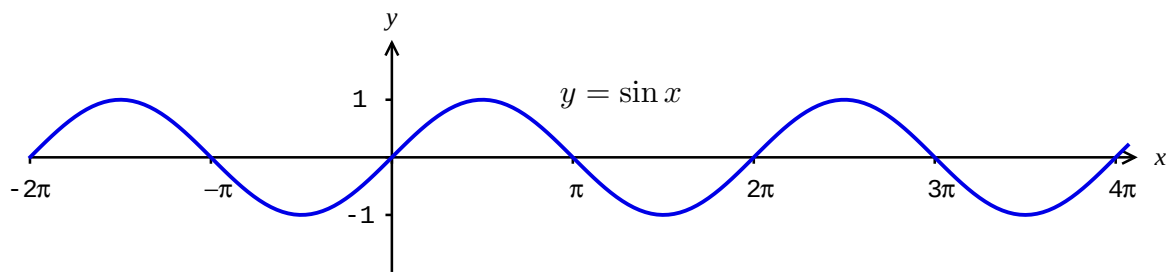
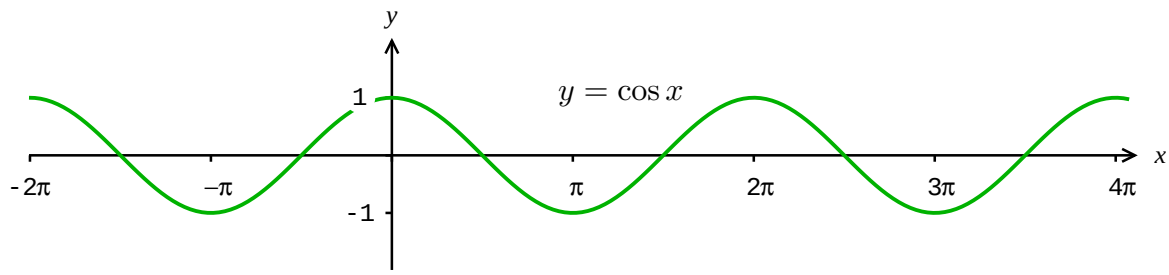
Ap.D: Trigonometry

For a more complete review of trignometry, see Appendix D of our text.
The two basic functions in trigonometry are the sine and cosine, graphed here:



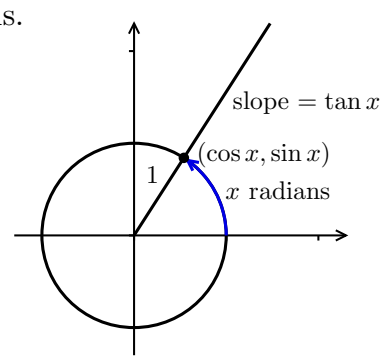


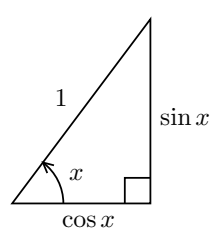
The other four trig functions are defined using sine and cosine:

$$\tan x = \frac{\sin x}{\cos x} \qquad \cot x = \frac{\cos x}{\sin x}$$

$$\sec x = \frac{1}{\cos x} \qquad \csc x = \frac{1}{\sin x}$$

$\sin x$ and $\cos x$ are defined for all real numbers x, but $\tan x$ and $\sec x$ are undefined whenever $\cos x = 0$, and $\cot x$ and $\csc x$ are undefined whenever $\sin x = 0$.
By definition, $\cos x$ and $\sin x$ are the coordinates of the point on the **unit circle** (i.e., the circle of radius one centered at the origin) x radians counterclockwise from the positive horizontal axis.



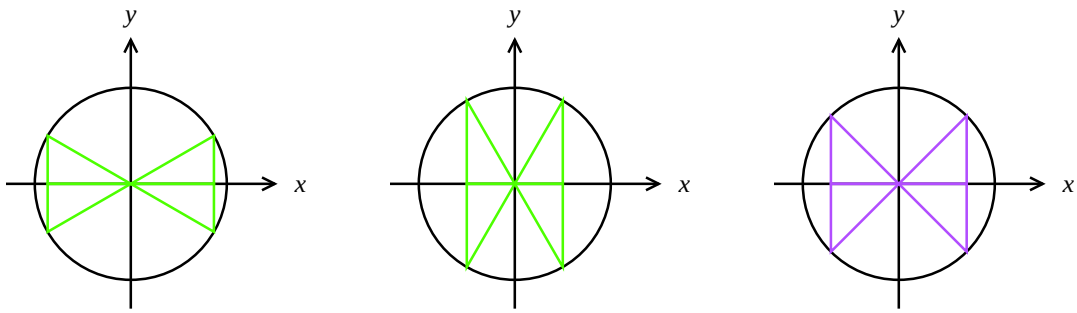


Consequently, the ray through the origin x radians from the positive horizontal axis has slope $\tan x$, and, when x is an acute angle, $\cos x$ and $\sin x$ are the legs of this right triangle with hypotenuse 1 and interior angle x

Known values of sine and cosine

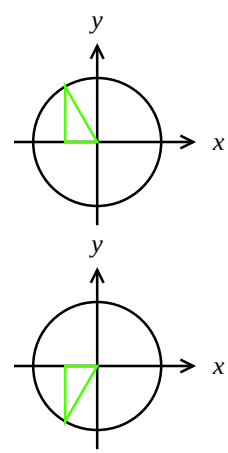
We already know the values of sine and cosine at the four points where the unit circle intersects the x and y axes. By placing these two triangles:

around the unit circle like this:



we find the sines and cosines at 12 more points on the unit circle (and the infinitely many angles that reach those points).

Ap.D.re1. Find all angles whose cosine is $-\frac{1}{2}$.



We recognize 1/2 as the short side of the 30-60-90 triangle, so for the cosine to be $-1/2$, the angle must be one of the two pictured at right. Find one angle that matches each drawing, for instance,

$$x = \pi - \pi/3 = 2\pi/3 \quad \text{and} \quad x = \pi + \pi/3 = 4\pi/3.$$

Then add all multiples of 2π to describe *all* angles that fit the drawings:

$$x = 2\pi/3 + 2\pi n \quad \text{and} \quad x = 4\pi/3 + 2\pi n \quad \text{(where } n \text{ is any integer).}$$

Ap.D.re2. Find all solutions x to the given equation.

a. $\sin x = -1/\sqrt{2}$ b. $\cos x = 0$ c. $\tan x = -1$

d. $\sec x = -2$ e. $\csc x = 2/\sqrt{3}$ f. $\cot x = \sqrt{3}$

Identities

You must know the derivatives of the six trig functions and these trig identities:

PYTHAGOREAN IDENTITIES
$\sin^2 x + \cos^2 x = 1$
$\tan^2 x + 1 = \sec^2 x$
$1 + \cot^2 x = \csc^2 x$
EVEN & ODD IDENTITIES
$\sin(-x) = -\sin x$
$\cos(-x) = \cos x$
SUM & DIFFERENCE IDENTITIES
$\sin(x+y) = \sin x \cos y + \cos x \sin y$
$\cos(x+y) = \cos x \cos y - \sin x \sin y$
$\sin(x-y) = \sin x \cos y - \cos x \sin y$
$\cos(x-y) = \cos x \cos y + \sin x \sin y$
DOUBLE ANGLE FORMULAS
$\sin(2x) = 2 \sin x \cos x$
$\cos(2x) = \cos^2 x - \sin^2 x$
$= 2\cos^2 x - 1$
$= 1 - 2\sin^2 x$
HALF ANGLE FORMULAS
$\cos^2 x = \frac{1}{2}\big(1 + \cos(2x)\big)$
$\sin^2 x = \frac{1}{2}\big(1 - \cos(2x)\big)$

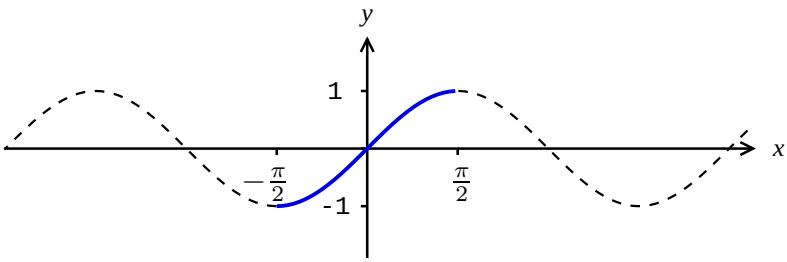
Ap.D.re3. The first pythagorean identity follows from the definition of sine and cosine. Derive the second and third pythagorean identities from the first by dividing by either $\sin^2 x$ or $\cos^2 x$.

Ap.D.re4. The formula for $\cos(x+y)$ is obtained by differentiating both sides of the formula for $\sin(x+y)$ with respect to x assuming y is a constant. Derive the other sum/difference/double/half-angle formulas from these two by differentiating with respect to x, replacing y with $-y$ and using the even/odd identities, or replacing y with x.

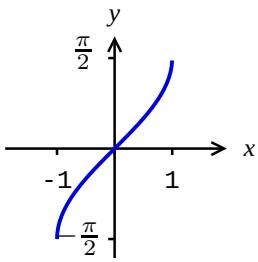
The inverse trig functions

To define inverses of the trig functions, we restrict each to a domain on which it takes each value in its range exactly once. The three we'll see most often in calculus are the inverses of the sine, tangent, and secant.

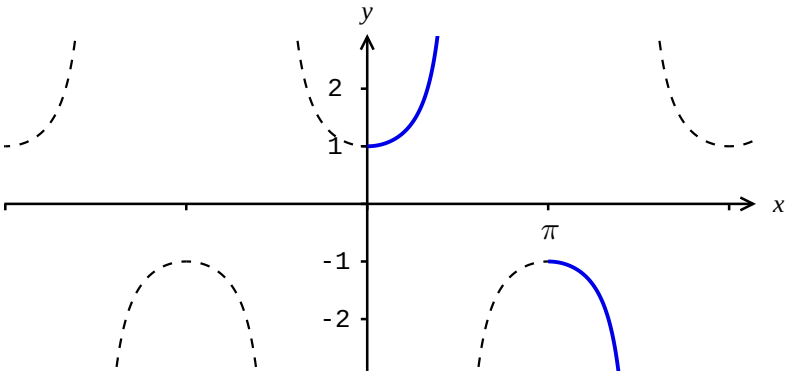
$y = \sin x$



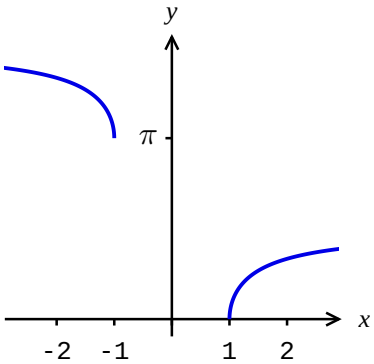
$y = \sin^{-1} x$



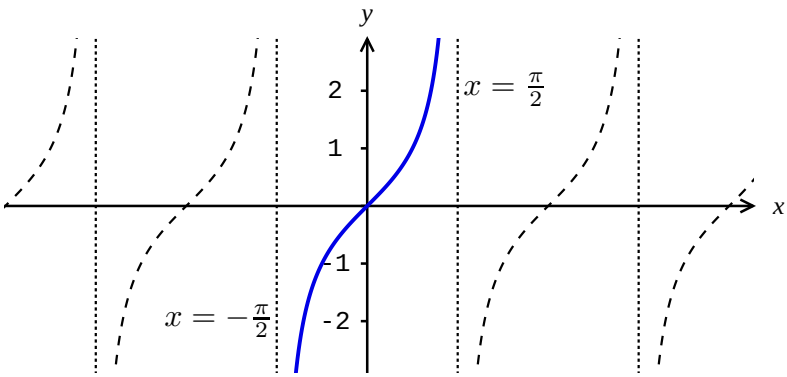
$y = \sec x$



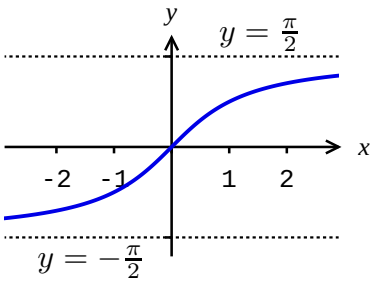
$y = \sec^{-1} x$



$y = \tan x$



$y = \tan^{-1} x$



Traditionally, each of the inverse trig functions has two names. The inverse function of $\sin x$ is called $\sin^{-1} x$ or $\arcsin x$, the inverse function of $\sec x$ is called $\sec^{-1} x$ or $\operatorname{arcsec} x$, etc..

As indicated in the graphs above,

	Domain	Range
$\sin^{-1}$	$[-1, 1]$	$\left[-\dfrac{\pi}{2}, \dfrac{\pi}{2}\right]$
$\sec^{-1}$	$(-\infty, -1] \cup [1, \infty)$	$\left[0, \dfrac{\pi}{2}\right) \cup \left[\pi, \dfrac{3\pi}{2}\right)$
$\tan^{-1}$	$(-\infty, \infty)$	$\left(-\dfrac{\pi}{2}, \dfrac{\pi}{2}\right)$

although there are few times in this course when you'll need to know these details.
Tip: if $x > 0$, and "arctrig" is any of the inverse trig functions, then $\operatorname{arctrig} x \in [0, \frac{\pi}{2}]$.

It is helpful to remember the definitions of the inverse trig functions in words:

If $-1 \le x \le 1$, then $\sin^{-1} x$ is the angle in $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ whose sine is x. That is,

$$\sin(\sin^{-1} x) = x \quad \text{for any } x \in [-1, 1].$$

If $|x| \ge 1$, then $\sec^{-1} x$ is the angle in $\left[0, \frac{\pi}{2}\right) \cup \left[\pi, \frac{3\pi}{2}\right)$ whose secant is x. That is,

$$\sec(\sec^{-1} x) = x \quad \text{for any } x \in (-\infty, -1] \cup [1, \infty).$$

If $-\infty < x < \infty$, then $\tan^{-1} x$ is the angle in $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ whose tangent is x. That is,

$$\tan(\tan^{-1} x) = x \quad \text{for any real number } x.$$

Ap.D.re5. Evaluate: $\sin^{-1}\left(-\frac{1}{2}\right)$

Answers

Ap.D.re2a. $x = -\pi/4 + 2\pi n$ or $x = -3\pi/4 + 2\pi n$ Ap.D.re2b. $x = \pi/2 + 2\pi n$ or $x = -\pi/2 + 2\pi n$
Ap.D.re2c. $x = 3\pi/4 + 2\pi n$ or $x = 11\pi/4 + 2\pi n$ Ap.D.re2d. (same as in example Ap.D.re1)
Ap.D.re2e. $x = \pi/3 + 2\pi n$ or $x = 2\pi/3 + 2\pi n$ Ap.D.re2f. $x = \pi/3 + 2\pi n$ or $x = 4\pi/3 + 2\pi n$ Ap.D.re5. $-\pi/6$

D: Differentiation

The **derivative** of the function $f(x)$, denoted $f'(x)$, is defined to be

$$f'(x) = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h} = \lim_{t\to x} \frac{f(t)-f(x)}{t-x}.$$

The **linearization** of a function $f(x)$ at $x = a$ is the linear function having the same value and derivative value as f at a and is given by

$$L(x) = f(a) + f'(a)(x-a).$$

Differentiation rules allow us to differentiate many functions without resorting to the definition. These rules come in two types.

I. A catalog of elementary functions and their derivatives.

$f(x)$	derivative of f
$f(x)$	$f'(x)$
x^n	nx^{n-1}
$\ln\lvert x\rvert$	x^{-1}
e^x	e^x
$\sin x$	$\cos x$
$\cos x$	$-\sin x$
$\tan x$	$\sec^2 x$
$\cot x$	$-\csc^2 x$
$\sec x$	$\sec x \tan x$
$\csc x$	$-\csc x \cot x$
$\sin^{-1} x$	$\frac{1}{\sqrt{1-x^2}}$
$\tan^{-1} x$	$\frac{1}{1+x^2}$
$\sec^{-1} x$	$\frac{1}{x\sqrt{x^2-1}}$

II. Combination laws.

Linearity Rules: If $f(x)$ and $g(x)$ are differentiable and c is a constant, then

$$\big(cf(x)\big)' = cf'(x)$$
$$\big(f(x)+g(x)\big)' = f'(x)+g'(x)$$

Product Rule: If $f(x)$ and $g(x)$ are differentiable, then so is their product, and

$$\big(f(x)g(x)\big)' = f'(x)g(x)+f(x)g'(x).$$

Quotient Rule: If $f(x)$ and $g(x)$ are differentiable, then so is their quotient, and

$$\left(\frac{f(x)}{g(x)}\right)' = \frac{f'(x)g(x)-f(x)g'(x)}{[g(x)]^2} \qquad \text{(provided } g(x)\neq 0\text{)}.$$

Chain Rule: If f and g are differentiable functions, then their composition is differentiable (where it exists), and

$$\Big[f\big(g(x)\big)\Big]' = f'\big(g(x)\big)g'(x)$$

When more than one combination law applies, remember this advice:

The last operation determines the first differentiation rule we must use.

D.re1. Find the indicated derivatives of the given function.

a. $f'(x)$ & $f''(x)$, where $f(x) = 7x^3 - \frac{3}{\sqrt[3]{x^2}} + \frac{2x}{3x+4}$

b. $g^{(3)}(x)$ & $g^{(4)}(x)$, where $g(x) = 7x^3 + 4x^2 + 2x + \frac{2}{4-x}$

D.re2. Find the quadratic function $k(x)$ satisfying

$$k(2) = 1 \qquad k'(2) = -1 \qquad k''(2) = 6.$$

D.re3. Find the derivative of the given function.

a. $x^2 e^x$ b. $x^2 e^x (2x - 3)$ c. $\dfrac{3x^3 + 2}{x^2}$

d. $\dfrac{x}{3x^2 + 1}$ e. $e^x \sqrt{x} \sin x$ f. $\dfrac{x^2}{3x^3 + 2}$

g. $\dfrac{e^x \sin x + 3 \cos x}{1 - \tan x}$

D.re4. Find the equation of the line tangent to $y = \dfrac{xe^x}{3x^2 - 1}$ at $x = 1$.

D.re5. Find the derivative of the given function.

a. $\dfrac{2x - 1}{2x + 1}$ b. $\dfrac{2x + 1}{2x^2 - 1}$ c. $\dfrac{2x + 1}{(2x - 1)^2}$

d. $\sin(x^2)$ e. $\sin^2 x$ f. $\sqrt{x^2 - 5x + 1}$

g. $\ln 2$ h. $\ln(2x)$ i. $\ln(e^{-x} 5x^2 \sin x)$

j. $\ln|\sec x|$ k. $\log_3 x$ l. $e^{\frac{1}{2} \ln x}$

m. $x^{x/2}$ n. $e^{x \sin x}$ o. 2^x

p. $e^{\tan^2 x}$ q. $x^2 \cos^2 x^3$ r. $(\sin^2 x) e^{\cos x}$

s. $\dfrac{e^x}{\sec \sqrt{x}}$ t. $\tan^{-1}(\sin^{-1} x)$ u. $(\tan^{-1} x)(\sin^{-1} x)$

v. $e^x \arccos x^3$ w. $\sec(\sec^{-1} x)$ x. $\cot(\arctan e^x)$

D.re6. Find the linearization of the function at the given point.

a. $e^x,\ a = 0$ b. $e^x,\ a = 1$

c. $\sin x,\ a = 0$ d. $\sin x,\ a = \pi/4$

e. $\sqrt{x},\ a = 9$ f. $e^x(\sin x - \cos x)$

For more differentiation practice, see the review problems from Chapter 3 listed on our syllabus under **Assigned Problems**.

Answers

D.re1a. $f'(x) = 21x^2 + 2x^{-5/3} + \frac{8}{3(3x+4)^2}$. $f''(x) = 42x - \frac{10}{3}x^{-8/3} - \frac{16}{(3x+4)^3}$. D.re1b. $g'''(x) = 42 + 12(4-x)^{-4}$. $g^{(4)}(x) = 48(4-x)^{-5}$. D.re2. $3x^2 - 13x + 15$ D.re3a. $(x^2 + 2x)e^x$ D.re3b. $(2x^3 + 3x^2 - 3x)e^x$ D.re3c. $3 - 4x^{-3}$ D.re3d. $\frac{1-3x^2}{(3x^2+1)^2}$ D.re3e. $e^x\sqrt{x}\sin x + \frac{1}{2}e^x\frac{1}{\sqrt{x}}\sin x + e^x\sqrt{x}\cos x$ D.re3f. $\frac{-3x^4+4x}{(3x^3+2)^2}$ D.re3g. $\frac{\left(e^x(\sin x+\cos x)-3\sin x\right)(1-\tan x)+(e^x\sin x+3\cos x)(\sec^2 x)}{(1-\tan x)^2}$ D.re4. $y - \frac{e}{2} = -\frac{e}{2}(x-1)$ D.re5a. $\frac{4}{(2x+1)^2}$ D.re5b. $\frac{-4x^2-4x-2}{(2x^2-1)^2}$ D.re5c. $\frac{-4x-6}{(2x-1)^3}$ D.re5d. $2x\cos(x^2)$ D.re5e. $2\sin x\cos x$ D.re5f. $\frac{1}{2}(2x-5)(x^2-5x+1)^{-1/2}$ D.re5g. 0 D.re5h. x^{-1} D.re5i. $-1 + \frac{2}{x} + \cot x$ D.re5j. $\tan x$ D.re5k. hint: $\log_3 x = (\ln x)/(\ln 3)$. Derivative $= \frac{1}{x\ln 3}$. D.re5l. hint: simplify before differentiating. Derivative is $\frac{1}{2}x^{-1/2}$. D.re5m. hint: $a^b = e^{\ln a^b} = e^{b\ln a}$. Derivative $= \frac{1}{2}e^{\frac{1}{2}x\ln x}(1+\ln x)$. D.re5n. $(x\cos x + \sin x)e^{x\sin x}$ D.re5o. hint: $2^x = e^{\ln(2^x)} = e^{x\ln 2}$. Derivative $= e^{x\ln 2}\ln 2$. D.re5p. $e^{\tan^2 x}2\tan x\sec^2 x$ D.re5q. $2x\cos^3(x^3) - 6x^4\cos(x^3)\sin(x^3)$ D.re5r. $(2\sin x\cos x - \sin^3 x)e^{\cos x}$ D.re5s. hint: $\frac{e^x}{\sec\sqrt{x}} = e^x\cos\sqrt{x}$. Derivative $= e^x(\cos(\sqrt{x}) - \frac{1}{2}x^{-1/2}\sin(\sqrt{x}))$. D.re5t. $\frac{1}{(1+(\sin^{-1}x)^2)\sqrt{1-x^2}}$ D.re5u. $\frac{\sin^{-1}x}{1+x^2} + \frac{\tan^{-1}x}{\sqrt{1-x^2}}$ D.re5v. $e^x\left(\cos^{-1}x^3 - \frac{3x^2}{\sqrt{1-x^6}}\right)$ D.re5w. hint: simplify before differentiating. Derivative $= 1$. D.re5x. hint: simplify before differentiating. Derivative is $-e^{-x}$. D.re6a. $1 + x$ D.re6b. xe D.re6c. x D.re6d. $\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}(x - \frac{\pi}{4})$ D.re6e. $3 + \frac{1}{6}(x-9)$ D.re6f. $e^{\pi/2}\Big(1 + 2(x - \pi/2)\Big)$

I: Integration

The **definite integral** from a to b of $f(x)\,dx$, written $\int_a^b f(x)\,dx$, is the limit of the Riemann sums of f on the interval $[a, b]$ as the length of the subintervals goes to zero.

