\pi}{4}$, $-\frac{\pi}{2\sqrt{2}}$; at 0, 0; at $\frac{\pi}{4}$, $\frac{\pi}{2\sqrt{2}}$

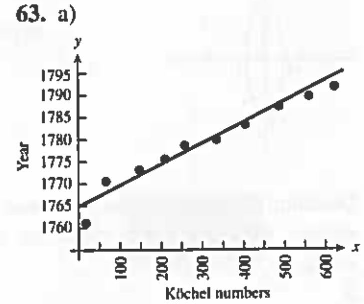
Section 12.8, pp. 975–979

1. $f(-3, 3) = -5$, local minimum
3. $f\left(\frac{2}{3}, \frac{4}{3}\right) = 0$, local maximum 5. $f(-2, 1)$, saddle point
7. $f\left(\frac{6}{5}, \frac{69}{25}\right)$, saddle point 9. $f(2, 1)$, saddle point
11. $f(2, -1) = -6$, local minimum 13. $f(1, 2)$, saddle point
15. $f(0, 0)$, saddle point
17. $f(0, 0)$, saddle point; $f\left(-\frac{2}{3}, \frac{2}{3}\right) = \frac{170}{27}$, local maximum
19. $f(0, 0) = 0$, local minimum; $f(1, -1)$, saddle point
21. $f(0, 0)$, saddle point; $f\left(\frac{4}{9}, \frac{4}{3}\right) = -\frac{64}{81}$, local minimum
23. $f(0, 0)$, saddle point; $f(0, 2) = -12$, local minimum; $f(-2, 0) = -4$, local maximum; $f(-2, 2)$, saddle point
25. $f(0, 0)$, saddle point; $f(1, 1) = 2$, $f(-1, -1) = 2$, local maxima
27. $f(0, 0) = -1$, local maximum
29. $f(n\pi, 0)$, saddle point; $f(n\pi, 0) = 0$ for every n

31. Absolute maximum: 1 at (0, 0); absolute minimum: -5 at (1, 2)
33. Absolute maximum: 4 at (0, 2); absolute minimum: 0 at (0, 0)
35. Absolute maximum: 11 at (0, -3); absolute minimum: -10 at (4, -2)
37. Absolute maximum: 4 at (2, 0); absolute minimum: $\frac{3\sqrt{2}}{2}$ at $\left(3, -\frac{\pi}{4}\right)$, $\left(3, \frac{\pi}{4}\right)$, $\left(1, -\frac{\pi}{4}\right)$, and $\left(1, \frac{\pi}{4}\right)$ 39. $a = -3, b = 2$
41. Hottest: $2\frac{1}{4}°$ at $\left(-\frac{1}{2}, \frac{\sqrt{3}}{2}\right)$ and $\left(-\frac{1}{2}, -\frac{\sqrt{3}}{2}\right)$; coldest: $-\frac{1}{4}°$ at $\left(\frac{1}{2}, 0\right)$
43. a) $f(0, 0)$, saddle point b) $f(1, 2)$, local minimum
c) $f(1, -2)$, local minimum; $f(-1, -2)$, saddle point
49. $\left(\frac{1}{6}, \frac{1}{3}, \frac{355}{36}\right)$
53. a) On the semicircle, max $f = 2\sqrt{2}$ at $t = \pi/4$, min $f = -2$ at $t = \pi$. On the quarter circle, max $f = 2\sqrt{2}$ at $t = \pi/4$, min $f = 2$ at $t = 0, \pi/2$.
b) On the semicircle, max $g = 2$ at $t = \pi/4$, min $g = -2$ at $t = 3\pi/4$. On the quarter circle, max $g = 2$ at $t = \pi/4$, min $g = 0$ at $t = 0, \pi/2$.
c) On the semicircle, max $h = 8$ at $t = 0, \pi$; min $h = 4$ at $t = \pi/2$. On the quarter circle, max $h = 8$ at $t = 0$, min $h = 4$ at $t = \pi/2$.
55. i) min $f = -1/2$ at $t = -1/2$; no max ii) max $f = 0$ at $t = -1, 0$; min $f = -1/2$ at $t = -1/2$ iii) max $f = 4$ at $t = 1$; min $f = 0$ at $t = 0$
57. $y = -\frac{20}{13}x + \frac{9}{13}$, $y|_{x=4} = -\frac{71}{13}$
59. $y = \frac{3}{2}x + \frac{1}{6}$, $y|_{x=4} = \frac{37}{6}$
61. $y = 0.122x + 3.58$