The **indefinite integral** of $f(x)\,dx$, written $\int f(x)\,dx$, is the collection of all **antiderivatives** of $f(x)$, that is, those functions $F(x)$ whose derivative $F'(x)$ equals $f(x)$.

$$\begin{aligned}
\int_a^b f(x)\,dx &= \text{a number.} \\
&= \text{the net signed area trapped between the graph of } f(x) \\
&\quad\ \text{and the } x\text{-axis from } x = a \text{ to } x = b. \\
&= \text{the net distance traveled from time } x = a \text{ to time } x = b \\
&\quad\ \text{by an object whose velocity at time } x \text{ is } f(x). \\
\int f(x)\,dx &= \text{a collection of functions.} \\
&= \text{the set of all antiderivatives of } f(x).
\end{aligned}$$

If $F(x)$ is an antiderivative of $f(x)$ on an interval, then, by the Mean Value Theorem, every antiderivative of f on that interval equals $F(x) + C$ for some constant C:

$$\int f(x)\,dx = F(x) + C, \text{ where } F'(x) = f(x)$$

According to the FTC below, evaluating a definite integral consists primarily of finding an antiderivative. We refer to both of those operations as **integration**.

Fundamental Theorem of Calculus: If $F(x)$ is any antiderivative of $f(x)$ and if f is continuous on $[a, b]$, then

$$\int_a^b f(t)\,dt = F(b) - F(a) \stackrel{\text{def}}{=} F(x)\Big|_a^b$$

I.re1. Integrate.

a. $\displaystyle\int (x^{0.7} - 2e^x)\,dx$ b. $\displaystyle\int_1^2 (1 - 6t^2 + 10t^4)\,dt$ c. $\displaystyle\int \frac{r^3 + 1}{2r}\,dr$

d. $\displaystyle\int_{-\pi/6}^{\pi/4} \sec x \tan x\,dx$ e. $\displaystyle\int_{-3}^{3} x^{11}\,dx$ f. $\displaystyle\int (u + 3)(u - 4)\,du$

A table of antiderivatives is practically the same as a table of derivatives read backwards:

$f(x)$	derivative of f
$f(x)$	$f'(x)$
x^n	nx^{n-1}
$\ln\lvert x\rvert$	x^{-1}
e^x	e^x
$\sin x$	$\cos x$
$\cos x$	$-\sin x$
$\tan x$	$\sec^2 x$
$\cot x$	$-\csc^2 x$
$\sec x$	$\sec x \tan x$
$\csc x$	$-\csc x \cot x$
$\sin^{-1} x$	$\frac{1}{\sqrt{1-x^2}}$
$\tan^{-1} x$	$\frac{1}{1+x^2}$
$\sec^{-1} x$	$\frac{1}{x\sqrt{x^2-1}}$

$f(x)$	antiderivative of f
$f(x)$	$\int f(x)\,dx$
x^n	$\begin{cases} \frac{1}{n+1}x^{n+1}+C & \text{if } n \neq -1 \\ \ln\lvert x\rvert + C & \text{if } n = -1 \end{cases}$
e^x	$e^x + C$
$\cos x$	$\sin x + C$
$\sin x$	$-\cos x + C$
$\sec^2 x$	$\tan x + C$
$\csc^2 x$	$-\cot x + C$
$\sec x \tan x$	$\sec x + C$
$\csc x \cot x$	$-\csc x + C$
$\frac{1}{\sqrt{1-x^2}}$	$\sin^{-1} x + C$
$\frac{1}{1+x^2}$	$\tan^{-1} + C$
$\frac{1}{x\sqrt{x^2-1}}$	$\sec^{-1} x + C$

There are fewer combination rules for antiderivatives than for derivatives. If f and g are functions and c is a constant, then

$$\int \big(f(x)+g(x)\big)\,dx = \int f(x)\,dx + \int g(x)\,dx$$

$$\int cf(x)\,dx = c\int f(x)\,dx$$

Generally, it's difficult to integrate a product or quotient, because

$$\left(\int f(x)\,dx\right)\left(\int g(x)\,dx\right) \neq \int f(x)g(x)\,dx, \quad \text{and}$$

$$\left(\int f(x)\,dx\right) \div \left(\int g(x)\,dx\right) \neq \int \big(f(x)/g(x)\big)\,dx.$$

For example, x^2 is an antiderivative for $2x$, and e^x is an antiderivative for itself, but x^2e^x is **not** an antiderivative for $2xe^x$, and $\dfrac{e^x}{x^2}$ is **not** an antiderivative for $\dfrac{e^x}{2x}$.

I.re2. Find $f(x)$ if

a. $f'(x) = \dfrac{1}{x}$ and $f(-2) = 1$

b. $f''(x) = \sin x - x^2$ and $f'(0) = 2$ and $f(0) = -1$

I.re3. Integrate.

a. $\displaystyle\int (x^2 - 3)\sqrt{x}\,dx$ b. $\displaystyle\int_2^1 \frac{x^2 - 3}{x^{-2}}\,dx$ c. $\displaystyle\int_0^1 e^p\,dp$

d. $\displaystyle\int_{-6}^{-2} \frac{ds}{s}$ e. $\displaystyle\int_{-1}^{4} e\,dw$ f. $\displaystyle\int (2\csc x\cot x - 3\cos x)\,dx$

g. $\displaystyle\int (x^2 - 1)^3\,dx$ h. $\displaystyle\int_0^{\pi/4} \frac{1}{\cos^2 t}\,dt$ i. $\displaystyle\int \frac{r^3 - 7r^2 + 1}{2r}\,dr$

j. $\displaystyle\int \frac{x^3 + 8}{x + 2}\,dx$ k. $\displaystyle\int_{-2}^{2} |x|\,dx$ l. $\displaystyle\int_0^{1/2} \frac{du}{\sqrt{1 - u^2}}$

m. $\displaystyle\int \left(\frac{2}{x} + \frac{3}{1 + x^2}\right) dx$ n. $\displaystyle\int \csc x(\csc x - \cot x)\,dx$

The **Substitution Rule** uses the chain rule to help us to simplify an integral. It says that an integral of the form $\int f\big(u(x)\big)u'(x)\,dx$ (for some function u of x) can be rewritten

$$\int f(u)\frac{du}{dx}\,dx = \int f(u)\,du$$

and thenceforth be treated as if u were the variable of integration. For this reason, substitution is also called a **change of variable**.

I.re4. Integrate.

a. $\displaystyle\int x^3 \sin(x^4)\,dx$ b. $\displaystyle\int (2x + 1)\sqrt{x^2 + x}\,dx$ c. $\displaystyle\int \frac{\sqrt{\ln x}}{x}\,dx$

d. $\displaystyle\int 2^{x+3}\,dx$ e. $\displaystyle\int \cos x\cot(\sin x)\sec^2(\sin x)\,dx$ f. $\displaystyle\int \frac{1}{1 + \sqrt{x}}\,dx$

g. $\displaystyle\frac{2e^{\sqrt{x}}}{\sqrt{x}}\,dx$ h. $\displaystyle\int \frac{e^x}{e^{2x} + 1}\,dx$ i. $\displaystyle\int (2t + 1)^{19}\,dt$

j. $\displaystyle\int x\sqrt{x + 2}\,dx$

For more integration practice, see the review problems from Chapter 5 listed on our syllabus under **Assigned Problems**.

Answers

I.re1a. $\frac{1}{1.7}x^{1.7} - 2e^x + C$ I.re1b. 49 I.re1c. $\frac{1}{6}r^3 + \frac{1}{2}\ln|r| + C$ I.re1d. $2 + \sqrt{2}$. I.re1e. 0 I.re1f. $\frac{1}{3}u^3 - \frac{1}{2}u^2 - 12u + C$ I.re2a. $f(x) = \ln|x| + 1 - \ln 2$ I.re2b. $f(x) = -\sin x - \frac{1}{12}x^4 + 3x - 1$ I.re3a. $\frac{2}{7}x^{3.5} - 2x^{1.5} + C$ I.re3b. 4/5 I.re3c. $e - 1$ I.re3d. $-\ln 3$ I.re3e. $5e$ I.re3f. $-2\csc x - 3\sin x + C$ I.re3g. $\frac{1}{7}x^7 - \frac{3}{5}x^5 + x^3 - x + C$

I.re3h. 1 I.re3i. $\frac{1}{6}r^3\frac{7}{4}r^2-\frac{1}{2}\ln|r|+C$ I.re3j. hint: use long division to rewrite integrand. $\frac{1}{3}x^3-x^2+4x+C$. I.re3k. 4 I.re3l. $\pi/6$ I.re3m. $2\ln|x|+3\tan^{-1}x+C$ I.re3n. $-\cot x+\csc x+C$ I.re4a. $-\frac{1}{4}\cos(x^4)+C$ I.re4b. $\frac{2}{3}(x^2+x)^{3/2}+C$ I.re4c. $\frac{2}{3}(\ln x)^{3/2}+C$ I.re4d. $\frac{1}{\ln 2}2^{x+3}+C$ I.re4e. $\ln|\tan(\sin x)|+C$ I.re4f. $2\sqrt{x}-2\ln(1+\sqrt{x})+C$ I.re4g. $4e^{\sqrt{x}}+C$ I.re4h. $\tan^{-1}e^x+C$. I.re4i. $\frac{1}{40}(2t+1)^{20}+C$ I.re4j. $\frac{2}{5}(x+2)^{5/2}-\frac{4}{3}(x+2)^{3/2}+C$

12.1: Three-dimensional space

Rectangular Coordinates

The xy-plane is sometimes called $\mathbb{R}^2$, **two-dimensional space**, or simply **two space**, because any point in the plane can be represented by its two rectangular coordinates (x, y) (also called Cartesian coordinates).
To define rectangular coordinates (x, y, z) for **three-dimensional space** (or $\mathbb{R}^3$, or **three space**), picture the z-axis perpendicular to the xy-plane and passing through the origin.

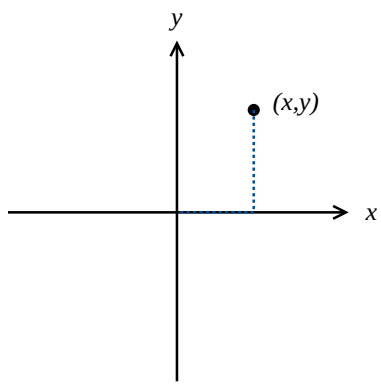


figure 12.1.1

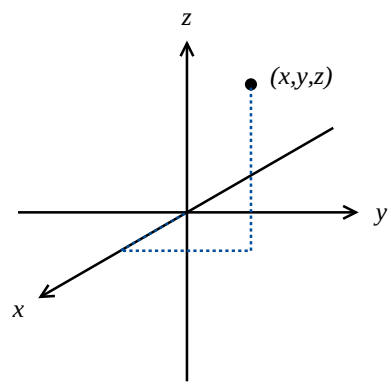
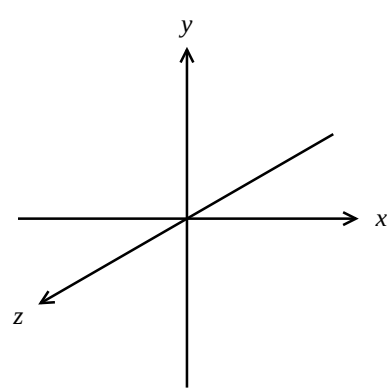


figure 12.1.2

Tip: draw the axes in three space from a point of view in the first octant (where x, y, and z are positive).

Some authors draw the same three axes rotated to look like the picture to the right of this paragraph. To see that this and figure 12.1.2 are the same after a rotation, check that both are a **right-handed system**: when pointing your right index finger in the direction of the positive x-axis and your right middle finger in the direction of the positive y-axis, your right thumb points in the direction of the positive z-axis.



Implicit equations of curves and surfaces

The graph in the xy-plane of an equation is often (but not always) a line or curve.
The graph in xyz-space of a single equation is often (but not always) a surface.
The graph of a system of two xyz-equations is often (b.n.a.) a curve.

12.1.re1. Find the equation(s) of the given coordinate plane or axes.

a. the xy-plane
b. the xz-plane
c. the yz-plane
d. the x-axis
e. the y-axis
f. the z-axis

Orthogonal Projection

The **orthogonal projection** of a point onto a line (or plane) is the the point on that line (or plane) closest to the given point. The line segment to the point from its projection is orthogonal to the line (or plane). Projecting onto one of the coordinate axes or planes is just a matter of reseting some coordinates equal to zero.

12.1.re2. Find the projection of the given point onto the coordinate line or plane.

a. $(2, -3, 4)$ onto the xy-plane
b. $(2, -3, 4)$ onto the xz-plane
c. $(2, -3, 4)$ onto the yz-plane
d. $(2, -3, 4)$ onto the x-axis
e. $(2, -3, 4)$ onto the y-axis
f. $(2, -3, 4)$ onto the z-axis

More on orthogonal projection onto lines in section 12.3.

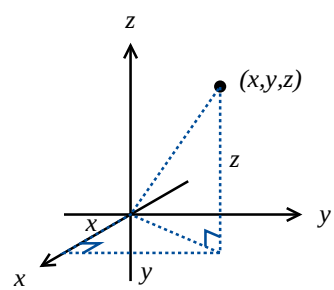
Distance in three space

The distance from a point (x, y, z) to the origin is

$$\sqrt{x^2 + y^2 + z^2}.$$

The distance from one point (x_0, y_0, z_0) to another (x_1, y_1, z_1) is

$$\sqrt{(x_1 - x_0)^2 + (y_1 - y_0)^2 + (z_1 - z_0)^2}.$$

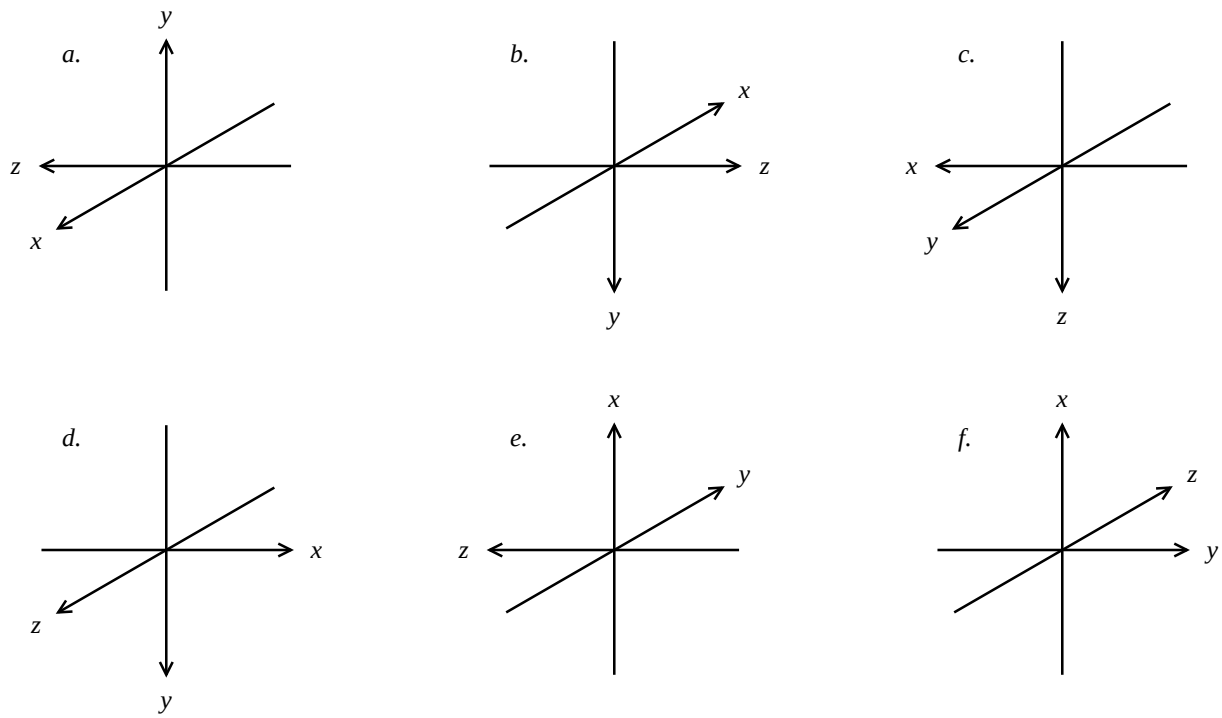


12.1.re3. Find an equation of the sphere centered at $(2, -1, 3)$ having radius 5.

12.1.re4. Find the center and radius of the given sphere.

a. $x^2 + (y - 3)^2 + (z + 2)^2 = 25$
b. $x^2 + 2x + y^2 - 4y + z^2 - 6z = 2$
c. $x^2 - x + y^2 + 3y + z^2 + 5z = \frac{1}{4}$

12.1.re5. Which of the figures below show a rotation of the axes pictured in figure 12.1.2?



Answers

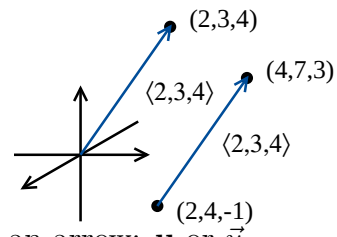
12.1.re1a. $z = 0$. 12.1.re1b. $y = 0$. 12.1.re1c. $x = 0$. 12.1.re1d. $y = 0$ and $z = 0$. 12.1.re1e. $x = 0$ and $z = 0$. 12.1.re1f. $x = 0$ and $y = 0$. 12.1.re2a. $(2, -3, 0)$ 12.1.re2b. $(2, 0, 4)$ 12.1.re2c. $(0, -3, 4)$ 12.1.re2d. $(2, 0, 0)$ 12.1.re2e. $(0, -3, 0)$ 12.1.re2f. $(0, 0, 4)$ 12.1.re3. $(x-2)^2 + (y+1)^2 + (z-3)^2 = 25$. 12.1.re4a. ctr $= (0, 3, -2)$, rad $= 5$. 12.1.re4b. ctr $= (-1, 2, 3)$, rad $= 4$. 12.1.re4c. ctr $= (\frac{1}{2}, -\frac{3}{2}, -\frac{5}{2})$, rad $= 3$. 12.1.re5. a,e,f.

12.2: Vectors

In Calculus III, a **vector** is a directed line segment in $\mathbb{R}^2$ or $\mathbb{R}^3$. The same vector can have different **initial** and **terminal** points.

12.2.re1. The vector from the initial point $(0,0,0)$ to the terminal point $(2,3,4)$ is denoted $\langle 2,3,4\rangle$. The vector with initial point $(2,4,-1)$ and terminal point $(4,7,3)$ is the same:

$$\langle 4-2, 7-4, 3-(-1)\rangle = \langle 2,3,4\rangle$$



A vector-valued variable is usually denoted either in bold or with an arrow: $\mathbf{u}$ or $\vec{u}$. To distinguish them from vectors, real numbers and real-valued variables are called **scalars** (which can be used either either a noun or an adjective).

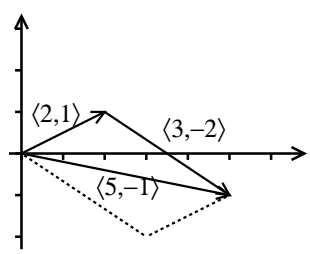
12.2.re2. x, y, and z are scalar variables. $\mathbf{r} = \langle x, y, z\rangle$ is a vector-valued variable.

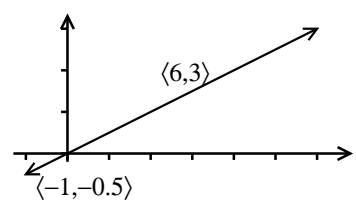
Vector Arithmetic

Vector Addition is the addition of vectors component by component.
Scalar Multiplication is the multiplication of every component of a vector by a scalar.

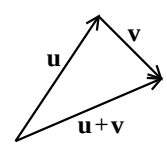
12.2.re3.

$$\langle 2,1\rangle + \langle 3,-2\rangle = \langle 5,-1\rangle$$
$$3\langle 2,1\rangle = \langle 6,3\rangle$$
$$-\tfrac{1}{2}\langle 2,1\rangle = \langle -1,-\tfrac{1}{2}\rangle$$

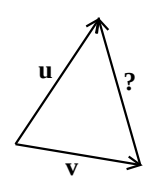




If the initial point of $\mathbf{v}$ is placed at the terminal point of $\mathbf{u}$, then $\mathbf{u}+\mathbf{v}$ reaches from the initial point of $\mathbf{u}$ to the terminal point of $\mathbf{v}$.



12.2.re4. Express the vector **?** shown in the figure in terms of $\mathbf{u}$ and $\mathbf{v}$. (Hint: what is $\mathbf{?}+\mathbf{v}$?)



If $\mathbf{u}$, $\mathbf{v}$, $\mathbf{w}$ are vectors and s and t are scalars, then

a. $\mathbf{u}+\mathbf{v} = \mathbf{v}+\mathbf{u}$
b. $\mathbf{u}+\mathbf{0} = \mathbf{u}$
c. $s(\mathbf{v}+\mathbf{w}) = s\mathbf{v}+s\mathbf{w}$
d. $s(t\mathbf{u}) = (st)\mathbf{u}$
e. $\mathbf{u}+(\mathbf{v}+\mathbf{w}) = (\mathbf{u}+\mathbf{v})+\mathbf{w}$
f. $\mathbf{u}+(-\mathbf{u}) = \mathbf{0}$
g. $(s+t)\mathbf{u} = s\mathbf{u}+t\mathbf{u}$
h. $1\mathbf{u} = \mathbf{u}$

Some Special Vectors

in $\mathbb{R}^3$: $\mathbf{0} = \langle 0, 0, 0\rangle$ $\mathbf{i} = \langle 1, 0, 0\rangle$ $\mathbf{j} = \langle 0, 1, 0\rangle$ $\mathbf{k} = \langle 0, 0, 1\rangle$

in $\mathbb{R}^2$: $\mathbf{0} = \langle 0, 0\rangle$ $\mathbf{i} = \langle 1, 0\rangle$ $\mathbf{j} = \langle 0, 1\rangle$

Always be careful to distinguish between the vector $\mathbf{0}$ and the scalar 0.
Every vector can be written uniquely as $a\mathbf{i} + b\mathbf{j} + c\mathbf{k}$ for some scalars a, b, c.