63. a)



b) $y = 0.0427K + 1764.8$ c) 1780

Section 12.9, pp. 987–989

1. $\left(\pm\frac{1}{\sqrt{2}}, \frac{1}{2}\right)$, $\left(\pm\frac{1}{\sqrt{2}}, -\frac{1}{2}\right)$ 3. 39 5. $(3, \pm 3\sqrt{2})$ 7. a) 8
b) 64 9. $r = 2$ cm, $h = 4$ cm 11. $l = 4\sqrt{2}$, $w = 3\sqrt{2}$
13. $f(0, 0) = 0$ is minimum, $f(2, 4) = 20$ is maximum

15. Minimum $= 0°$, maximum $= 125°$ **17.** $\left(\frac{3}{2}, 2, \frac{5}{2}\right)$ **19.** 1

21. $(0, 0, 2)$, $(0, 0, -2)$

23. $f(1, -2, 5) = 30$ is maximum, $f(-1, 2, -5) = -30$ is minimum

25. 3, 3, 3 **27.** $\frac{2}{\sqrt{3}}$ by $\frac{2}{\sqrt{3}}$ by $\frac{2}{\sqrt{3}}$ units **29.** $\left(\pm\frac{4}{3}, -\frac{4}{3}, -\frac{4}{3}\right)$

31. $U(8, 14) = \$128$ **33.** $f\left(\frac{2}{3}, \frac{4}{3}, -\frac{4}{3}\right) = \frac{4}{3}$ **35.** $(2, 4, 4)$

37. Maximum is $1 + 6\sqrt{3}$ at $(\pm\sqrt{6}, \sqrt{3}, 1)$, minimum is $1 - 6\sqrt{3}$ at $(\pm\sqrt{6}, -\sqrt{3}, 1)$

39. Maximum is 4 at $(0, 0, \pm 2)$, minimum is 2 at $(\pm\sqrt{2}, \pm\sqrt{2}, 0)$

Section 12.10, p. 993

1. Quadratic: $x + xy$; cubic: $x + xy + \frac{1}{2}xy^2$

3. Quadratic: xy; cubic: xy

5. Quadratic: $y + \frac{1}{2}(2xy - y^2)$;
cubic: $y + \frac{1}{2}(2xy - y^2) + \frac{1}{6}(3x^2y - 3xy^2 + 2y^3)$

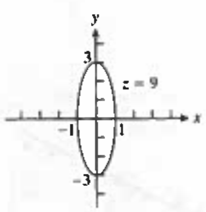
7. Quadratic: $\frac{1}{2}(2x^2 + 2y^2) = x^2 + y^2$; cubic: $x^2 + y^2$

9. Quadratic: $1 + (x + y) + (x + y)^2$;
cubic: $1 + (x + y) + (x + y)^2 + (x + y)^3$

11. Quadratic: $1 - \frac{1}{2}x^2 - \frac{1}{2}y^2$, $E(x, y) \le 0.00134$

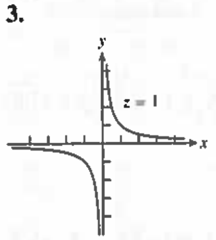
Chapter 12 Practice Exercises, pp. 994–998

1.