12.2.re5. Express the given vector in terms of $\mathbf{i}$, $\mathbf{j}$, and $\mathbf{k}$.

a. $\langle 2, 3, -1\rangle$ b. $\langle -1, 0, 1\rangle$ c. $\langle 3, \pi\rangle$

Magnitude

If $\mathbf{u} = \langle u_1, u_2, u_3\rangle$, the **magnititude** or **length** of $\mathbf{u}$ is

$$|\mathbf{u}| = \sqrt{u_1^2 + u_2^2 + u_3^2}$$

If $\mathbf{u}$ is a vector and s is a scalar, then

$$|s\mathbf{u}| = |s||\mathbf{u}|$$

A **unit vector** is a vector of length 1. If $\mathbf{u} \neq \mathbf{0}$, the unit vector

$$\frac{1}{|\mathbf{u}|}\mathbf{u}$$

is the **normalization** of $\mathbf{u}$.

12.2.re6. Let

$$\mathbf{u} = \langle 2, 1, -2\rangle \qquad \mathbf{v} = \langle 5, 2, -1\rangle \qquad \mathbf{w} = \langle 3, -4\rangle \qquad \mathbf{p} = 5\mathbf{i} - 4\mathbf{j} - 2\mathbf{k} \qquad \mathbf{q} = -2\mathbf{i} + 4\mathbf{j} - \mathbf{k}$$

and find the following, if they exist.

a. $\mathbf{u} - 2\mathbf{v}$ b. $3\mathbf{u} - \mathbf{v} + 2\mathbf{p}$ c. $3\mathbf{u} - 2\mathbf{w}$ d. $\mathbf{u} - 2$
e. $\mathbf{v} - 2\mathbf{i}$ f. $|\mathbf{v}|$ g. $|2\mathbf{u}|$ h. $|-2\mathbf{w}| + |\mathbf{w}|$
i. the normalization of $\mathbf{u}$
j. a vector of length 3 in the opposite direction of $\mathbf{w}$

Vectors in Physics

Vectors are used to model things that have *magnitude* and *direction*.

12.2.re7. A ship sails due west with speed 4 knots. Taking the magnitude of velocity to be speed, and assuming $\mathbf{j}$ and $\mathbf{i}$ represent the directions north and east, resp., find a vector representing the velocity of the ship.

12.2.re8. Find the vector representing a force with magnitude 3 Newtons in the direction of $\langle 7, -4, -4\rangle$.

Answers

12.2.re4. $\mathbf{u}$ - $\mathbf{v}$. 12.2.re5a. $2\mathbf{i} + 3\mathbf{j} - \mathbf{k}$. 12.2.re5b. $-\mathbf{i} + \mathbf{k}$. 12.2.re5c. $3\mathbf{i} + \pi\mathbf{j}$. 12.2.re6a. $\langle -8, -3, 0\rangle$. 12.2.re6b. $\langle 11, -7, -9\rangle$. 12.2.re6c. dne. 12.2.re6d. dne. 12.2.re6e. $3\mathbf{i} + 2\mathbf{j} - \mathbf{k}$. 12.2.re6f. $\sqrt{30}$. 12.2.re6g. 6. 12.2.re6h. 15. 12.2.re6i. $\langle \frac{2}{3}, \frac{1}{3}, -\frac{2}{3}\rangle$. 12.2.re6j. $\langle -\frac{9}{5}, \frac{12}{5}\rangle$. 12.2.re7. $-4\mathbf{i}$. 12.2.re8. $\langle \frac{7}{3}, -\frac{4}{3}, \frac{4}{3}\rangle$.

12.3: The Dot Product

Definition. The **dot product** of the vectors $\mathbf{u} = \langle u_1, u_2, u_3\rangle$ and $\mathbf{w} = \langle w_1, w_2, w_3\rangle$ is

$$\mathbf{u}\cdot\mathbf{w} = u_1w_1 + u_2w_2 + u_3w_3.$$

The dot product is sometimes called the **inner product** or **scalar product**.

12.3.re1. a. $\langle 4, 3\rangle \cdot \langle 1, 5\rangle = 4\cdot 1 + 3\cdot 5 = 19$.
b. $\langle 2, -4, 3\rangle \cdot \langle 3, 3, 2\rangle = 2\cdot 3 - 4\cdot 3 + 3\cdot 2 = 0$.

If $\mathbf{u}$, $\mathbf{v}$, $\mathbf{w}$ are vectors and s is a scalar and θ is the angle between $\mathbf{u}$ and $\mathbf{v}$, then

a. $\mathbf{0}\cdot\mathbf{u} = 0$
b. $\mathbf{u}\cdot\mathbf{v} = \mathbf{v}\cdot\mathbf{u}$
c. $\mathbf{u}\cdot(\mathbf{v}+\mathbf{w}) = \mathbf{u}\cdot\mathbf{v} + \mathbf{u}\cdot\mathbf{w}$
d. $(s\mathbf{u})\cdot\mathbf{v} = s(\mathbf{u}\cdot\mathbf{v}) = \mathbf{u}\cdot(s\mathbf{v})$
e. $\mathbf{u}\cdot\mathbf{u} = |\mathbf{u}|^2$
f. $\mathbf{u}\cdot\mathbf{v} = |\mathbf{u}||\mathbf{v}|\cos\theta$
g. $\mathbf{u}\perp\mathbf{v}$ iff $\mathbf{u}\cdot\mathbf{v} = 0$
h. $\mathbf{u}\cdot\mathbf{v} > 0$ iff $0 \le \theta < \pi/2$
$\mathbf{u}\cdot\mathbf{v} < 0$ iff $\pi/2 < \theta \le \pi$

Orthogonal Projection

If $\mathbf{u}$ and $\mathbf{v}$ are vectors, and $\mathbf{v} \neq \mathbf{0}$, then the vector

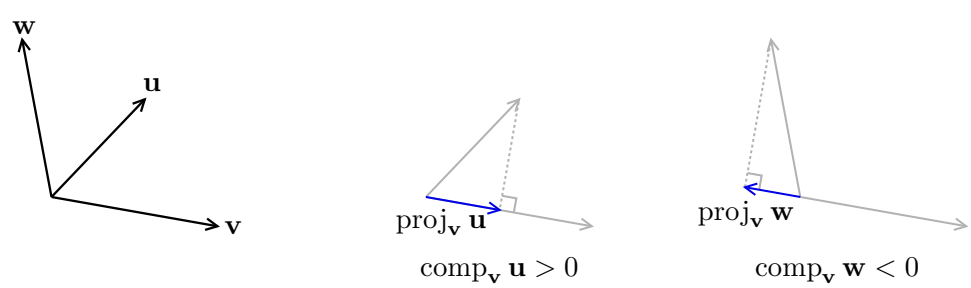
$$\operatorname{proj}_{\mathbf{v}}\mathbf{u} = \frac{\mathbf{u}\cdot\mathbf{v}}{\mathbf{v}\cdot\mathbf{v}}\mathbf{v}$$

is called the **orthogonal projection** or **vector projection** of $\mathbf{u}$ onto $\mathbf{v}$, and the scalar

$$\operatorname{comp}_{\mathbf{v}}\mathbf{u} = \frac{\mathbf{u}\cdot\mathbf{v}}{|\mathbf{v}|}$$

is the called the **scalar projection** of $\mathbf{u}$ onto $\mathbf{v}$, or the **component** of $\mathbf{u}$ in the direction of $\mathbf{v}$.

The vector projection $\operatorname{proj}_{\mathbf{v}}\mathbf{u}$ is the "shadow" of $\mathbf{u}$ cast on the line spanned by $\mathbf{v}$ by a ray of light orthogonal to $\mathbf{v}$. The scalar projection $\operatorname{comp}_{\mathbf{v}}\mathbf{u}$ is the signed length of $\operatorname{proj}_{\mathbf{v}}\mathbf{u}$.



$\operatorname{comp}_{\mathbf{v}}\mathbf{u} > 0$	iff $\mathbf{u}\cdot\mathbf{v} > 0$	iff $\operatorname{proj}_{\mathbf{v}}\mathbf{u}$ is in same direction as $\mathbf{v}$
$\operatorname{comp}_{\mathbf{v}}\mathbf{u} < 0$	iff $\mathbf{u}\cdot\mathbf{v} < 0$	iff $\operatorname{proj}_{\mathbf{v}}\mathbf{u}$ is in opposite direction as $\mathbf{v}$

12.3.re2. Let

$$\mathbf{u} = \langle 1, 2, 2\rangle \qquad \mathbf{v} = \langle 3, 0, -2\rangle \qquad \mathbf{w} = \langle 2, 1, -1\rangle \qquad \mathbf{p} = \langle -2, 1, 0\rangle$$

and find the following, if they exist.

a. $\operatorname{proj}_{\mathbf{u}} \mathbf{v}$ b. $\operatorname{comp}_{\mathbf{u}} \mathbf{v}$ c. $\operatorname{proj}_{\mathbf{v}} \mathbf{u}$ d. $\operatorname{comp}_{\mathbf{v}} \mathbf{u}$
e. $\operatorname{proj}_{\mathbf{u}} \mathbf{w}$ f. $\operatorname{proj}_{\mathbf{u}} \mathbf{p}$ g. $\mathbf{u} \cdot (\mathbf{w} - \operatorname{proj}_{\mathbf{u}} \mathbf{w})$ h. $\mathbf{w} \cdot (\mathbf{w} - \operatorname{comp}_{\mathbf{u}} \mathbf{w})$

12.3.re3. Use orthogonal projection to find the point on the line closest to the given point.

a. $(2, 3)$, $y = x$ b. $(1, -1)$, $2y + 3x = 0$

Work

The work done by a constant force $\mathbf{F}$ moving an object along a straight line is

$$W = \mathbf{F} \cdot \mathbf{D}$$

where $\mathbf{D}$ is the change in position, or **displacement**, of the object.

12.3.re4. Find the work done by a force of magnitude 3 N in the direction of $\langle 8, -4, 1\rangle$ in moving an object in a straight line from the point $(1, 0, 1)$ to the point $(3, 1, -1)$. (Assume coordinates are measured in meters).

Answers

12.3.re2a. $\langle -\frac{1}{9}, -\frac{2}{9}, -\frac{2}{9}\rangle$. 12.3.re2b. $-\frac{1}{3}$. 12.3.re2c. $\langle -\frac{3}{13}, 0, \frac{2}{13}\rangle$. 12.3.re2d. $-\frac{1}{\sqrt{13}}$.
12.3.re2e. $\langle \frac{2}{9}, \frac{4}{9}, \frac{4}{9}\rangle$. 12.3.re2f. $\mathbf{0}$. 12.3.re2g. 0. 12.3.re2h. dne. 12.3.re3a. $(\frac{5}{2}, \frac{5}{2})$.
12.3.re3b. $(\frac{10}{13}, -\frac{15}{13})$. 12.3.re4. $\mathbf{F} = \frac{3}{9}\langle 8, -4, 1\rangle$. $\mathbf{D} = \langle 2, 1, -2\rangle$. Work $= \frac{10}{3}$ ft-lbs.

12.4: The Cross Product

Definition. The **cross product** of the vectors $\mathbf{u} = \langle u_1, u_2, u_3\rangle$ and $\mathbf{w} = \langle w_1, w_2, w_3\rangle$ is the determinant

$$\mathbf{u} \times \mathbf{w} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ u_1 & u_2 & u_3 \\ w_1 & w_2 & w_3 \end{vmatrix},$$

which can be calculated by expansion along the top row:

$$= \mathbf{i}\begin{vmatrix} u_2 & u_3 \\ w_2 & w_3 \end{vmatrix} - \mathbf{j}\begin{vmatrix} u_1 & u_3 \\ w_1 & w_3 \end{vmatrix} + \mathbf{k}\begin{vmatrix} u_1 & u_2 \\ w_1 & w_2 \end{vmatrix}$$

12.4.re1. $(2\mathbf{i} + \mathbf{j} - 3\mathbf{k}) \times (-\mathbf{i} - \mathbf{j} + \mathbf{k}) =$

$$\begin{aligned}\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & 1 & -3 \\ -1 & -1 & 1 \end{vmatrix} &= \mathbf{i}\begin{vmatrix} 1 & -3 \\ -1 & 1 \end{vmatrix} - \mathbf{j}\begin{vmatrix} 2 & -3 \\ -1 & 1 \end{vmatrix} + \mathbf{k}\begin{vmatrix} 2 & 1 \\ -1 & -1 \end{vmatrix} \\ &= \mathbf{i}(1\cdot 1 - (-1)(-3)) - \mathbf{j}(2\cdot 1 - (-1)(-3)) + \mathbf{k}(2(-1) - (-1)1) \\ &= \langle -2, 1, -1\rangle\end{aligned}$$

12.4.re2. Find the following, if they exist.

a. $\langle 2, -4, 1\rangle \times \langle 1, 0, 1\rangle$ b. $\langle 1, 0, 1\rangle \times \langle 2, -4, 1\rangle$ c. $\langle 2, -4, 1\rangle \times \langle -4, 8, -2\rangle$
d. $\langle 2, -4, 1\rangle \times \mathbf{k}$ e. $\langle 2, -4, 1\rangle \times \mathbf{i}$ f. $\mathbf{i} \times \mathbf{j}$

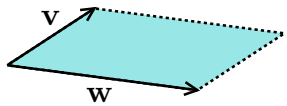
If $\mathbf{u}$, $\mathbf{v}$, $\mathbf{w}$ are vectors and s is a scalar and θ is the angle between $\mathbf{u}$ and $\mathbf{v}$, then

a. $\mathbf{u}\cdot(\mathbf{u}\times\mathbf{v}) = \mathbf{v}\cdot(\mathbf{u}\times\mathbf{v}) = 0$
b. $|\mathbf{u}\times\mathbf{v}| = |\mathbf{u}||\mathbf{v}|\sin\theta$
c. $\mathbf{u}\times\mathbf{v} = -(\mathbf{v}\times\mathbf{u})$
d. $(s\mathbf{u})\times\mathbf{v} = s(\mathbf{u}\times\mathbf{v}) = \mathbf{u}\times(s\mathbf{v})$
e. $\mathbf{u}\times(\mathbf{v}+\mathbf{w}) = \mathbf{u}\times\mathbf{v} + \mathbf{u}\times\mathbf{w}$
f. $\mathbf{u}$, $\mathbf{v}$, $\mathbf{u}\times\mathbf{v}$ is a right-handed system,

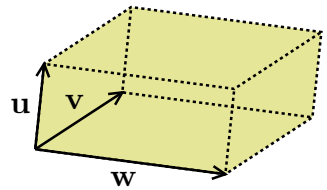
a. means that $\mathbf{u}\times\mathbf{v}$ is orthogonal to both $\mathbf{u}$ and $\mathbf{v}$. b. implies that $|\mathbf{u}\times\mathbf{v}|$ = the area of the parallelogram with sides $\mathbf{u}$ and $\mathbf{v}$, and that $\mathbf{u}\times\mathbf{v} = \mathbf{0}$ iff $\mathbf{u} = \mathbf{0}$, $\mathbf{v} = \mathbf{0}$, or $\mathbf{u} \parallel \mathbf{v}$.
f. means that when you point to $\mathbf{u}$ with your open right hand, and then curl your fingers closed in the direction of $\mathbf{v}$, your thumb points in the direction of $\mathbf{u}\times\mathbf{v}$. For instance, $\mathbf{i}$, $\mathbf{j}$, $\mathbf{k}$ is a right-handed system.
The **triple product** of the vectors $\mathbf{u} = \langle u_1, u_2, u_3\rangle$, $\mathbf{v} = \langle v_1, v_2, v_3\rangle$, and $\mathbf{w} = \langle w_1, w_2, w_3\rangle$ is

$$\mathbf{u}\cdot(\mathbf{v}\times\mathbf{w}) = \begin{vmatrix} u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{vmatrix}$$

$|\mathbf{u}\cdot(\mathbf{v}\times\mathbf{w})|$ is the volume of the parallelepiped with sides $\mathbf{u}$, $\mathbf{v}$, $\mathbf{w}$.



Area $= |\mathbf{v} \times \mathbf{w}|$



Volume $= |\mathbf{u} \cdot (\mathbf{v} \times \mathbf{w})|$

12.4.re3. Find the area of the parallelogram with the given vertices.

a. $(0,0,0)$, $(3,1,2)$, $(2,-1,4)$, $(5,0,6)$

b. $(7,-1)$, $(12,2)$, $(3,1)$, $(8,4)$

12.4.re4. Find the volume of the parallelepiped with edges $\mathbf{u}$, $\mathbf{v}$, $\mathbf{w}$.

$$\mathbf{u} = \mathbf{i} + \mathbf{j} \quad \mathbf{v} = \mathbf{j} - \mathbf{k} \quad \mathbf{w} = \mathbf{i} - 2\mathbf{j} + \mathbf{k}$$

12.4.re5. Let

$$\mathbf{u} = \langle 1,2,2 \rangle \qquad \mathbf{v} = \langle 3,0,-2 \rangle \qquad \mathbf{w} = \langle 2,1,-1 \rangle$$

and find the following, if they exist.

a. $\mathbf{u} \times \mathbf{v}$ b. $\mathbf{v} \times \mathbf{w}$ c. $\mathbf{v} \times (\mathbf{w} \times \mathbf{w})$ d. $\mathbf{u} \cdot (\mathbf{v} \times \mathbf{w})$

e. $\mathbf{u} \cdot (\mathbf{v} \times \mathbf{u})$ f. $(\mathbf{u} \cdot \mathbf{v}) \cdot (\mathbf{v} \cdot \mathbf{w})$ g. $(\mathbf{u} \times \mathbf{v}) \times (\mathbf{v} \times \mathbf{w})$ h. $\operatorname{comp}_{\mathbf{v} \times \mathbf{w}} \mathbf{u}$

Torque

The torque of a force vector $\mathbf{F}$ and position vector $\mathbf{r}$ is defined to be $\vec{\tau} = \mathbf{r} \times \mathbf{F}$. Torque can be thought of as the magnitude and direction of a turning force acting on a (right-handed) bolt at the origin when the force $\mathbf{F}$ is applied to a wrench $\mathbf{r}$.

12.4.re6. A wrench 0.5 m long lies along the line $y = x$ in quadrant I in the xy-plane and grips a bolt at the origin. A force of magnitude 2 N in the direction $3\mathbf{i} + 4\mathbf{j}$ applied to the end of the wrench. Find the magnitude of the torque applied to the bolt.

Answers

12.4.re2a. $\langle -4,-1,4 \rangle$. 12.4.re2b. $\langle 4,1,-4 \rangle$. 12.4.re2c. $\mathbf{0}$. 12.4.re2d. $\langle -4,-2,0 \rangle$. 12.4.re2e. $\langle 0,1,4 \rangle$. 12.4.re2f. $\mathbf{k}$. 12.4.re3a. $5\sqrt{5}$. 12.4.re3b. 22. 12.4.re4. 2. 12.4.re5a. $\langle -4,8,-6 \rangle$. 12.4.re5b. $\langle 2,-1,3 \rangle$. 12.4.re5c. $\mathbf{0}$. 12.4.re5d. 6. 12.4.re5e. 0. 12.4.re5f. dne. 12.4.re5g. $\langle 18,0,-12 \rangle$. 12.4.re5h. $6/\sqrt{14}$. 12.4.re6. $\frac{1}{5\sqrt{2}}$ Nm.

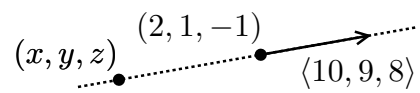
12.5: Equations of lines and planes

Lines

A line is determined by a point on the line a vector parallel the line.

12.5.re1. Find the equation(s) of the line passing through $(2, 1, -1)$ parallel to $\langle 10, 9, 8\rangle$.

Solution one: a point (x, y, z) lies on the line iff the vector from $(2, 1, -1)$ to (x, y, z) is parallel to $\langle 10, 9, 8\rangle$:



$$\langle x, y, z\rangle - \langle 2, 1, -1\rangle = t\langle 10, 9, 8\rangle$$
$$\langle x, y, z\rangle = \langle 2, 1, -1\rangle + t\langle 10, 9, 8\rangle$$

for some value of t, so the vector-valued function

$$\mathbf{r}(t) = \langle 2, 1, -1\rangle + t\langle 10, 9, 8\rangle,$$

or $\langle 2 + 10t, 1 + 9t, -1 + 8t\rangle$, will trace out the line as t goes from $-\infty$ to ∞. This is called a **(parametric) vector form** of the line.

Solution two: setting x, y, and z equal the components of $\mathbf{r}$ gives

$$x = 2 + 10t \qquad y = 1 + 9t \qquad z = -1 + 8t$$

Together, these three equations are called a **parametric form** of the line.

Solution three: solving for t in terms of x, y, and z and setting these expressions equal gives

$$\frac{x-2}{10} = \frac{y-1}{9} = \frac{z+1}{8}$$

This pair of equations is called a **symmetric form** of the line.

12.5.re2. Find an equation of the given line.

a. Through $(2, 0, 3)$, parallel $\langle 1, 2, 3\rangle$.
b. Through the points $(2, 0, 3)$ and $(9, 1, 5)$.
c. Through $(2, 8, 3)$, parallel $\langle 0, 4, 5\rangle$.

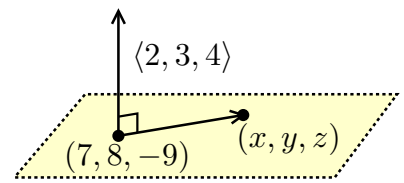
12.5.re3. Find a point on and a vector parallel to the given line. (There are many correct answers.)

a. $x = 2 - t,\ y = 10 - t,\ z = 2 + 5t$.
b. $\frac{x-3}{4} = y - 2 = 2z + 5$
c. $\mathbf{r} = \langle 2 + 3t, 3 - t, 8\rangle$.

Planes

A plane is determined by a point on the plane a vector orthogonal to the plane (called a **normal vector**).

12.5.re4. Find an equation of the plane passing through $(7, 8, -9)$ normal to $\langle 2, 3, 4\rangle$.



Solution: a point (x, y, z) lies on the plane iff the vector from $(7, 8, -9)$ to (x, y, z) is orthogonal to $\langle 2, 3, 4\rangle$:

$$\langle 2, 3, 4\rangle \perp \langle x - 7, y - 8, z + 9\rangle,$$

which is true iff their dot product is zero. Therefore, the plane is the solution set to the equation

$$2(x - 7) + 3(y - 8) + 4(z + 9) = 0.$$

There are other equations for the same plane, obtainable from this one by some algebra, e.g.

$$2x + 3y + 4z = 2.$$

12.5.re5. Find an equation of the given plane.

a. Through the point $(2, 1, 0)$ and normal to $\langle 1, 2, 3\rangle$.
b. Through the points $(2, 1, 0)$, $(3, 2, 1)$, and $(9, 1, 5)$.
c. Through the points $(2, 1, 0)$ and parallel the plane $x - 5z = 10$.
d. The plane containing $(0, 1, -1)$ and the line $x = 1 + t,\ y = 2t - 1,\ z = 3t$.

12.5.re6. Find a point on and a vector normal to the given plane. (There are many correct answers.)

a. $3x + 10y + 5z = 6$. b. $2x + 1 = 4y - z$

12.5.re7. Does the given equation(s) describe a line or a plane?

a. $x = 4t,\ y = 2 - t,\ z = 5 - 6t$. b. $\frac{x-2}{3} = 2y - 2 = \frac{z+1}{5}$ c. $4x - y - 6z = 0$

12.5.re8. Find the point of intersection, if there is one.

a. The line $x = 1 - t,\ y = 2t - 1,\ z = 3 + 2t$ and the plane $2x + 3y - z = -6$
b. The lines $x = 1 + 3t,\ y = 2 - 4t,\ z = 4 + t$ and $x = 3 + 2s\ y = -s + 1,\ z = -2s + 1$.
c. The lines $x = 7 + 3t,\ y = -6 - 4t,\ z = 6 + t$ and $x = -1 + 2s\ y = s + 1,\ z = 3s + 1$.

Answers

12.5.re2a. vector form: $\mathbf{r} = \langle 2 + t, 2t, 3 + 3t\rangle$. symmetric form: $x - 2 = \frac{y}{2} = \frac{z-3}{3}$. 12.5.re2b. vector form: $\mathbf{r} = \langle 2 + 7t, t, 3 + 2t\rangle$. symmetric form: $\frac{x-2}{7} = y = \frac{z-3}{2}$. 12.5.re2c. vector form: $\mathbf{r} = \langle 2, 8 + 4t, 3 + 5t\rangle$. symmetric form: $x = 2$; $\frac{y-8}{4} = \frac{z-3}{5}$. 12.5.re3a. $(2, 10, 2)$, $\langle -1, -1, 5\rangle$. 12.5.re3b. $(3, 2, -\frac{5}{2})$, $\langle 4, 1, \frac{1}{2}\rangle$. 12.5.re3c. $(5, 2, 8)$ (when $t = 1$), $\langle 3, -1, 0\rangle$. 12.5.re5a. $x - 2 + 2(y - 1) + 3z = 0$, or $x + 2y + 3z = 4$. 12.5.re5b. $5x+2y-7z = 12$. 12.5.re5c. $x-5z = 2$. 12.5.re5d. $4x+y-2z = 3$. 12.5.re6a. $(2, 0, 0)$, $\langle 3, 10, 5\rangle$. 12.5.re6b. $(1, 0, -3)$, $\langle 2, -4, 1\rangle$. 12.5.re7a. line (in parametric form). 12.5.re7b. line (in symmetric form). 12.5.re7c. plane. 12.5.re8a. $(x, y, z) = (2, -3, 1)$ (at $t = -1$). 12.5.re8b. none. 12.5.re8c. both $= (1, 2, 4)$ at $t = -2$, $s = 1$.

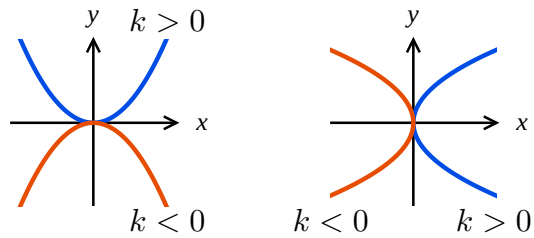
12.6: Cylinders and Quadratic Surfaces

See http://kunklet.people.cofc.edu/MATH111/transformations.pdf for a review of how changes to an equation change the corresponding graph.

Elementary conic sections in the xy-plane

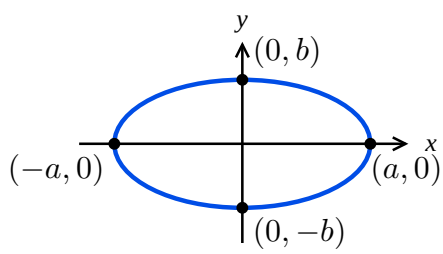
1. Parabolas

$y = kx^2$ $\qquad$ $x = ky^2$



2. Ellipses

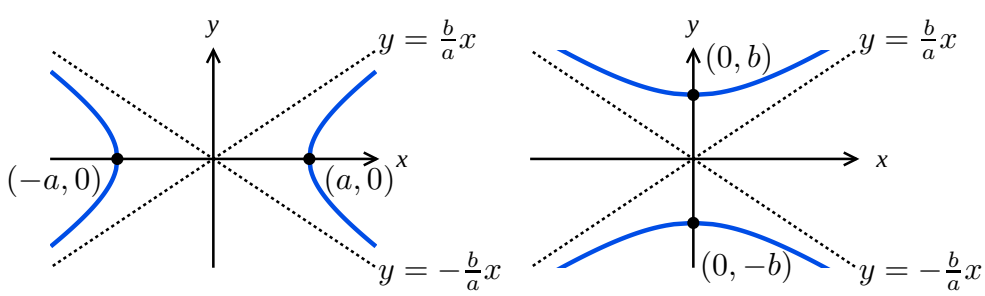
$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$



semimajor axis $= \max\{a, b\}$
semiminor axis $= \min\{a, b\}$

3. Hyperbolas

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1 \qquad\qquad \frac{y^2}{b^2} - \frac{x^2}{a^2} = 1$$



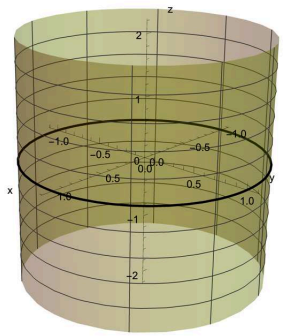
$(x \neq 0)$ $\qquad$ $(y \neq 0)$

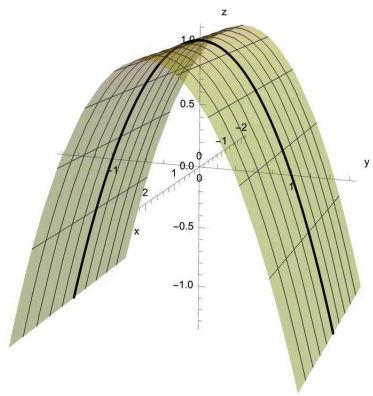
For practice with conics, see http://kunklet.people.cofc.edu/MATH221/stew1005prob.pdf.

Cylinders

A cylinder is a surface obtained by dragging a planar curve in the direction perpendicular to its plane. Any equation in x, y, or x, z, or y, z generates a cylinder in xyz space.

12.6.re1. The graph of $x^2 + y^2 = 1$ is a circle in the xy plane, where $z = 0$. Since the equation is independent of z, its graph is the (right circular) cylinder made up of copies of the same circle at all other z-values. (below left)





12.6.re2. The graph of $z = 1 - y^2$ is a parabola in the yz-plane $x = 0$. Dragging this curve in the x-direction generates the graph of the equation in xyz-space. (above right)

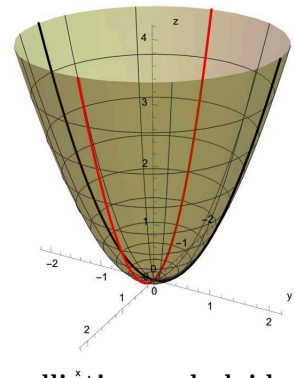
12.6.re3. Sketch the graph of the given equation.

a. $z = \sin y$ b. $xy = -1$ c. $y^2 - z^2 = 4$ d. $x = 2z - z^2$

Quadratic surfaces

12.6.re4. $z = x^2 + y^2$

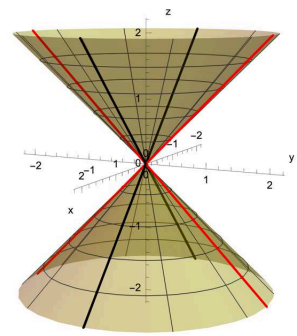
When	equation is	cross-section is
$x = 0$	$z = y^2$	a parabola;
$y = 0$	$z = x^2$	a parabola;
$z = \text{const.}$	$x^2 + y^2 = \text{const.}$	a circle.



elliptic paraboloid

12.6.re5. $x^2 + y^2 = z^2$

When	equation is	cross-section is
$x = 0$	$y^2 = z^2$	a pair of lines ($z = \pm y$);
$y = 0$	$x^2 = z^2$	a pair of lines ($z = \pm x$);
$z = \text{const.}$	$x^2 + y^2 = \text{const.}$	a circle.

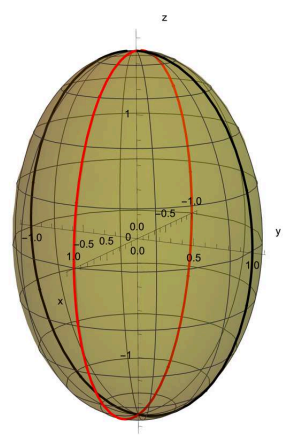


elliptical cone

12.6.re6. $x^2 + y^2 + \dfrac{4}{9}z^2 = 1$

When	equation is	cross-section is
$x = 0$	$y^2 + \frac{4}{9}z^2 = 1$	an ellipse;
$y = 0$	$x^2 + \frac{4}{9}z^2 = 1$	an ellipse;
$z = \text{const.}$	$x^2 + y^2 = \text{const.}$	a circle.

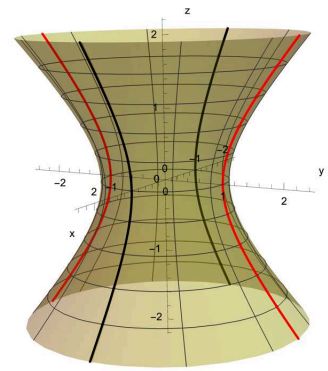
Tip: graph of $x^2+y^2+\left(\frac{z}{3/2}\right)^2 = 1$ is obtained from unit sphere $x^2 = y^2 = z^2 = 1$ by scaling in z-direction by a factor of $3/2$.



ellipsoid

12.6.re7. $x^2 + y^2 - z^2 = 1$

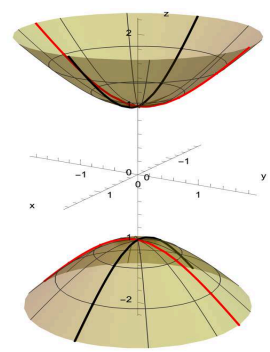
When	equation is	cross-section is
$x = 0$	$y^2 - z^2 = 1$	a hyperbola ($y \neq 0$);
$y = 0$	$x^2 - z^2 = 1$	a hyperbola ($x \neq 0$);
$z = \text{const.}$	$x^2 + y^2 = \text{const.}$	a circle.



hyperboloid of one sheet

12.6.re8. $-x^2 - y^2 + z^2 = 1$

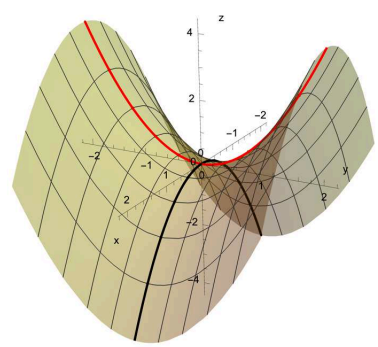
When	equation is	cross-section is
$x = 0$	$-y^2 + z^2 = 1$	a hyperbola ($z \neq 0$);
$y = 0$	$-x^2 + z^2 = 1$	a hyperbola ($z \neq 0$);
$z = \text{const.}$	$x^2 + y^2 = \text{const.}$	a circle.



hyperboloid of two sheets

12.6.re9. $z = y^2 - x^2$

When	equation is	cross-section is
$x = \text{const.}$	$z = y^2 + \text{const.}$	a parabola;
$y = \text{const.}$	$z = -x^2 + \text{const.}$	a parabola;
$z = \text{const.}$	$-x^2 + y^2 = \text{const.}$	a hyperbola.



hyperbolic parabolid

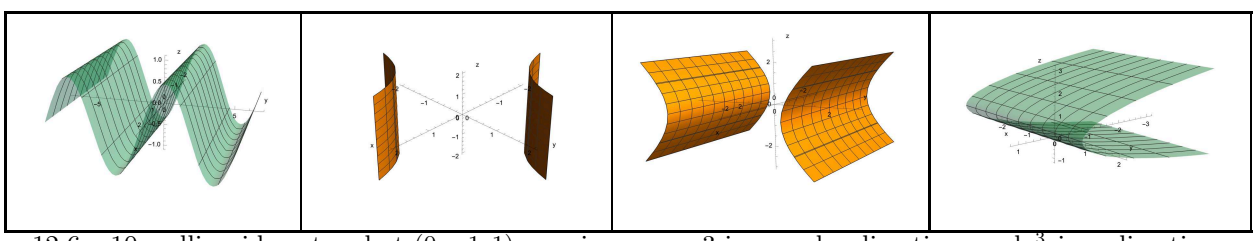
See the Table 1 in 12.6 for a summary of quadratic surfaces in general. These six quadratic surfaces can always be draw in the positions shown above after shifting (possibly by completing the square) and rotating axes (as in example 12.1.re5)

12.6.re10. Describe and sketch the graph of the given equation.

a. $x^2 + 4y^2 + 8y + z^2 - 2z = 4$	b. $4x^2 - z^2 = y^2$	c. $-x^2 + y^2 - z^2 = 9$
d. $-x^2 + y^2 + 2y - z^2 = 0$	e. $x^2 + 4y^2 = 4$	f. $-x^2 + y^2 + z^2 = 9$
g. $-x^2 + 4x + y^2 - z^2 = 3$	h. $x^2 = z^2 - y$	i. $2 = -x + y^2 + 4z^2$
j. $0 = x^2 - 4y^2 + 4z^2$	k. $0 = x^2 - 4z^2$	

Answers

12.6.re3. a.,b.,c.,d.:



12.6.re10a. ellipsoid centered at $(0, -1, 1)$. semi-axes are 3 in x and z directions and $\frac{3}{2}$ in y direction. 12.6.re10b. circular cone; axis of symmetry is x-axis 12.6.re10c. hyperboloid; $y \neq 0$, two sheets; axis is y-axis. 12.6.re10d. hyperboloid; $y \neq -1$ two sheets; centered at $(0, -1, 0)$. 12.6.re10e. equation without z, hence a cylinder. semi-axis in x direction is 2; in y direction is 1. axis of symmetry is z-axis. 12.6.re10f. hyperboloid; y and z not both $= 0$, so of one sheet; axis is x-axis 12.6.re10g. hyperboloid of one sheet (x, z not both 0) centered at $(2, 0, 0)$. 12.6.re10h. hyperbolic paraboloid. includes the cross-sections $x = 0; y = z^2$ and $z = 0; y = -x^2$. 12.6.re10i. elliptic paraboloid with axis of symmetry x-axis and vertex $(-2, 0, 0)$. $x + 2 \geq 0$. 12.6.re10j. elliptic cone with axis of symmetry y-axis. Elliptical cross-sections at $y =$ const. are twice as long in x direction as in z direction. 12.6.re10k. equation without y, hence a cylinder; in fact, it consists of the two planes $x = 2z$ and $x = -2z$.

13.1: Vector-valued functions and the representation of curves by equations

Vector-valued functions

When $x = x(t)$, $y = y(t)$, and $z = z(t)$ are scalar-valued functions of the scalar variable t, then $\mathbf{r}(t) = \langle x(t), y(t), z(t)\rangle$ is a **vector-valued function** of t. Its domain is the set of t-values at which $x(t)$, $y(t)$, and $z(t)$ are all well-defined.

13.1.re1. Find the domain of vector-valued function.

a. $\langle \sqrt{t-2}, \ln(5-t)\rangle$ b. $\langle e^t, \frac{t+1}{t-1}, \sqrt{t}\rangle$ c. $\langle \frac{1}{\sqrt{2t-3}}, \sin(t^2), \frac{2t-1}{t^2-5t+6}\rangle$

Limits of vector-valued functions are computed component-wise.

13.1.re2. $\lim_{t\to 1}\langle e^{t-1}, \frac{t^2-1}{t-1}, \frac{\ln t}{t^2-1}\rangle = \langle \lim_{t\to 1} e^{t-1}, \lim_{t\to 1}\frac{t^2-1}{t-1}, \lim_{t\to 1}\frac{\ln t}{t^2-1}\rangle$.
The first of these three limits equals 1 by continuity. The second is the same as $\lim_{t\to 1}\frac{(t+1)(t-1)}{t-1} = 2$. The third, by l'Hospital's Rule, is $\lim_{t\to 1}\frac{t^{-1}}{2t} = \frac{1}{2}$. Therefore, the limit of $\mathbf{r}(t)$ is $\langle 1, 2, \frac{1}{2}\rangle$.

13.1.re3. Calculate the limit

a. $\lim_{t\to 0}\langle \frac{\sin t}{t}, \frac{1-\cos t}{t}, \frac{1-\cos t}{t^2}\rangle$

b. $\lim_{t\to 4}\langle \sin(t\pi), \cos\big((t+1)\frac{\pi}{2}\big), \frac{2-\sqrt{t}}{t-4}\rangle$

c. $\lim_{h\to 0}\langle \frac{(t+h)^3-t^3}{h}, \frac{\ln(t+h)-\ln t}{h}, \frac{e^{-t-h}-e^{-t}}{h}\rangle$

(Hint for part c: what is $\lim_{h\to 0}\frac{f(t+h)-f(t)}{h}$?)

Representations of curves by equations

	PARAMETRIC EQUATIONS	IMPLICIT EQUATIONS
$\mathbb{R}^2$	$x = x(t)$ $y = y(t)$	$f(x, y) = 0$
$\mathbb{R}^3$	$x = x(t)$ $y = y(t)$ $z = z(t)$	$f(x, y, z) = 0$ $g(x, y, z) = 0$

13.1.re4. The unit circle in $\mathbb{R}^2$ can be expressed implicitly by $x^2+y^2 = 1$ and parametrically by $x = \cos t$, $y = \sin t$.

We sometimes express the parametric functions for x, y, and z along a curve in a single vector-valued function $\mathbf{r}(t)$, as in the next example.

13.1.re5. The line in $\mathbb{R}^3$ passing through the point $(0, 1, -2)$ parallel to $\langle 3, -4, 5\rangle$ can be expressed parametrically by

$$x = 3t, \qquad y = 1 - 4t, \qquad z = -2 + 5t,$$

or

$$\mathbf{r}(t) = \langle 3t, 1 - 4t, -2 + 5t\rangle.$$

The same line can be expressed implicitly by the two equations of its symmetric form

$$\frac{x}{3} = \frac{y-1}{-4} = \frac{z+2}{5},$$

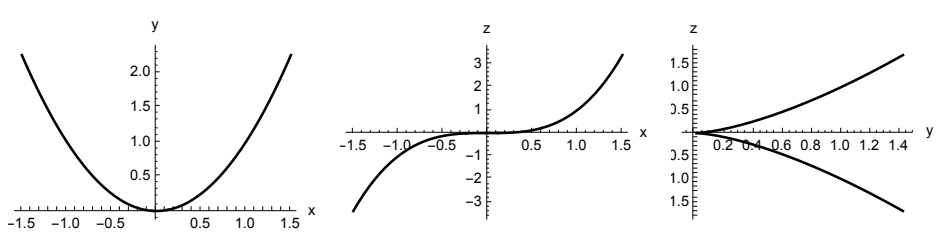
or, if you prefer, $\frac{x}{3} + \frac{y-1}{4} = 0$, $\frac{y-1}{4} + \frac{z+2}{5} = 0$.

Sketching curves in space

Sketching curves in space by hand is a worthwhile exercise (though, in practice, best left to machines). It often helps to identify the equation of a surface to which the curve belongs, that is, one of the equations of its implicit representation.

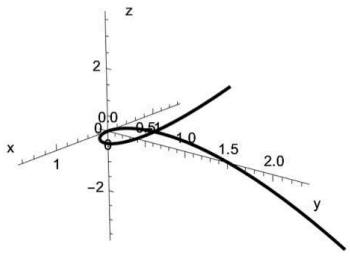
13.1.re6. Sketch the curve given parametrically by $\langle t, t^2, t^3\rangle$.

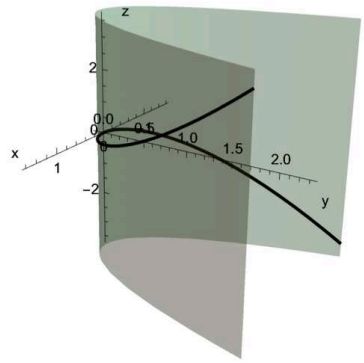
It's difficult to capture the shape of this curve in a single drawing. It might help to eliminate the parameter t to obtain an xy equation, an xz equation, and a yz equation. Then draw these curves in the three coordinate planes. These are views of the curve from the positive z-, negative y- and positive x-axes.



$y = x^2$ $\qquad$ $z = x^3$ $\qquad$ $y = \sqrt[3]{z^2}$

Based on these, we produce a sketch like the graph below left. To make the drawing clearer, include the cylinder $y = x^2$ on which the curve lies, below right.





13.1.re7. Describe and sketch the given parametrically by $\mathbf{r}(t)$.