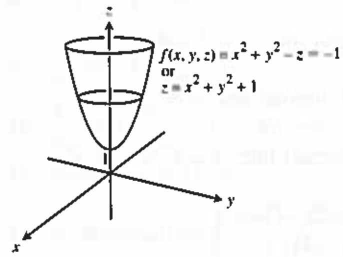
Domain: all points in the xy-plane; range: $z \ge 0$. Level curves are ellipses with major axis along the y-axis and minor axis along the x-axis.

3.



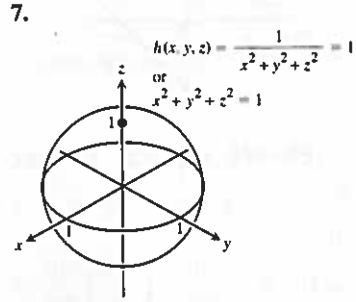
Domain: all (x, y) such that $x \ne 0$ and $y \ne 0$; range: $z \ne 0$. Level curves are hyperbolas with the x- and y-axes as asymptotes.

5.



Domain: all (x, y, z) such that $(x, y, z) \ne (0, 0, 0)$; range: all real numbers. Level surfaces are paraboloids of revolution with the z-axis as axis.

7.



Domain: all (x, y, z) such that $(x, y, z) \ne (0, 0, 0)$; range: positive real numbers. Level surfaces are spheres with center $(0, 0, 0)$ and radius $r > 0$.

9. -2 **11.** $1/2$ **13.** 1 **15.** Let $y = kx^2$, $k \ne 1$

17. a) Does not exist b) Not continuous at $(0, 0)$

19. $\frac{\partial g}{\partial r} = \cos\theta + \sin\theta$, $\frac{\partial g}{\partial \theta} = -r\sin\theta + r\cos\theta$

21. $\frac{\partial f}{\partial R_1} = -\frac{1}{R_1^2}$, $\frac{\partial f}{\partial R_2} = -\frac{1}{R_2^2}$, $\frac{\partial f}{\partial R_3} = -\frac{1}{R_3^2}$

23. $\frac{\partial P}{\partial n} = \frac{RT}{V}$, $\frac{\partial P}{\partial R} = \frac{nT}{V}$, $\frac{\partial P}{\partial T} = \frac{nR}{V}$, $\frac{\partial P}{\partial V} = -\frac{nRT}{V^2}$

25. $\frac{\partial^2 g}{\partial x^2} = 0$, $\frac{\partial^2 g}{\partial y^2} = \frac{2x}{y^3}$, $\frac{\partial^2 g}{\partial y\,\partial x} = \frac{\partial^2 g}{\partial x\,\partial y} = -\frac{1}{y^2}$

27. $\frac{\partial^2 f}{\partial x^2} = -30x + \frac{2 - 2x^2}{(x^2 + 1)^2}$, $\frac{\partial^2 f}{\partial y^2} = 0$, $\frac{\partial^2 f}{\partial y\,\partial x} = \frac{\partial^2 f}{\partial x\,\partial y} = 1$

29. Answers will depend on the upper bound used for $|f_{xx}|$, $|f_{xy}|$, $|f_{yy}|$. With $M = \sqrt{2}/2$, $|E| \le 0.0142$. With $M = 1$, $|E| \le 0.02$.

31. $L(x, y, z) = y - 3z$, $L(x, y, z) = x + y - z - 1$

33. Be more careful with the diameter.

35. $dI = 0.038$, % change in $V = -4.17\%$, % change in $R = -20\%$, % change in $I = 15.83\%$

37. a) 5% **39.** $\left.\frac{dw}{dt}\right|_{t=0} = -1$

41. $\left.\frac{\partial w}{\partial r}\right|_{(r,s)=(\pi,0)} = 2$, $\left.\frac{\partial w}{\partial s}\right|_{(r,s)=(\pi,0)} = 2 - \pi$

43. $\left.\frac{df}{dt}\right|_{t=1} = -(\sin 1 + \cos 2)\sin 1 + (\cos 1 + \cos 2)\cos 1 - 2(\sin 1 + \cos 1)\sin 2$