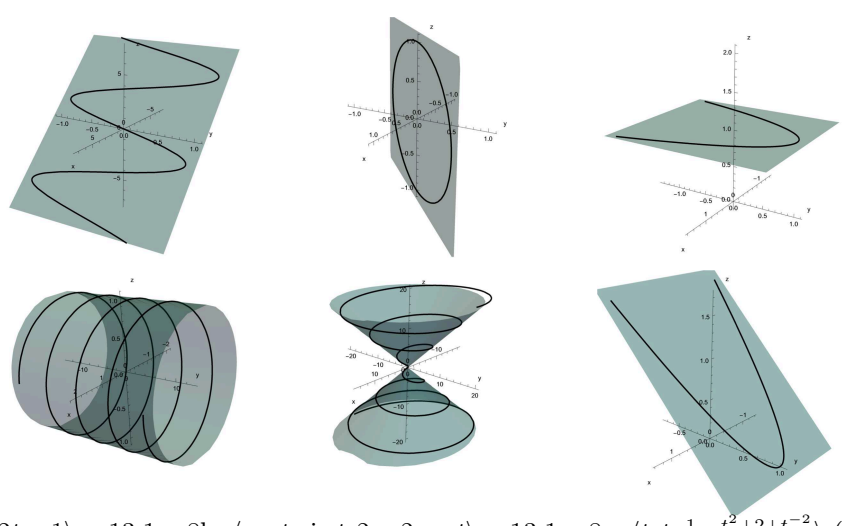
a. $\langle t, \sin t, -t\rangle$ b. $\langle \sin t, \sin t, -\cos t\rangle$ c. $\langle t, 1-t^2, 1\rangle$
d. $\langle 2\cos t, t, \sin t\rangle$ e. $\langle t\sin t, t\cos t, t\rangle$ f. $\langle t, 1-t^2, t^2\rangle$

13.1.re8. Find a parametric representation of the curve given implicitly by the system of equations.

a. $x + y = 1,\ x^2 - y^2 = z$ b. $z = (x-1)^2 + y^2,\ x^2 + y^2 = 1$
c. $xy = 1,\ z = e^{(x+y)^2}$ d. $(x-3)^2 + z^2 = 1,\ x^2 - y^2 + z^2 = 2,\ y > 0$

Answers

13.1.re1a. $[2, 5)$. 13.1.re1b. $[0, 1) \cup (1, \infty)$. 13.1.re1c. $(\frac{3}{2}, 2) \cup (2, 3) \cup (3, \infty)$. 13.1.re3a. $\langle 1, 0, \frac{1}{2}\rangle$. 13.1.re3b. $\langle 0, 0, -\frac{1}{4}\rangle$. 13.1.re3c. $\langle 3t^2, \frac{1}{t}, -e^{-t}\rangle$. 13.1.re7a. A sinusoidal curve long the line $x + z = 0$; $y = 0$. 13.1.re7b. An ellipse in the plane $x = y$ whose shadow in the xz-plane is the unit circle. 13.1.re7c. The parabola $y = 1 - x^2$ in the plane $z = 1$. 13.1.re7d. A helix on the elliptical cylinder $\frac{1}{4}x^2 + z^2 = 1$. 13.1.re7e. A helix on the cone $x^2 + y^2 = z^2$. 13.1.re7f. The parabola in the plane $y + z = 1$. Graphs a-f below.



13.1.re8a. $\langle t, 1-t, 2t-1\rangle$. 13.1.re8b. $\langle \cos t, \sin t, 2 - 2\cos t\rangle$. 13.1.re8c. $\langle t, t^{-1}, e^{t^2+2+t^{-2}}\rangle$ $(t \neq 0)$. 13.1.re8d. $\langle 3 + \cos t, \sqrt{8 + 6\cos t}, \sin t\rangle$.

13.2: Calculus on vector-valued functions

Differentiation

If $\mathbf{r}(t) = \langle x(t), y(t), z(t)\rangle$, then $\frac{d\mathbf{r}}{dt} = \left\langle \frac{dx}{dt}, \frac{dy}{dt}, \frac{dz}{dt} \right\rangle$.

13.2.re1. $\frac{d}{dt}\langle e^t, \sec t, \tan^{-1} t\rangle = \langle e^t, \sec t \tan t, \frac{1}{t^2+1}\rangle$

See page 858 of of text for important rules of vector differentiation.

13.2.re2. Find the derivative of vector-valued function.

a. $\langle \sqrt{t-2}, \ln|t-5|\rangle$ b. $\langle e^t, \frac{t+1}{t-1}, \sqrt{t^2+1}\rangle$ c. $\langle \frac{1}{\sqrt[3]{3t-1}}, \sinh(t^2), \cosh^2 t\rangle$

Integration

Like differentiation, integration of vector-valued functions is performed component-wise. The Fundatmental Theorem of Calculus for vector-valued functions says that

$$\int_a^b \frac{d\mathbf{r}}{dt}\, dt = \mathbf{r}(b) - \mathbf{r}(a),$$

provided $\frac{d\mathbf{r}}{dt}$ is continuous.

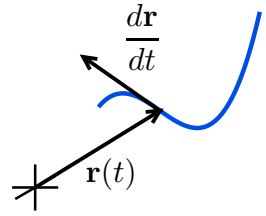
13.2.re3.
a. $\int \langle t^2+1, \sec t\rangle\, dt = \langle \frac{1}{3}t^3 + t + C_1, \ln|\sec t + \tan t| + C_2\rangle$, or $\langle \frac{1}{3}t^3 + t, \ln|\sec t + \tan t|\rangle + \mathbf{C}$, where $\mathbf{C}$ is a constant vector in $\mathbb{R}^2$.
b. $\int_{-1}^1 \langle t^2+t, 2te^{-t^2}\rangle\, dt = \langle \frac{1}{3}t^3 + \frac{1}{2}t^2, -e^{-t^2}\rangle\Big|_{-1}^{1} = \langle \frac{1}{3}+\frac{1}{2}, -e^{-1}\rangle - \langle -\frac{1}{3}+\frac{1}{2}, -e^{-1}\rangle = \langle \frac{2}{3}, 0\rangle$

13.2.re4. Integrate.

a. $\int \langle \frac{t}{t^2-4}, \sin t \cos^3 t, te^t\rangle\, dt$ b. $\int \langle \frac{1}{t^2-4}, e^x \sec^2(e^x), \tan t\rangle\, dt$
c. $\int_0^1 \langle \frac{1}{t^2+1}, \sin(\pi t), (t+1)^4\rangle\, dt$

Tangent & unit tangent vectors

If $\dfrac{d\mathbf{r}}{dt} \neq \mathbf{0}$, then $\dfrac{d\mathbf{r}}{dt}$ is a tangent vector to the curve parametrized by $\mathbf{r}$, and



$$\mathbf{T} = \frac{\mathbf{r}'(t)}{|\mathbf{r}'(t)|}$$

is the **unit tangent** vector to the curve.

The vector-valued function $\mathbf{r}$ is said to be **smooth** if $\dfrac{d\mathbf{r}}{dt}$ is continuous and never equal $\mathbf{0}$.

13.2.re5. For the given $\mathbf{r}$, find $\frac{d\mathbf{r}}{dt}$, $\mathbf{T}$, and the line tangent to the curve parametrized by $\mathbf{r}$ at the point corresponding to the given time.

a. $\mathbf{r} = \langle t^2+1, t^3-t, t\rangle$, $t=1$

b. $\mathbf{r} = \langle \frac{1}{2}t^2, \ln|t|\rangle$, $t=-1$

c. $\mathbf{r} = e^t\mathbf{i} + e^t\sin t\mathbf{j} + e^t\cos t\mathbf{k}$, $t=0$

Answers

13.2.re2a. $\langle \frac{1}{2}(t-2)^{-1/2}, (t-5)^{-1}\rangle$. 13.2.re2b. $\langle e^t, -2(t-1)^{-2}, t(t^2+1)^{-1/2}\rangle$. 13.2.re2c. $\langle -(3t-1)^{-4/3}, 2t\cosh(t^2), 2\cosh t\sinh t\rangle$. 13.2.re4a. $\langle \frac{1}{2}(\ln|t^2-4|, -\frac{1}{4}\cos^4 t, te^t - e^t\rangle + \mathbf{C}$. 13.2.re4b. $\langle \frac{1}{4}(\ln|t-2| - \ln|t+2|), \tan(e^t), \ln|\sec t|\rangle + \mathbf{C}$. 13.2.re4c. $\langle \frac{\pi}{4}, \frac{2}{\pi}, \frac{31}{5}\rangle$. 13.2.re5a. $\frac{d\mathbf{r}}{dt} = \langle 2t, 3t^2-1, 1\rangle$. $\mathbf{T} = \frac{1}{\sqrt{1+4t^2+(3t^2-1)^2}}\langle 2t, 3t^2-1, 1\rangle$. Line is $\langle 2,0,1\rangle + t\langle 2,2,1\rangle$. 13.2.re5b. $\frac{d\mathbf{r}}{dt} = \langle t, t^{-1}\rangle$. $\mathbf{T} = \frac{1}{\sqrt{t^2+t^{-2}}}\langle t, t^{-1}\rangle$. Line is $\langle \frac{1}{2}, 0\rangle - t\langle 1,1\rangle$. 13.2.re5c. $\frac{d\mathbf{r}}{dt} = e^t\langle 1, \sin t+\cos t, \cos t - \sin t\rangle$. $\mathbf{T} = \frac{1}{\sqrt{3}}\langle 1, \sin t+\cos t, \cos t-\sin t\rangle$. Line is $\langle 1,0,1\rangle + t\langle 1,1,1\rangle$.

13.3: Arc length, curvature, and the **TNB** *frame*

Arc length

The total length of the curve parametrized by $\mathbf{r}(t) = \langle x(t), y(t), z(t)\rangle$ from $t = a$ to $t = b$ is

$$\int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2}\, dt = \int_a^b \left|\frac{d\mathbf{r}}{dt}\right| dt$$

The length of a curve is also called its **arc length**, but arc length can also refer to a variable s that increases from 0 at the beginning of the curve to its total length at the end of the curve. If you drove a car along the curve, and if you set the trip odometer to zero at start of the curve, then the odometer will display the value of s as you travel.
At time t, the current value of s is

$$s = \int_a^t \left|\frac{d\mathbf{r}}{dt^*}\right| dt^*$$

(where t^* is dummy variable of integration). As a consequence, the particle's speed

$$\frac{ds}{dt} = \left|\frac{d\mathbf{r}}{dt}\right|.$$

The direction of $\dfrac{d\mathbf{r}}{dt}$ = the direction of the particle's motion, and
the magnitude of $\dfrac{d\mathbf{r}}{dt}$ = the speed of the particle.

13.3.re1. Find the length of the helix parametrized by $\mathbf{r} = \langle \sin t, \cos t, t\rangle$ for $0 \le t \le \pi$.
Solution: $\frac{ds}{dt} = |\langle \cos t, -\sin t, 1\rangle| = \sqrt{\cos^2 t + \sin^2 t + 1} = \sqrt{2}$. Therefore, the arc length equals $\int_0^\pi \sqrt{2}\, dt = \sqrt{2}\pi$.

13.3.re2. Find the length of the given curve.

a. $\mathbf{r} = \langle \frac{1}{2}t^2, \frac{4}{3}t^{3/2}, 2t\rangle \quad 0 < t < 1$

b. $\mathbf{r} = \langle \ln t, \frac{1}{2}t^2, \sqrt{2}t\rangle \quad 1 < t < e$

c. $y = x^2$, $z = \frac{2}{3}x^3$, from $(-1, 1, -\frac{2}{3})$ to $(1, 1, \frac{2}{3})$

T, **N**, **B**, *and* κ

The **Unit Tangent**, **Unit Normal**, and **Binormal** are three mutually orthogonal unit vectors given by

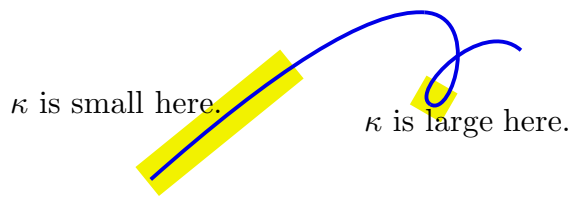
$$\mathbf{T} = \frac{d\mathbf{r}}{dt} \Big/ \left|\frac{d\mathbf{r}}{dt}\right| \qquad \mathbf{N} = \frac{d\mathbf{T}}{dt} \Big/ \left|\frac{d\mathbf{T}}{dt}\right| \qquad \mathbf{B} = \mathbf{T} \times \mathbf{N}$$

(as long as $\frac{d\mathbf{r}}{dt} \neq \mathbf{0}$). **T**, **N**, **B** is a right-handed system. You can think of them as a set of coordinate axes that travels along the curve, twisting so that **T** is always tangent to the curve and **N** always points in the direction the curve is turning. There's nice animation of the **TNB** "frame" moving along a curve in space at https://youtu.be/JZGFcwipHYY.

Curvature is the scalar given by

$$\begin{aligned}\kappa &= \left|\frac{d\mathbf{T}}{ds}\right| = \left|\frac{d\mathbf{T}}{dt}\right| \Big/ \frac{ds}{dt} \\ &= \frac{|\mathbf{r}'(t) \times \mathbf{r}''(t)|}{|\mathbf{r}'(t)|^3}\end{aligned}$$

κ is the speed at which **T** turns when we travel along the curve with the constant speed 1. The curvature of a straight line is zero, and the curvature of a circle is the reciprocal of its radius. Along most curves, curvature is not constant.



While **r** and its derivatives depend on the motion of the particle tracing out the curve, **T**, **N**, **B**, and κ are geometric properties of the curve itself.

13.3.re3. Find $\mathbf{T}$, $\mathbf{N}$, $\mathbf{B}$, and κ along the curve given by $\mathbf{r} = \langle \frac{1}{2}t^2, \frac{4}{3}t^{3/2}, 2t\rangle$ $(t > 0)$.

Two tips in calculations such as these:

> If c is a scalar and $\mathbf{u}$ a vector, then $\frac{d}{dt}(c\mathbf{u}) = \frac{dc}{dt}\mathbf{u} + c\frac{d\mathbf{u}}{dt}$.
> If c is positive, then $\mathbf{u}$ and $c\mathbf{u}$ have the same normalization.

Solution. $\frac{d\mathbf{r}}{dt} = \langle t, 2t^{1/2}, 2\rangle$, and $\frac{ds}{dt}$, the length of $\frac{d\mathbf{r}}{dt}$, is $\sqrt{t^2+4t+4} = \sqrt{(t+2)^2} = t+2$ (since $t+2>0$). Normalize $\frac{d\mathbf{r}}{dt}$ to obtain

$$\mathbf{T} = (t+2)^{-1}\langle t, 2t^{1/2}, 2\rangle$$

Differentiate using the product rule:

$$\frac{d\mathbf{T}}{dt} = (t+2)^{-1}\langle 1, t^{-1/2}, 0\rangle - (t+2)^{-2}\langle t, 2t^{1/2}, 2\rangle$$

We can obtain $\mathbf{N}$ by normalizing

$$\begin{aligned}(t+2)^2\frac{d\mathbf{T}}{dt} &= (t+2)\langle 1, t^{-1/2}, 0\rangle - \langle t, 2t^{1/2}, 2\rangle \\ &= \langle 2, 2t^{-1/2} - t^{1/2}, -2\rangle,\end{aligned}$$

the magnitude of which is

$$\begin{aligned}\sqrt{4 + (2t^{-1/2} - t^{1/2})^2 + 4} &= \sqrt{4 + (4t^{-1} - 4 + t) + 4} \\ &= \sqrt{4t^{-1} + 4 + t} = 2t^{-1/2} + t^{1/2}\end{aligned}$$

Therefore,

$$\mathbf{N} = (2t^{-1/2} + t^{1/2})^{-1}\langle 2, 2t^{-1/2} - t^{1/2}, -2\rangle$$

Now $\mathbf{B} = \mathbf{T} \times \mathbf{N} =$

$$\begin{aligned}&(t+2)^{-1}(2t^{-1/2} + t^{1/2})^{-1}\Big(\langle t, 2t^{1/2}, 2\rangle \times \langle 2, 2t^{-1/2} - t^{1/2}, -2\rangle\Big) \\ &= \left(t^{-1/2}(t+2)^2\right)^{-1}\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ t & 2t^{1/2} & 2 \\ 2 & 2t^{-1/2} - t^{1/2} & -2 \end{vmatrix} \\ &= \frac{t^{1/2}}{(t+2)^2}\left(\mathbf{i}\begin{vmatrix} 2t^{1/2} & 2 \\ 2t^{-1/2} - t^{1/2} & -2 \end{vmatrix} - \mathbf{j}\begin{vmatrix} t & 2 \\ 2 & -2 \end{vmatrix} + \mathbf{k}\begin{vmatrix} t & 2t^{1/2} \\ 2 & 2t^{-1/2} - t^{1/2} \end{vmatrix}\right) \\ &= \frac{t^{1/2}}{(t+2)^2}\Big(-(2t^{1/2} + 4t^{-1/2})\mathbf{i} + (2t+4)\mathbf{j} - (2t^{1/2} + t^{3/2})\mathbf{k}\Big)\end{aligned}$$

or, remarkably,

$$\left\langle \frac{-2}{t+2}, \frac{2t^{1/2}}{t+2}, \frac{-t}{t+2} \right\rangle.$$

Finally, we can calculate κ either from

$$\left| \frac{d\mathbf{T}}{dt} \right| \div \frac{ds}{dt} = \left(\frac{2t^{-1/2} + t^{1/2}}{(t+2)^2} \right) \div (t+2) = \frac{t^{-1/2}}{(t+2)^2}$$

or by calculating

$$\begin{aligned} \mathbf{r}' \times \mathbf{r}'' &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ t & 2t^{1/2} & 2 \\ 1 & t^{-1/2} & 0 \end{vmatrix} \\ &= \langle -2t^{-1/2}, 2, -t^{1/2} \rangle \\ &= t^{-1/2} \langle -2, 2t^{1/2} - t \rangle, \end{aligned}$$

the length of which is $t^{-1/2}(t+2)$, which we divide by $\left(\frac{ds}{dt}\right)^3$ to obtain again

$$\kappa = t^{-1/2}(t+2) \div (t+2)^3 = \frac{t^{-1/2}}{(t+2)^2}.$$

Note that

$$\mathbf{B} = (t+2)^{-1} \langle -2, 2t^{1/2}, -t \rangle$$

is a positive-scalar multiple of $\mathbf{r}' \times \mathbf{r}''$ and so could be obtained by normalizing this cross product. We'll see in section 13.4 that this is always the case.

13.3.re4. Find $\mathbf{T}$, $\mathbf{N}$, $\mathbf{B}$, and κ along the given curve.

a. $\mathbf{r} = \langle 4\sin t, 3t, -4\cos t \rangle$

b. $\mathbf{r} = \langle \ln t, \frac{1}{2}t^2, \sqrt{2}t \rangle$

c. $\mathbf{r} = \langle \cos^3 t, \sin^3 t, 0 \rangle \quad 0 < t < \pi/2$

d. $\mathbf{r} = \langle 3\sin t, 4\sin t, 5\cos t \rangle$

The normal and osculating planes

As **TNB** travel along the curve, two planes travel along with them. At any point on the curve, the **normal plane** passes through that point and is orthogonal to **T**, and the **osculating plane** passes through that point and is orthogonal to **B**.

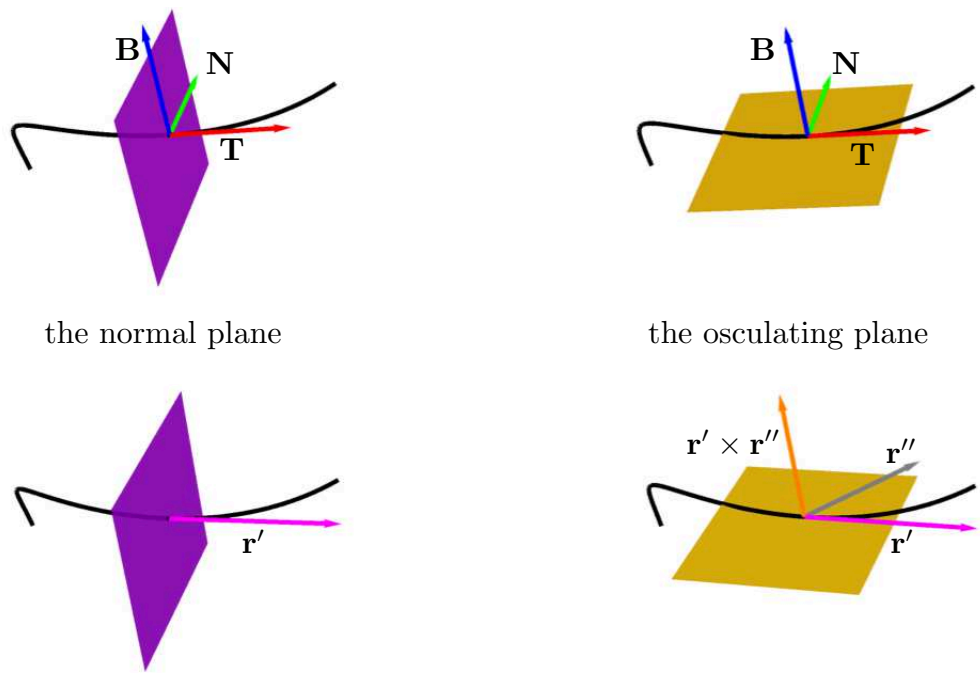


Figure 13.3.1

It is unnecessary to compute **T** to find the normal plane, since $\mathbf{r}'$ is also orthogonal to this plane. Likewise, it is unnecessary to compute **B** to find the osculating plane, since, as we'll see in 13.4, $\mathbf{r}'(t) \times \mathbf{r}''(t)$ is also orthogonal to this plane.

13.3.re5. Find the normal and osculating planes to the curve at the given point..

a. $\mathbf{r} = \langle t, t\sin t, t\cos t\rangle$, at $(2\pi, 0, 2\pi)$
b. $\mathbf{r} = \langle t^2, t^2\sin t^2, t^2\cos t^2\rangle$, at $(2\pi, 0, 2\pi)$
c. $y = z^2$ and $xy = 1$, at $(\frac{1}{4}, 4, -2)$

Answers

13.3.re2a. 5/2 13.3.re2b. $\frac{e^2+1}{2}$ 13.3.re2c. 10/3 13.3.re4a. $\mathbf{T} = \frac{1}{5}\langle 4\cos t, 3, 4\sin t\rangle$, $\mathbf{N} = \langle -\sin t, 0, \cos t\rangle$, $\mathbf{B} = \frac{1}{5}\langle 3\cos t, 4, 3\sin t\rangle$, $\kappa = 4/25$. 13.3.re4b. $\mathbf{T} = (t^2+1)^{-1}\langle 1, t^2, \sqrt{2}t\rangle$, $\mathbf{N} = (t^2+1)^{-1}\langle -\sqrt{2}t, \sqrt{2}t, 1-t^2\rangle$, $\mathbf{B} = (t^2+1)^{-2}\langle -t^2, -1, \sqrt{2}t\rangle$, $\kappa = 2^{1/2}t^{-2}(t^2+1)^{-2}$. 13.3.re4c. $\mathbf{T} = \langle -\cos t, \sin t, 0\rangle$, $\mathbf{N} = \langle \sin t, \cos t, 0\rangle$, $\mathbf{B} = -\mathbf{k}$, $\kappa = \frac{1}{3}\sec t\csc t$. 13.3.re4d. $\mathbf{T} = \frac{1}{5}\langle 3\cos t, 4\cos t, -5\sin t\rangle$, $\mathbf{N} = -\frac{1}{5}\langle 3\sin t, 4\sin t, \cos t\rangle$, $\mathbf{B} = \langle -4/5, 3/5, 0\rangle$, $\kappa = \frac{1}{5}$. 13.3.re5a. n.p.: $x + 2\pi y + z = 4\pi$; o.p.: $(-2\pi^2 - 1)(x - 2\pi) + \pi y + z - 2\pi = 0$. 13.3.re5b. same as in a. 13.3.re5c. n.p.: $\frac{1}{4}(x-\frac{1}{4})+4(y-4)+(z+2) = 0$. o.p: $-2(x-\frac{1}{4})+\frac{3}{8}(y-4)-(z+2) = 0$.

13.4: Velocity and acceleration

It $\mathbf{r}(t)$ represents the position of an object at time t, then its first two derivatives are named **velocity** and **acceleration**. The magnitude of velocity is **speed**. We sometimes "suppress the t" when writing these functions, e.g., when we write $\mathbf{r}$ instead of $\mathbf{r}(t)$.

$$\begin{aligned}
\mathbf{r} &= \text{position (vector)}\\
\mathbf{v} &= \frac{d\mathbf{r}}{dt} = \text{velocity (vector)}\\
\mathbf{a} &= \frac{d\mathbf{v}}{dt} = \text{acceleration (vector)}\\
\frac{ds}{dt} &= |\mathbf{v}| = \text{speed (scalar)}
\end{aligned}$$

13.4.re1. Find velocity, acceleration, and speed for the given position.

a. $\mathbf{r} = \langle 4\sin t, 3t, -4\cos t\rangle$ b. $\mathbf{r} = \langle \ln t, \frac{1}{2}t^2, \sqrt{2}t\rangle$ c. $\mathbf{r} = \langle \cos^3 t, \sin^3 t, t\rangle$

Initial value problems

13.4.re2. Find position $\mathbf{r}$ if $\mathbf{a}(t) = \langle te^t, 2t, 1\rangle$, $\mathbf{v}(0) = \langle -1, 1, 0\rangle$, and $\mathbf{r}(0) = \langle -1, 0, -1\rangle$.

Solution: Integrate once to find $\mathbf{v}$ and again to find $\mathbf{r}$. Use the given values of $\mathbf{v}$ and $\mathbf{r}$ to solve for constants of integration. (Integrate te^t by parts.)

$$\begin{aligned}
\mathbf{v} &= \langle te^e - e^t, t^2, t\rangle + \mathbf{C}\\
\langle -1, 1, 0\rangle &= \langle -1, 0, 0\rangle + \mathbf{C} \quad\Longrightarrow\quad \mathbf{C} = \langle 0, 1, 0\rangle\\
\mathbf{v} &= \langle te^e - e^t, t^2, t\rangle + \langle 0, 1, 0\rangle\\
&= \langle te^e - e^t, t^2 + 1, t\rangle\\
\mathbf{r} &= \langle te^e - 2e^t, \tfrac{1}{3}t^3 + t, \tfrac{1}{2}t^2\rangle + \mathbf{D}\\
\langle -1, 0, -1\rangle &= \langle -2, 0, 0\rangle + \mathbf{D} \quad\Longrightarrow\quad \mathbf{D} = \langle 1, 0, -1\rangle\\
\mathbf{r} &= \langle te^e - 2e^t, \tfrac{1}{3}t^3 + t, \tfrac{1}{2}t^2\rangle + \langle 1, 0, -1\rangle\\
&= \langle te^e - 2e^t + 1, \tfrac{1}{3}t^3 + t, \tfrac{1}{2}t^2 - 1\rangle
\end{aligned}$$

13.4.re3. A constant force of magnitude 15 in the direction of $-3\mathbf{i} + 4\mathbf{k}$ acts on an object of mass 1/2. If, at time 0, the object's position and velocity are $2\mathbf{i} - \mathbf{k}$ and $\mathbf{j} - \mathbf{i}$ respectively, find the object's position at time t. Hint: Newton's second law of motion states that **force** = mass × **acceleration**.

13.4.re4. An acrobat is at to be shot from a cannon with speed $32\sqrt{2}$ ft/sec at an upward angle $\pi/4$ radians. So that we may correctly position the net to catch her, find the (horizontal) distance from the cannon at which the acrobat will descend to altitude of 12 ft. Assume the acrobat is launched from altitude zero and that, due to gravity, her acceleration is -32 ft/sec^2 downward.

Tip: place the cannon at the origin in the xy-plane, firing into the first quadrant. At what x will $y = 12$?

Tangential and normal components of acceleration

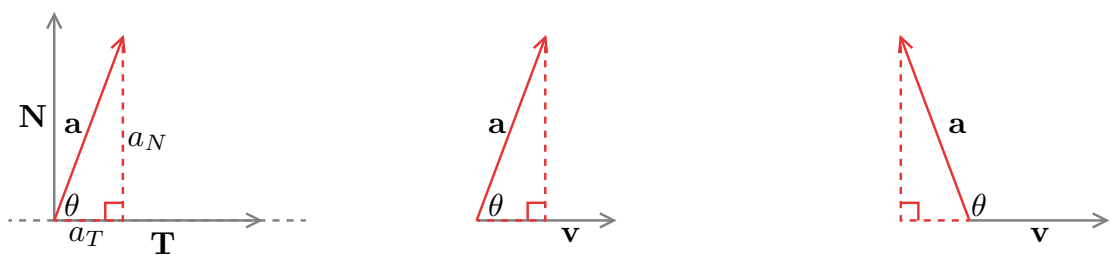
The tangential and normal components of $\mathbf{a}$ are scalars a_T and a_N for which

$$\mathbf{a} = a_T\mathbf{T} + a_N\mathbf{N}. \tag{13.4.1}$$

Consequently, $\mathbf{v}$ and $\mathbf{a}$ lie in the same plane as $\mathbf{T}$ and $\mathbf{N}$, the osculating plane (as seen in Figure 13.3.1). a_T and a_N are the components of $\mathbf{a}$ in the directions $\mathbf{T}$ and $\mathbf{N}$ seen in section 12.3 and can be calculated any of these formulas

$$\begin{aligned} a_T &= \frac{d^2s}{dt^2} \\ &= |\mathbf{a}|\cos\theta = \frac{\mathbf{v}\cdot\mathbf{a}}{|\mathbf{v}|} \end{aligned} \qquad \begin{aligned} a_N &= \kappa\left(\frac{ds}{dt}\right)^2 \\ &= |\mathbf{a}|\sin\theta = \frac{|\mathbf{v}\times\mathbf{a}|}{|\mathbf{v}|} \\ &= \sqrt{|\mathbf{a}|^2 - a_T^2} \end{aligned}$$

Note that a_T is positive [negative] if the object is speeding up [slowing down]. Unless speed or curvature is zero, a_N is positive: $\mathbf{a}$ and $\mathbf{N}$ lie on the same side of the line containing $\mathbf{v}$ in the osculating plane.



$a_T > 0$ speeding up. $a_T < 0$ slowing down.

13.4.re5. Find a_T and a_N for the given $\mathbf{r}$.

a. $\mathbf{r} = \langle 4\sin t, 3t, -4\cos t\rangle$ b. $\mathbf{r} = \langle \ln t, \frac{1}{2}t^2, \sqrt{2}t\rangle$ c. $\mathbf{r} = \langle \frac{1}{2}t^2, \frac{4}{3}t^{3/2}, 2t\rangle$ (see 13.3.re3)

Calculating the **TNB** *frame from* $\mathbf{r}'$ *and* $\mathbf{r}''$

It's possible to find $\mathbf{N}$ and $\mathbf{B}$ without differentiating $\mathbf{T}$(which is easily found by normalizing $\mathbf{r}'$). Since $\mathbf{r}''$ lies in the osculating plane on the same side of $\mathbf{T}$ as $\mathbf{N}$, one can find $\mathbf{B}$ by normalizing $\mathbf{r}' \times \mathbf{r}''$. And since **TNB** form a right-handed system, $\mathbf{N}$ must equal $\mathbf{B} \times \mathbf{T}$:

$$\mathbf{T} = \frac{\mathbf{r}'}{|\mathbf{r}'|} \qquad \mathbf{N} = \mathbf{B} \times \mathbf{T} \qquad \mathbf{B} = \frac{\mathbf{r}' \times \mathbf{r}''}{|\mathbf{r}' \times \mathbf{r}''|}$$

13.4.re6. Suppose that, at a particular time, $\mathbf{v} = \langle -2, 1, 2\rangle$ and $\mathbf{a} = \langle 1, 1, -1\rangle$. Find the following at that time.

a. a_T b. a_N c. $\mathbf{T}$ d. $\mathbf{N}$ e. $\mathbf{B}$

Answers

13.4.re1a. $\mathbf{v} = \langle 4\cos t, 3, 4\sin t\rangle$, $\mathbf{a} = \langle -4\sin t, 0, 4\cos t\rangle$, $\frac{ds}{dt} = 5$ 13.4.re1b. $\mathbf{v} = \langle t^{-1}, t, \sqrt{2}\rangle$, $\mathbf{a} = \langle -t^{-2}, 1, 0\rangle$, $\frac{ds}{dt} = t^{-1} + t$. 13.4.re1c. $\mathbf{v} = \langle -3\cos^2 t\sin t, 3\sin^2 t\cos t, 1\rangle$, $\mathbf{a} = \langle 6\sin^2 t\cos t - 3\cos^3 t, 6\cos^2 t\sin t - 3\sin^3 t, 0\rangle$, $\frac{ds}{dt} = \sqrt{1 + 9\cos^2 t\sin^2 t}$. 13.4.re3. $\mathbf{a} = 6\langle -3, 0, 4\rangle$, $\mathbf{v} = 6\langle -3t, 0, 4t\rangle + \langle -1, 1, 0\rangle$, $\mathbf{r} = \langle -9t^2 - t + 2, t, 12t^2 - 1\rangle$. 13.4.re4. 48 ft (at time $t = 3/2$). 13.4.re5a. $a_T = 0$, $a_N = 4$ 13.4.re5b. $a_T = \frac{t - t^{-3}}{t + t^{-1}}$, $a_N = \frac{\sqrt{2}}{t}$. 13.4.re5c. $a_T = 1$, $a_N = t^{-1/2}$. 13.4.re6a. -1. 13.4.re6b. $\sqrt{2}$. 13.4.re6c. $\frac{1}{3}\langle 2, 1, -2\rangle$. 13.4.re6d. $\frac{1}{3\sqrt{2}}\langle 1, -4, -1\rangle$. 13.4.re6e. $\frac{1}{\sqrt{2}}\langle -1, 0, -1\rangle$.

14.1: Real-valued functions of several real variables

Recall that a **function** is a rule that assigns to each elements of a set, called its **domain**, a unique element of another set, called its **range**. We may write

$$f : U \to V$$

to indicate that f maps elements of the set U to elements of the set V.
In Calculus III, we consider functions with domain $\subset \mathbb{R}^m$ and range $\subset \mathbb{R}^n$ for some m and n. Generally, a function's domain is easier to determine than its range.

14.1.re1.

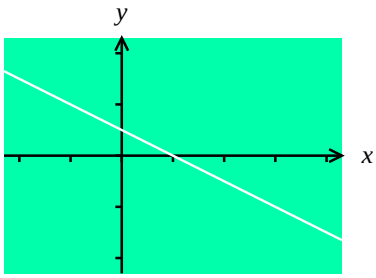
function	**domain**	**range**
$f(x) = \dfrac{1}{x-2}$	$(-\infty, 2) \cup (2, \infty) \subset \mathbb{R}^1$	$(-\infty, 0) \cup (0, \infty) \subset \mathbb{R}^1$
$g(x) = \ln x$	$(0, \infty) \subset \mathbb{R}^1$	$\mathbb{R}^1$
$\mathbf{u}(t) = \langle (t-1)^2, \sin t, t \rangle$	$\mathbb{R}^1$	a curve $\subset \mathbb{R}^3$
$v(x, y) = \dfrac{x^2 - y^2}{x + 2y - 1}$	$\{(x, y) \mid x + 2y \neq 1\} \subset \mathbb{R}^2$	$\mathbb{R}^1$
$w(x, y, z) = \sqrt{1 - x^2 - 4z^2}$	$\{(x, y, z) \mid x^2 + 4z^2 \leq 1\} \subset \mathbb{R}^3$	$[0, 1] \subset \mathbb{R}^1$

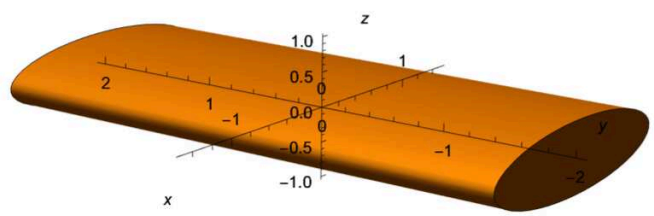
14.1.re2. Sketch and describe the domain of the given function.

a. $u(x, y) = \frac{x^2 - y^2}{x + 2y - 1}$ b. $v(x, y, z) = \sqrt{1 - x^2 - 4z^2}$

Solutions.

a. The plane minus the line $x + 2y = 1$. b. The cylinder $x^2 - 4z^2 = 1$ and its interior.





14.1.re3. Describe and sketch the domain of the given function.

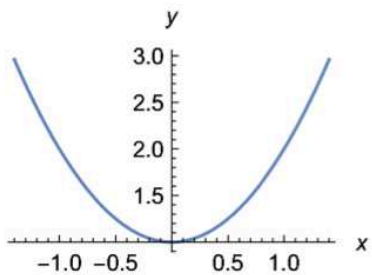
a. $\nu(x, y) = \sqrt{y - x} + \sin^{-1}(x + y)$ b. $\omega(x, y) = \ln\big((x - y)(y - 2)\big)$

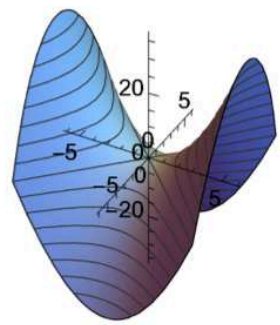
c. $\alpha(x, y, z) = \dfrac{x^2 + xz + z^2}{(x - y)(z - x^2 - y^2)}$ d. $\beta(x, y, z) = \sqrt{1 - y^2 - z^2} + \dfrac{1}{\sqrt{x^2 + y^2 + z^2 - 4}}$

Graphs of functions on $\mathbb{R}$ *and* $\mathbb{R}^2$

The **graph of an equation** $g(x, y) = 0$ [or $g(x, y, z) = 0$] is its **solution set**: the set of all points in the xy-plane [or in xyz-space] whose coordinates satisfy the equation. The graph of a function $f(x)$ [or $f(x, y)$] is the graph of the equation $y = f(x)$ [or $z = f(x, y)$], that is, the set of all the points $\big(x, f(x)\big)$ in the plane [or the set of all points $\big(x, y, f(x, y)\big)$ in space].

14.1.re4. The graph of $f(x) = x^2$ is the parabola $y = x^2$, below left. The graph of $g(x, y) = x^2 - y^2$ is the hyperbolic paraboloid $z = x^2 - y^2$, below right.





14.1.re5. Sketch the graph of the given function.

a. $f(x) = \dfrac{x^2 - 1}{x}$

b. $g(x, y) = -4x^2 - 8x - y^2$

c. $h(x, y) = 1 - \sqrt{x^2 + y^2}$

d. $k(x, y) = 4 - 2x - 3y$

Level curves and surfaces; contour maps

The **level curves** of a function $f(x, y)$ are the graphs of equations of the form $f(x, y) = k$, where k is a constant. That is, level curves are the curves along which $f(x, y)$ is constant. The **level surfaces** of a function $f(x, y, z)$ are the graphs of equations $f(x, y, z) = k$, that is, the surfaces along which $f(x, y, z)$ is constant.

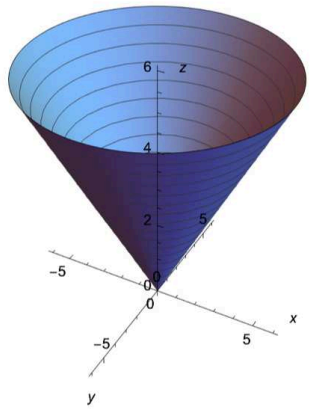
Each point in the domain of $f(x, y)$ lies on exactly one level curve of f. Consequently, the level curves of $f(x, y)$ are non-overlapping and completely fill the domain of f. Likewise, the level surfaces of $f(x, y, z)$ are nonoverlapping and completely fill the domain of $f(x, y, z)$.

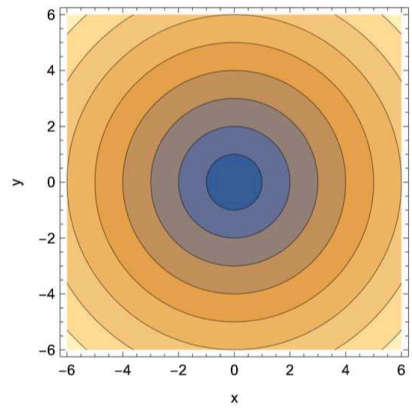
A **contour map** for a function is a graph of a representative sample of is level curves. Typically, a contour map displays the curves

$$f(x, y) = k_0, \quad f(x, y) = k_1, \quad f(x, y) = k_2, \quad \ldots f(x, y) = k_n$$

for some equally spaced numbers $k_0, k_1, \ldots k_n$.

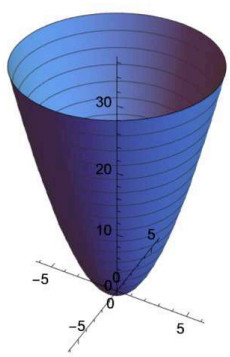
14.1.re6. The domain of $f(x, y) = \sqrt{x^2 + y^2}$ is the entire xy-plane, and its level curves are circles centered at the origin. Each point in the plane lies on exactly one of the level curves of f. The graph of $f(x, y)$ appears below left. A contour map for $f(x, y)$ for $f(x, y) = 0, 1, 2, \ldots, 8$ appears below right.



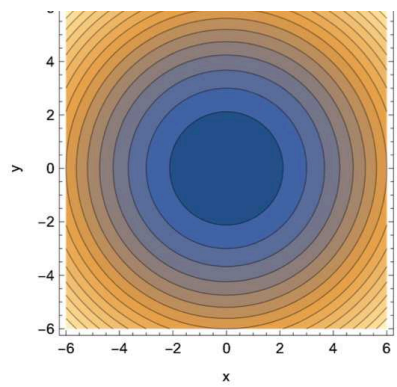


If the contour map displays the level curves for equally spaced values of f, then we judge where and in which directions the function is increasing rapidly per unit

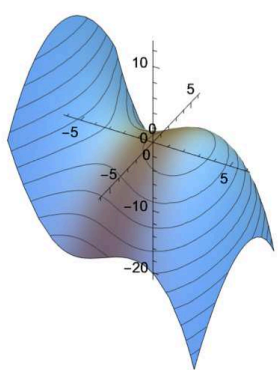
14.1.re7. More graphs and contour maps:



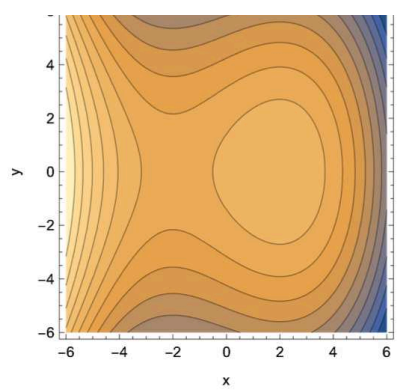
graph of $x^2 + y^2$



contour map of $x^2 + y^2$

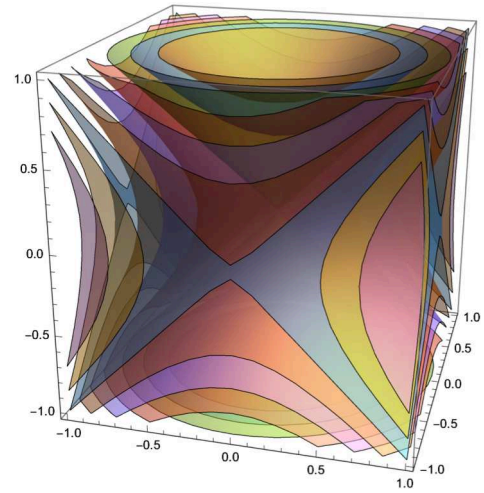


graph of $x - \frac{1}{12}x^3 - \frac{1}{4}y^2 + \frac{1}{2}$



contour map of $x - \frac{1}{12}x^3 - \frac{1}{4}y^2 + \frac{1}{2}$

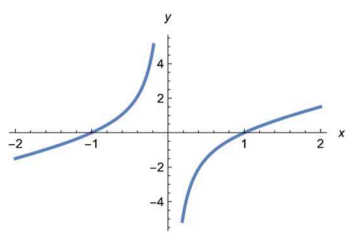
14.1.re8. We can't display the graph of $x^2 + y^2 - z^2$ in three dimensions, but we can still see a contour map:

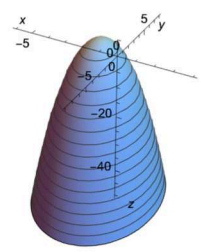


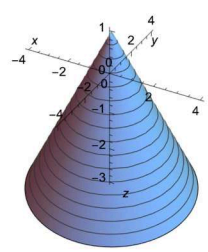
Answers

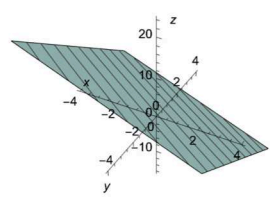
14.1.re3a. $y \geq x$ and $-1 \leq x+y \leq 1$. The region in the plane between $x+y=-1$ and $x+y=1$ and above $y=x$. 14.1.re3b. $(x-y)(y-2) > 0$. The region above $y=2$ and below $y=x$ plus the region below $y=2$ and above $y=x$ (excluding those lines). 14.1.re3c. $z \neq x^2+y^2$ and $x \neq y$. All of $\mathbb{R}^3$, minus the circular paraboloid $z = x^2+y^2$ and the plane $x=y$. 14.1.re3d. $y^2+z^2 \leq 1$ and $x^2+y^2+z^2 > 4$. The part of the cylinder $y^2+z^2=1$ (and its interior) that lies outside the sphere $x^2+y^2+z^2=4$.

14.1.re5. Top row $= \{a, b\}$. Bottom row $= \{c, d\}$.









14.2: Limits and Continuity of functions of several variable

Showing that a limit fails to exist

It is fortunate that most limits we have to take in Calc III are limits on one variable only, because genuine limits of functions of two or more variables can be difficult to evaluate.

14.2.re1. Show that the limit does not exist.

a. $\displaystyle\lim_{(x,y)\to(0,0)} \frac{x^2+2y^2}{x^2+y^2}$ b. $\displaystyle\lim_{(x,y)\to(0,0)} \frac{xy}{(x+y)^2}$

Solutions:

a. If it exists, the limit of $\frac{x^2+2y^2}{x^2-y^2}$ should be the same no matter how (x,y) approaches $(0,0)$. Consider the limits when we let (x,y) approach $(0,0)$ along the lines $x=0$ and $y=0$ (which go through the origin).

$$\text{along the } x\text{-axis:} \quad \lim_{(x,0)\to(0,0)} \frac{x^2+2y^2}{x^2-y^2} = \lim_{x\to 0} \frac{x^2}{x^2} = 1$$

$$\text{along the } y\text{-axis:} \quad \lim_{(0,y)\to(0,0)} \frac{x^2+2y^2}{x^2-y^2} = \lim_{y\to 0} \frac{2y^2}{-y^2} = -2$$

Since these are not the same, the limit $\displaystyle\lim_{(x,y)\to(0,0)} \frac{x^2+2y^2}{x^2-y^2}$ cannot exist.

b. Compute the limit along these paths that pass through the origin.

$$\text{along the } x\text{-axis:} \quad \lim_{(x,0)\to(0,0)} \frac{xy}{(x+y)^2} = \lim_{x\to 0} \frac{0}{y^2} = 0$$

$$\text{along the } y\text{-axis:} \quad \lim_{(0,y)\to(0,0)} \frac{xy}{(x+y)^2} = \lim_{y\to 0} \frac{0}{x^2} = 0$$

$$\text{along the line } x=y\text{:} \quad \lim_{(x,x)\to(0,0)} \frac{xx}{(x+x)^2} = \lim_{x\to 0} \frac{x^2}{4x^2} = \tfrac{1}{4}$$

Since we found two paths with two different limits, the limit does not exist.

14.2.re2. Show that $\lim_{(x,y)\to(0,0)}$ of the given function does not exist.

a. $\dfrac{x^4-3y^4}{(x^2+y^2)^2}$ b. $\dfrac{xy}{(x-y)^2}$ c. $\dfrac{x^2y^{1/3}}{x^3+y}$

If the limit

$$\lim_{(x,y)\to(a,b)} f(x,y) \tag{14.2.1}$$

is known to exist, then

$$\lim_{(x,y)\to(a,b)} f(x,y) = \lim_{x\to a}\lim_{y\to b} f(x,y) = \lim_{y\to b}\lim_{x\to a} f(x,y).$$

But example 14.2.re1b demonstrates that

$$\lim_{x\to a}\lim_{y\to b} f(x,y) = \lim_{y\to b}\lim_{x\to a} f(x,y)$$

is no guarantee that (14.2.1) exists.

Continuity

A function $f(x, y)$ is said to be **continuous at the point** (a, b) if

$$\lim_{(x,y)\to(a,b)} f(x, y) = f(a, b)$$

A function is said to be **continuous** it is it continuous at every point in its domain. (Continuity for functions of more than two variables is defined similarly.)

A **monomial** in x, y is a function of the form $x^m y^n$ for nonnegative integers m and n

A **polynomial** is a sum of scalar multiples of monomials.

A **rational function** is the ratio of two polynomials.

(Monomials, polynomials, and rational functions of more than two variables are defined similarly.)

14.2.re3. x^0, x^2y^{13}, and $x^7y^2z^9$ are monomials. $x^2y^{3/2}$ is not.

$x^2 + 2xy - y^2 + 2y - 1$ and $(x + y - 2z)^4$ are polynomials; e^{xy} is not.

$\dfrac{x^2 - y^2 + y}{1 + 2x - 3y + z^3}$ is a rational function; $\dfrac{\sin x + \cos y}{\sqrt{x + 1}}$ is not.

Fact: Polynomials and rational functions are continuous.

Fact (from calculus I): These functions (of one variable) are all continuous:

power functions x^p	the absolute value	trig functions
inverse trig functions	exponential functions	logarithms

Fact: If f and g are continuous, then so are

$$f + g \qquad f - g \qquad f \cdot g \qquad f \div g \qquad f \circ g.$$

Consequently, other than piecewise defined functions, most functions we can write down are automatically continuous, meaning that their limits can be computed simply by evaluating the function.

Tools to evaluate limits in several variables

Other than continuity, we have only these two facts to allow to take limits in two or more variables.

Fact: if $f(x,y) = g(x,y)$ for all (x,y) other than (a,b), then

$$\lim_{(x,y)\to(a,b)} f(x,y) = \lim_{(x,y)\to(a,b)} g(x,y),$$

meaning that either both exist are equal or both fail to exist.

Squeeze Theorem: If $g(x,y) \le f(x,y) \le h(x,y)$ for all (x,y) other than (a,b), and if

$$\lim_{(x,y)\to(a,b)} g(x,y) = \lim_{(x,y)\to(a,b)} h(x,y) = L,$$

then

$$\lim_{(x,y)\to(a,b)} f(x,y)$$

must also exist and equal L.

14.2.re4. Evaluate the limit as $(x,y) \to (0,0)$, or determine that it does not exist.

a. $\dfrac{x^2 - 4xy + 4y^2}{(xy+1)(x-2y)}$ b. $\dfrac{e^{x+y+1}}{1+\cos(2x)}$ c. $\dfrac{2x^2 + 7xy + 3y^2}{x^2 + 2xy - 3y^2}$

d. $\dfrac{2x^2 + 4x - xy - 2y}{2xy - y^2 - 2x + y}$ e. $|x-y|\sin\left(\dfrac{x}{y}\right)$ f. $\dfrac{x^2 + 5xy + 6y^2}{x^2 + 3xy - x - 3y}$

Answers

14.2.re2a. Compare limits along $x = 0$ and $y = 0$. 14.2.re2b. Compare limits at (0,0) along $y = 0$ and $y = -x$. 14.2.re2c. Compare along $x = 0$ and $y = x^3$. 14.2.re4a. 0. 14.2.re4b. $e/2$, by continuity. 14.2.re4c. DNE. 14.2.re4d. -2. 14.2.re4e. 0, by Squeeze. 14.2.re4f. 0.

14.3: Partial Derivatives

The **partial derivative** of $f(x, y, z)$ with respect to x is the limit

$$f_x(x, y, z) = \lim_{h \to 0} \frac{f(x + h, y, z) - f(x, y, z)}{h}.$$

The partial derivatives (or, more simply, "partials") of f with respect to y and z are defined similarly.

$$f_y(x, y, z) = \lim_{h \to 0} \frac{f(x, y + h, z) - f(x, y, z)}{h}$$
$$f_z(x, y, z) = \lim_{h \to 0} \frac{f(x, y, z + h) - f(x, y, z)}{h}$$

Because of its similarity to the derivative seen in calc I, f_x (or f_y or f_z) can be calculated by treating all variables other than x (or y or z) as constants and differentiating with the familiar rules.

14.3.re1. If $f(x, y, z) = x^3y^2z + 2x + 3y + x\sin(yz)$, then

$$\begin{aligned} f_x &= 3x^2y^2z + 2 + \sin(yz) \\ f_y &= 2x^3yz + 3 + xz\cos(yz) \\ f_z &= x^3y^2 + xy\cos(yz) \end{aligned}$$

Higher order partials are indicated with more subscripts:

$$\begin{aligned} f_{xx} &= (f_x)_x = 6xy^2z \\ f_{yx} &= (f_y)_x = 6x^2yz + z\cos(yz) \\ f_{zy} &= (f_z)_y = 2x^3y + x\cos(yz) - xyz\sin(yz) \end{aligned}$$

For most functions we see, the order of differentiation isn't important:

Clairaut's Theorem on the equality of mixed partials: if $f_{xy}(x, y)$ and $f_{yx}(x, y)$ are continuous on an open disk in the plane, then $f_{xy} = f_{yx}$ on that disk.

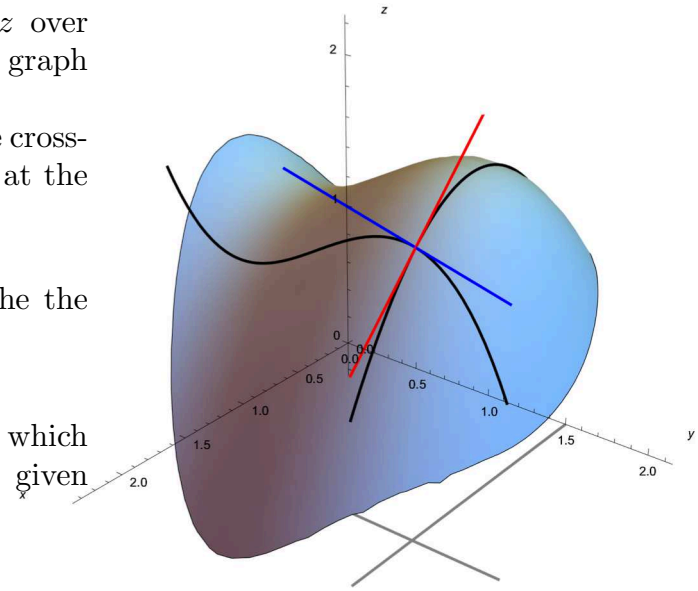
14.3.re1, continued. $f_{xy} = (f_x)_y = \big(3x^2y^2z + 2 + \sin(yz)\big)_y = 6x^2yz + z\cos(yz) = f_{yx}$ found above.

14.3.re2. Find all first and second order partials of the given function.

a. $f = (x + y)(y - 2z)$ b. $g = e^{x+2y} - \cos(xz) + \ln(yz^2)$ c. $h = \tan^{-1}(x + 2y)$

Geometric interpretation of the partial derivative

$f_x(x_0, y_0)$ tells us the slope (change in z over change in x) of the cross-section to the graph of $f(x, y)$ at $y = y_0$ at the point (x_0, y_0).
Similarly, $f_y(x_0, y_0)$ tells us the slope of the cross-section to the graph of $f(x, y)$ at $x = x_0$ at the point (x_0, y_0).



14.3.re3. To write the line tangent to the the curve

$$z = x^2 + 3y^2, \quad y = 1$$

at the point $(2, 1, 7)$, we calculate $z_x = 2x$, which equals 4 at $x = 2$. The tangent line is given implicitly by the equations

$$z - 7 = 4(x - 2), \quad y = 1.$$

To parametric equations for the line, set $t = z - 7 = 4(x - 2)$ and solve for x and z:

$$x = \frac{1}{4}t + 2 \quad z = 7 + t \quad y = 1.$$

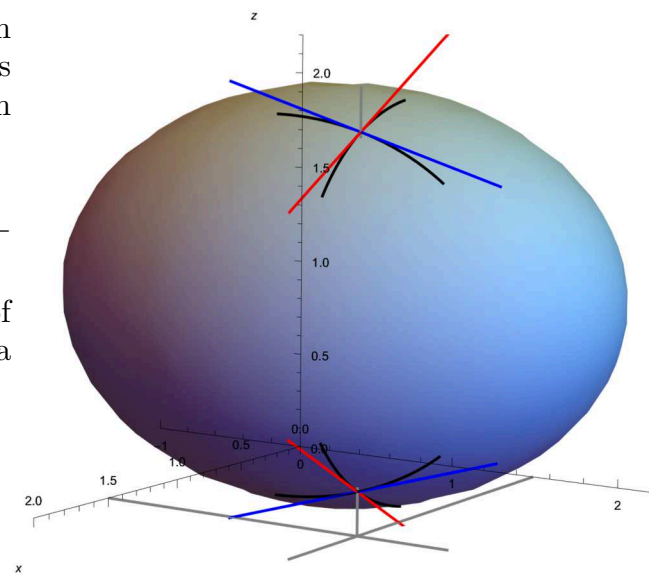
14.3.re4. a. Find implicit and parametric equations of line tangent to the curve $z = e^{x^2 y - 1}$, $y = 2$ at the point $(1, 2, e)$.
b. Repeat for the curve $z = e^{x^2 y - 1}$, $x = 2$ at the point $(2, \frac{1}{4}, 1)$.

Alternate notation for partial derivatives

$$f_x = D_x f = \frac{\partial f}{\partial x}$$
$$f_{xx} = D_x^2 f = \frac{\partial}{\partial y}\frac{\partial f}{\partial x} = \frac{\partial^2 f}{\partial x^2}$$
$$f_{xy} = D_y D_x f = \frac{\partial}{\partial x}\frac{\partial f}{\partial x} = \frac{\partial^2 f}{\partial y \partial x}$$

Implicit differentiation

Along a surface given implicitly by the equation $f(x, y, z) = 0$ we can express $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$ in terms of x, y, and z by implicit differentiation as in calculus I.



14.3.re5. Find $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$ along the surface $3x + e^{xz} = y^2 z$.
Solution. To find z_x, differentiate both sides of the equation as though y was constant and z a function of x:

$$3 + e^{xz}(xz)_x = y^2 z_x$$
$$3 + e^{xz}(1z + xz_x) = y^2 z_x$$

Now solve for z_x:

$$3 + ze^{xz} + xz_x e^{xz} = y^2 z_x \qquad 3 + ze^{xz} = z_x(y^2 - xe^{xz})$$
$$3 + ze^{xz} = y^2 z_x - xz_x e^{xz} \qquad \frac{3 + ze^{xz}}{y^2 - xe^{xz}} = z_x$$

Find z_y similarly: differentiate as though x was constant and z a function of y:

$$xz_y e^{xz} = 2yz + y^2 z_y \qquad (xe^{xz} - y^2)z_y = 2yz$$
$$xz_y e^{xz} - y^2 z_y = 2yz \qquad z_y = \frac{2yz}{xe^{xz} - y^2}$$

14.3.re6. Find z_x and z_y along the graph of the given equation.

a. $e^{x+2y+3z} = x^2 + y^2 + z^2$ b. $x^2 y - y^3 z + x \ln z = y$ c. $\dfrac{x^2 + 2xz}{y^2 - xz^3} = y$

Answers

14.3.re2a. $f_x = y - 2z$. $f_y = x + 2y - 2z$. $f_z = -2x - 2y$. $f_{xx} = 0$. $f_{xy} = 1$. $f_{xz} = -2$. $f_{yy} = 2$. $f_{yz} = -2$. $f_{zz} = 0$. 14.3.re2b. $g_x = e^{x+2y} + z\sin(xz)$. $g_y = 2e^{x+2y} + y^{-1}$. $g_z = x\sin(xz) + 2z^{-1}$. $g_{xx} = e^{x+2y} + z^2\cos(xz)$. $g_{xy} = 2e^{x+2y}$. $g_{xz} = \sin(xz) + xz\cos(xz)$. $g_{yy} = 4e^{x+2y} - y^{-2}$. $g_{yz} = 0$. $g_{zz} = x^2\cos(xz) - 2z^{-2}$. 14.3.re2c. $h_x = (1 + (x + 2y)^2)^{-1}$. $h_y = 2(1 + (x + 2y)^2)^{-1}$. $h_{xx} = -2(x + 2y)(1 + (x + 2y)^2)^{-2}$. $h_{xy} = -4(x + 2y)(1 + (x + 2y)^2)^{-2}$. $h_{yy} = -8(x + 2y)(1 + (x + 2y)^2)^{-2}$. 14.3.re4. a. imp: $z - e = 4e(x - 1)$, $y = 2$. para: $x = \frac{1}{4e}t + 1$, $y = 2$, $z = e + t$. b. imp: $z - 1 = 4(y - \frac{1}{4})$, $x = 2$. para: $x = 2$, $y = \frac{t+1}{4}$, $z = 1 + t$. 14.3.re6a. $z_x = \frac{-2x+e^{x+2y+3z}}{2z-3e^{x+2y+3z}}$, $z_y = \frac{-2y+2e^{x+2y+3z}}{2z-3e^{x+2y+3z}}$. 14.3.re6b. $z_x = \frac{2xy+\ln z}{y^3-xz^{-1}}$, $z_y = \frac{x^2-3y^2z-1}{y^3-xz^{-1}}$. 14.3.re6c. $z_x = \frac{2x+2z+yz^3}{-2x-3xyz^2}$, $z_y = \frac{3y^2-xz^3}{2x+3xyz^2}$.

14.4: Tangent planes, linear approximation, and differentiability

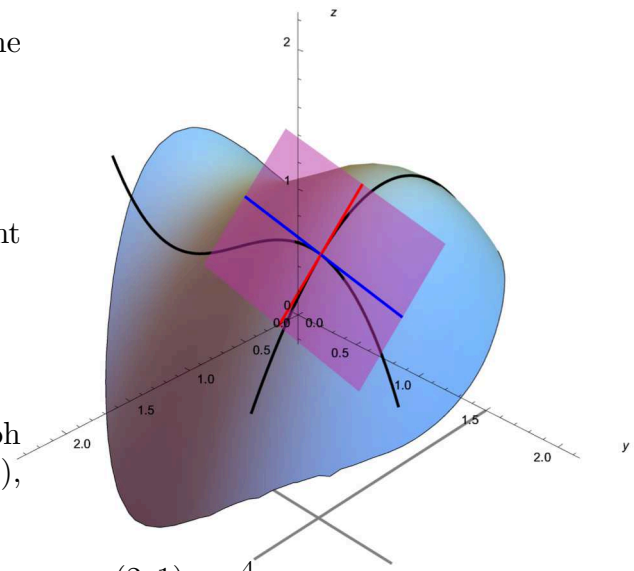
Tangent planes

The plane tangent to the graph of $f(x, y)$ at the point $(a, b, f(a, b))$ is given by

$$z - f(a, b) = f_x(a, b)(x - a) + f_y(a, b)(y - b).$$

Note the similarity to the equation of the tangent line from calculus I:

$$y - f(a) = f'(a)(x - a).$$



14.4.re1. To find the plane tangent to the graph of $r(x, y) = e^{x+2y} - \sin(\pi xy)$ at $(x, y) = (2, 1)$, evaluate r and its first partials at $(2, 1)$:

$$\begin{aligned} r(x, y) &= e^{x+2y} - \sin(\pi xy) & r(2, 1) &= e^4 \\ r_x(x, y) &= e^{x+2y} - \pi y \cos(\pi xy) & r_x(2, 1) &= e^4 - \pi \\ r_y(x, y) &= 2e^{x+2y} - \pi x \cos(\pi xy) & r_y(2, 1) &= 2e^4 - 2\pi \end{aligned}$$

The tangent plane is given by the equation

$$z - e^4 = (e^4 - \pi)(x - 2) + (2e^4 - 2\pi)(y - 1)$$

14.4.re2. Find an equation of the plane tangent to the graph of the given function at the given point (x, y).

a. $(x + y + 1)(y - 2x - 1)$ at $(1, 0)$

b. $e^{x+2y} - \ln(x^2 y)$ at $(-1, 2)$

c. $\dfrac{x + y}{x - 2y}$ at $(-2, 2)$.

Linear approximation of functions of one variable

In calculus I, if f is a function of one variable whose derivative exists at $x = a$, we say f is **differentiable** at $x = a$. When that occurs,

$$f(x) \approx f(a) + f'(a)(x - a)$$

for x is near a. The precise meaning of $\approx$ is given elsewhere, but this means that, near $x = a$, $f(x)$ can be well approximated by its **linearization**

$$L(x) = f(a) + f'(a)(x - a).$$

Linear approximation of functions of two or more variables

The **linearization** of $f(x, y)$ at the point (a, b) is defined to be the function

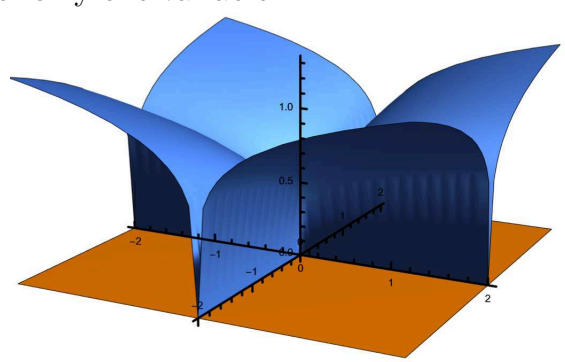
$$L(x, y) = f(a, b) + f_x(a, b)(x - a) + f_y(a, b)(y - b).$$

When we say that $f(x, y)$ is **differentiable** at (a, b), we mean not only that f_x and f_y exist at that point, but that $L(x, y)$ provides a good approximation to $f(x, y)$:

$$f(x, y) \approx f(a, b) + f_x(a, b)(x - a) + f_y(a, b)(y - b) \text{ if } (x, y) \text{ is near } (a, b).$$

While they may appear different, this definition of differentiability and our earlier use of that word in calc I coincide when f is a function of only one variable.

Not every function with partial derivatives is differentiable. For instance, here's the graph of a continuous function f which fails to be differentiable at $(0, 0)$, even though both f_x and f_y exist there.



But, if a function possess *continuous* partial derivatives, it must be differentiable:

Theorem: If f_x and f_y exist in a disk about the point (a, b) and are continuous at (a, b), then f is differentiable at (a, b)

More generally,

Theorem: If f is a function of several real variables and if its first partial derivatives are continuous in some neighborhood about the point $\mathbf{a}$ and are continuous at $\mathbf{a}$, then f is differentiable at $\mathbf{a}$.

14.4.re3. Explain why the function is differentiable on its entire domain, and find its linearization at the given point.

a. $f = \dfrac{x-y}{x+y}$ at $(2, -1)$

b. $h = \tan^{-1}(\sin x + \cos y)$ at $(\frac{\pi}{6}, \frac{\pi}{3})$.

c. $g = e^{x+2y-3z}$ at $(1, 1, 1)$.

Answers

14.4.re2a. $z+6 = -7(x-1) - y$. 14.4.re2b. $z - e^3 + \ln 2 = (2+e^3)(x+1) + (2e^3 - \frac{1}{2})(y-2)$. 14.4.re2c. $6z + x + y = 0$. 14.4.re3a. f and its partial derivatives are rational functions, which are know to be continuous. $L(x, y) = 3 - 2(x-2) - 4(y+1)$. 14.4.re3b. Trig functions, inverse trig functions, and rational functions are continuous. The partials of h are combinations of these. $L(x, y) = \frac{\pi}{4} + \frac{\sqrt{3}}{4}(x - \frac{\pi}{6}) - \frac{\sqrt{3}}{4}(y - \frac{\pi}{3})$. 14.4.re3c. Exponentials, polynomials, and any combinations of these are continuous, so g has continuous partials on all of $\mathbb{R}^3$. $L(x, y, z) = 1 + (x-1) + 2(y-1) - 3(z-1)$, or $1 + x + 2y - 3z$.

14.5: The Chain Rule

If f is a differentiable function of x and y, and x and y are differentiable functions of t, then f is a differentiable function of t, and

$$\boxed{\frac{df}{dt} = \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt}}$$

(In derivatives, we use d for functions of one variable and ∂ for several variables.)

14.5.re1. If $f(x, y) = xy - \cos x + \ln|\sec y|$ and x and y are differentiable functions of t, then

$$\frac{df}{dt} = (y + \sin x)\frac{dx}{dt} + (x + \tan y)\frac{dy}{dt}$$

Furthermore, if $x = e^{2t}$ and $y = t^3$, then

$$\begin{aligned}\frac{df}{dt} &= (t^3 + \sin e^{2t})2e^{2t} + (e^{2t} + \tan t^3)3t^2 \\ &= 2e^{2t}t^3 + e^{2t}3t^2 + 2e^{2t}\sin e^{2t} + 3t^2\tan t^3, \\ &= \frac{d}{dt}\big(e^{2t}t^3 - \cos e^{2t} + \ln|\sec t^3|\big)\end{aligned}$$

The chain rule works the same regardless of the number of variables involved. For instance, if $f = f(x, y, z)$, then

$$\boxed{\frac{df}{dt} = \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt} + \frac{\partial f}{\partial z}\frac{dz}{dt}}$$

If x, y, and z are functions of s and t, then

$$\boxed{\begin{aligned}\frac{\partial f}{\partial s} &= \frac{\partial f}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial s} + \frac{\partial f}{\partial z}\frac{\partial z}{\partial s} \\ \frac{\partial f}{\partial t} &= \frac{\partial f}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial t} + \frac{\partial f}{\partial z}\frac{\partial z}{\partial t}\end{aligned}}$$

14.5.re2. Supose $f(x,y)$, $x(u,v)$, and $y(u,v)$ are differentiable. Use the table of values to find f_u and f_v at $u=3$, $v=1$:

x	y	f	f_x	f_y
1	-1	10	-6	8

u	v	x	y	x_u	x_v	y_u	y_v
3	1	1	-1	7	5	3	9

Remember that $f(x,y) = f\big(x(u,v), y(u,v)\big)$, so f and its derivatives are evaluated at functions of x and y. At $u=3$ and $v=1$ (where $x=1$, $y=-1$),

$$\begin{aligned} f_u &= f_x x_u + f_y y_u \\ &= -6\cdot 7 + 8\cdot 3 = -18 \\ f_v &= f_x x_v + f_y y_v \\ &= -6\cdot 5 + 8\cdot 9 = 42 \end{aligned}$$

14.5.re3. The temperature at the point (x,y) is $T(x,y) = x^2 + 2xy$ degrees. Suppose that, at one moment, a particle traveling in the plane is at the point $(-3,2)$, moving in the positive x direction 0.5 units/second and in the negative y direction 1.5 units/second. At what rate is the temperature at the particle's position changing at that moment? Is the temperature increasing or decreasing then?

If the particle is at position $\langle x(t), y(t)\rangle$ at time t, then the temperature at its position is $T\big(x(t), y(t)\big)$. At the moment in question,

$$\begin{aligned} T_x &= 2x + 2y = -6 + 4 = -2 \\ T_y &= 2x = -6 \\ \frac{dT}{dt} &= T_x x_t + T_y y_t = -2\cdot\frac{1}{2} + -6\cdot\frac{-3}{2} = -1 + 9 = 8 \end{aligned}$$

That is, temperature experienced by the particle is increasing 8 degrees per second.

14.5.re4. Find f_u and f_v.

a. $f = (x - y)(2x + 3y),\ x = \ln(u - v),\ y = e^u v$

b. $f = \cos(p + 2q),\ p = uv,\ q = uv^{-1}$.

c. $f = a(b - c)^2,\ a = \sqrt{u + v},\ b = \cos v,\ c = \sin u$

14.5.re5. The dimensions of a rectangular box are changing with time. At one particular moment, the length, width, and height of the box are 2 cm, 3 cm 4 cm, resp., and are changing 0.4 cm/sec, -0.3 cm/sec, 0.2 cm/sec. At that time, at what rates are the following changing?

a. the volume of the box

b. the surface area of the box

14.5.re6. Suppose $f(x, y)$, $x(u, v)$, and $y(u, v)$ are differentiable. Use the first two tables to complete the third.

x	y	f	f_x	f_y
2	1	π	$-.2$	.3
0	4	2	.1	$-.4$

u	v	x	y	x_u	x_v	y_u	y_v
3	1	2	1	9	8	7	6
8	9	0	4	3	2	1	-1

u	v	f	f_u	f_v
3	1			
8	9			

Answers

14.5.re4a. $f_u = (4x + y)\left(\frac{1}{u-v}\right) + (x - 6y)ve^{uv}$. $f_v = (4x + y)\left(\frac{-1}{u-v}\right) + (x - 6y)ue^{uv}$. 14.5.re4b. $f_u = -\sin(p + 2q)v - 2\sin(p + 2q)v^{-1}$. $f_v = -\sin(p + 2q)u + 2\sin(p + 2q)uv^{-2}$. 14.5.re4c. $f_u = \frac{1}{2}(b - c)^2(u + v)^{-1/2} - 2a(b - c)\cos u$. $f_v = \frac{1}{2}(b - c)^2(u + v)^{-1/2} - 2a(b - c)\sin v$. 14.5.re5a. 3.6 cm^3/sec

14.5.re5b. 4.0 cm^2/sec 14.5.re6.

u	v	f	f_u	f_v
3	1	π	0.3	0.2
8	9	2	-0.1	0.6

14.6: The Gradient

The **gradient** of a differentiable function $f(x, y)$ of two variables is the vector

$$\nabla f(x, y) = \langle f_x(x, y), f_y(x, y)\rangle.$$

The gradient of a differentiable function $f(x, y, z)$ of three variables is the vector

$$\nabla f(x, y, z) = \langle f_x(x, y, z), f_y(x, y, z), f_z(x, y, z)\rangle.$$

While f is a scalar-valued function, ∇f is a **vector-valued** function having the same number of variables as f and the same number of components as variables. That is,

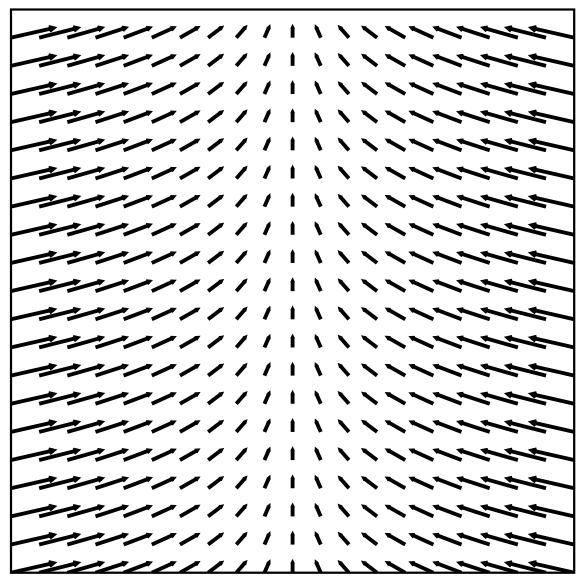
$$f : \mathbb{R}^n \to \mathbb{R}^1 \qquad \nabla f : \mathbb{R}^n \to \mathbb{R}^n.$$

∇f is an example of a **vector field**: a (vector-valued) function from $\mathbb{R}^n$ into $\mathbb{R}^n$. (§16.1)

14.6.re1. If $f(x, y) = y - x^2$, the its gradient is

$$\nabla f = \langle -2x, 1\rangle.$$

Here's a plot of the values of ∇f at some select points in the plane.



Directional derivatives and the gradient

If f is a function of two or three variables, if $\mathbf{r}$ denotes either $\langle x, y\rangle$ or $\langle x, y, z\rangle$, and if $\mathbf{u}$ is a unit vector, the **derivative of f in the direction $\mathbf{u}$** is the limit

$$D_{\mathbf{u}} f(\mathbf{r}) = \lim_{h\to 0} \frac{f(\mathbf{r} + h\mathbf{u}) - f(\mathbf{r})}{h}.$$

For instance, if $f = f(x, y)$ and $\mathbf{u} = \langle \frac{3}{5}, -\frac{4}{5}\rangle$, then

$$D_{\mathbf{u}} f(x, y) = \lim_{h\to 0} \frac{f(x + \frac{3}{5}h, y - \frac{4}{5}h) - f(x, y)}{h}.$$

The directional derivative $D_{\mathbf{u}} f(x, y, z)$ is the rate of increase of f at the point (x, y, z) when we move away from that point in the direction of $\mathbf{u}$. The partial derivatives we studied earlier are directional derivatives.

14.6.re2. Identify the three unit vectors $\mathbf{u}$, $\mathbf{v}$, $\mathbf{w}$ in $\mathbb{R}^3$ for which

$$\frac{\partial}{\partial x} = D_{\mathbf{u}} \qquad \frac{\partial}{\partial y} = D_{\mathbf{v}} \qquad \frac{\partial}{\partial z} = D_{\mathbf{w}}.$$

In practice, we can calculate directional derivatives without using their definition as limits:

Fact. If f is differentiable at the point $\mathbf{p}$ and $\mathbf{u}$ is a unit vector, then $D_{\mathbf{u}}f(\mathbf{p}) = \mathbf{u} \cdot \nabla f(\mathbf{p})$.

14.6.re3. Find ∇f for the given f. Then calculate $D_{\mathbf{u}}f$ and $D_{\mathbf{v}}f$ at the given point $\mathbf{p}$ for the given unit vector $\mathbf{u}$ and $\mathbf{v}$.

a. $f = y^2 - x^3$; $\mathbf{p} = (1, -1)$; $\mathbf{u} = \frac{1}{5}\langle -4, 3\rangle$; $\mathbf{v} = \frac{1}{\sqrt{13}}\langle 2, -3\rangle$
b. $f = xy - \ln(2x - y)$; $\mathbf{p} = (1, -1)$; $\mathbf{u} = \frac{1}{\sqrt{2}}\langle 1, 1\rangle$, $\mathbf{v} = -\mathbf{u}$.
c. $f = \frac{x+y}{z}$; $\mathbf{p} = (1, -1, 2)$; $\mathbf{u} = \frac{1}{3}\langle 2, -1, -2\rangle$; $\mathbf{v} = \frac{1}{2\sqrt{5}}\langle 0, 1, 2\rangle$

An interpretation of the gradient

Fact: If f is differentiable at the point $\mathbf{p}$, then $\pm|\nabla f(\mathbf{p})|$ are the greatest and least directional derivatives of f at $\mathbf{p}$, and they occur in the directions $\pm\nabla f(\mathbf{p})$.

14.6.re4. The temperature at the point (x, y, z) is $T(x, y, z) = xyz + 2xy - yz + 3xz$. I'm standing at the point $(0, 1, 2)$ and find it too hot. In which direction should I move to see the rapid decrease in temperature. If T is measured in °C and x, y, and z are measured in meters, at what rate will temperature decrease when I move away from $(0, 1, 2)$ in that direction?

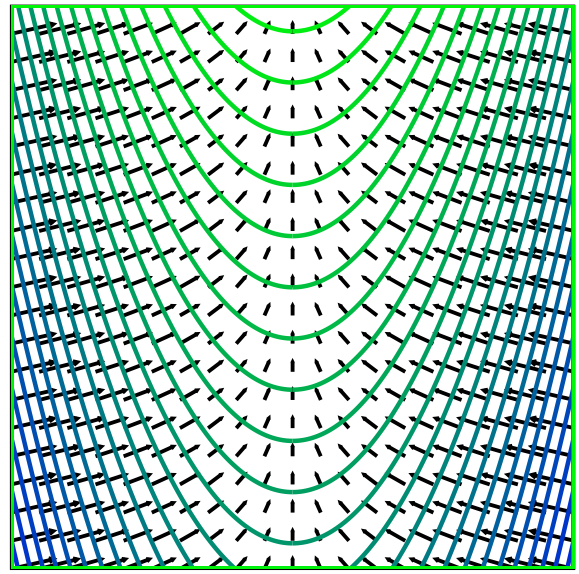
14.6.re5. Find the direction of greatest and least rate of change of f at the given point, and the derivative of f in that direction at p. Give the direction at a unit vector.

a. $f = y^2 - x^3$; $\mathbf{p} = (1, -1)$.
b. $f = xy - \ln(2x - y)$; $\mathbf{p} = (1, -1)$.
c. $f = \frac{x+y}{z}$; $\mathbf{p} = (1, -1, 2)$.

Level curves/surfaces and the gradient

Fact: If f is differentiable, then $\nabla f(\mathbf{p})$ is orthogonal at $\mathbf{p}$ to the level curve or surface of f that passes through $\mathbf{p}$.

14.6.re1, continued. Here's the same plot of $\nabla(y-x^2)$ at selected points in the plane superimposed on a contour map of level curves of $y-x^2$. Note the orthogonality of the gradient to the level curves.



14.6.re6. Find an equation to the plane tangent to the surface $x^2-y^3+4z^2=-3$ at the point $(1,2,-1)$.

Solution: The surface is orthogonal to $\nabla(x^2-y^3+4z^2)=\langle 2x,-3y^2,8z\rangle=\langle 2,-12,-8\rangle$ at $(1,2,-1)$. Plane is $2(x-1)-12(y-2)-8(z+1)=0$, or $x-6y-4z=-7$.

14.6.re7. Find an equation to the plane tangent to the given surface at the given point.

a. $e^{x+y-2z}-e^{2x-y+z}=0$; $(0,0,0)$.

b. $\sin x-\cos(x+z)-y=0$; $(\pi,0,\pi/2$.

c. $1+\tan^{-1}(x+y)=e^z+\frac{\pi}{4}$; $(\frac{3}{2},-\frac{1}{2},0)$

Answers

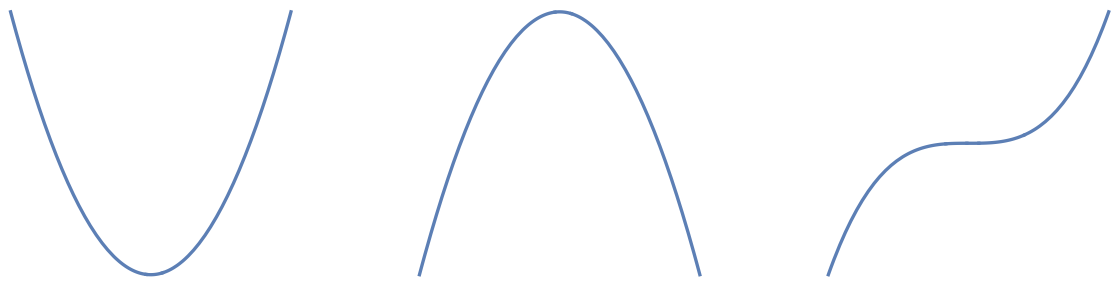
14.6.re2. $\frac{\partial}{\partial x}=D_{\mathbf{i}}$ $\frac{\partial}{\partial y}=D_{\mathbf{j}}$ $\frac{\partial}{\partial z}=D_{\mathbf{k}}$. 14.6.re3a. $\nabla f=\langle -3x^2,2y\rangle$; $D_{\mathbf{u}}f(\mathbf{p})=\frac{6}{5}$. $D_{\mathbf{v}}f(\mathbf{p})=0$. 14.6.re3b. $\nabla f=\langle y-\frac{2}{2x-y},x+\frac{1}{2x-y}\rangle$; $D_{\mathbf{u}}f(\mathbf{p})=-\frac{1}{3\sqrt{2}}$. $D_{\mathbf{v}}f(\mathbf{p})=\frac{1}{3\sqrt{2}}$. 14.6.re3c. $\nabla f(x,y,z)=\langle z^{-1},z^{-1},-(x+y)z^{-2}\rangle$. $D_{\mathbf{u}}f(\mathbf{p})=\frac{1}{6}$. $D_{\mathbf{v}}f(\mathbf{p})=\frac{1}{2\sqrt{5}}$. 14.6.re4. Most rapid decrease in temperature occurs in the direction $\frac{1}{\sqrt{105}}\langle -10,2,1\rangle$. (This is the normalization of $-\nabla T$ at $(0,1,2)$.) In that direction, temperature decreases $\sqrt{105}^\circ$C/m. 14.6.re5a. Maximum and minimum derivatives are $\pm\sqrt{13}$. Max occurs in the direction $\frac{-1}{\sqrt{13}}\langle 3,2\rangle$ and the minimum in the direction $\frac{1}{\sqrt{13}}\langle 3,2\rangle$. 14.6.re5b. Maximum and minimum derivatives are $\pm\frac{1}{3}\sqrt{41}$. Max occurs in the direction $\frac{1}{\sqrt{41}}\langle -5,4\rangle$ and the minimum in the direction $\frac{1}{\sqrt{41}}\langle 5,-4\rangle$ 14.6.re5c. Maximum and minimum derivatives are $\pm\frac{1}{\sqrt{2}}$. Max occurs in the direction $\frac{1}{\sqrt{2}}\langle 1,1,0\rangle$ and the minimum in the opposite direction. 14.6.re7a. $-x+2y=3z$. 14.6.re7b. $2x+y+z=5\pi/2$ 14.6.re7c. $x+y-2z=1$

14.7: Extrema of multivariate functions

An **extremum** of a function is a maximum or a minimum value of a function. The plural of extremum is **extrema**. There are two types of extrema: local and absolute.

Local extrema

These are function values that are the maximum or minimum of nearby values of the function.



A local min a local max not a local extremum

As we saw in calculus I, the local extrema of a $f(x)$ can occur only at its critical points: points p in the domain where $f'(p)$ is zero or undefined. But, not every critical point is the location of an extremum. The second derivative test can sometimes tell whether a local extremum occurs at a given critical point.

A local min a local max a saddle point

For functions of two variables, much the same thing is true:

Fact: The local extrema of $f(x, y)$ can occur only at its **critical points**: points $\mathbf{p}$ in its domain at which both $f_x(\mathbf{p})$ and $f_y(\mathbf{p})$ are either zero or undefined.

The Second Derivative Test: Suppose $f_x(\mathbf{p}) = f_y(\mathbf{p}) = 0$ for some point $\mathbf{p} = (x, y)$, and define D to be the determinant

$$\begin{vmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{vmatrix} = f_{xx}f_{yy} - f_{xy}^2.$$

If $D(\mathbf{p}) < 0$, then f has a saddle point at $\mathbf{p}$

If $D(\mathbf{p}) > 0$, then f has a local extremum at $\mathbf{p}$:

a local min if $f_{xx}(\mathbf{p})$ or $f_{yy}(\mathbf{p}) > 0$, and

a local max if $f_{xx}(\mathbf{p})$ or $f_{yy}(\mathbf{p}) < 0$.

14.7.re1. Find the locations of all local extrema and saddle points of $f(x,y) = x^2+2y^2-x^2y$.
Search for critical points by setting f_x and f_y equal 0:

$$\begin{aligned} f_x(x,y) &= 2x - 2xy = 0 \\ &= 2x(1-y) = 0 \;\Rightarrow\; x = 0 \text{ or } y = 1. \\ f_y(x,y) &= 4y - x^2 = 0 \\ & \qquad\qquad x = 0 \;\Rightarrow\; y = 0 \\ & \qquad\qquad y = 1 \;\Rightarrow\; x = \pm 2 \end{aligned}$$

So, the critical points are $(0,0)$, $(2,1)$, and $(-2,1)$.
Now use the Second Derivative Test at the critical points. You can simplify your calculations by factor a 2 out of both rows.

$$D = \begin{vmatrix} f_{xx} & f_{xy} \\ f_{xy} & f_{yy} \end{vmatrix} = \begin{vmatrix} 2-2y & -2x \\ -2x & 4 \end{vmatrix} = 2 \cdot 2 \begin{vmatrix} 1-y & -x \\ -x & 2 \end{vmatrix} = 4(2(1-y) - x^2)$$

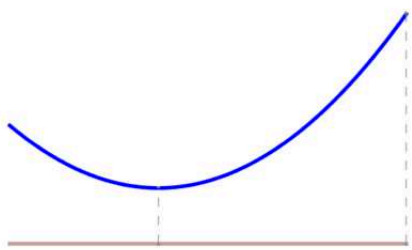
critical point	D	f_{xx}	conclusion
$(0,0)$	$8 > 0$	$2 > 0$	Local Minimum
$(2,1)$	$-16 < 0$	irrelevant	Saddle Point
$(-2,1)$	$-16 < 0$	irrelevant	Saddle Point

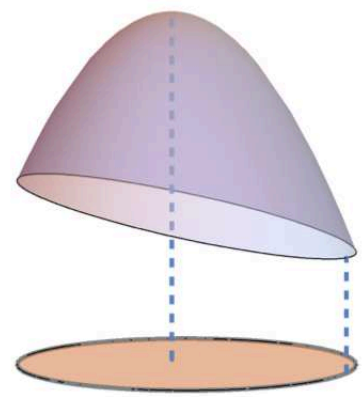
14.7.re2. Find the locations of all local extrema and saddle points.

a. $x(\frac{4}{3}x^2 - 1) - (x+y)^2$ b. $(x+y)^3 - (x-2)^2 - 3x - 3y$ c. xye^{x-y}

Absolute extrema on a domain

The absolute max and min of a function on a region R in its domain are its largest and smallest values on R. Just as we saw in calculus I, the absolute extrema of a function on a region must occur at a boundary point or at a critical point interior to the region. The second derivative test isn't relevant in such a problem.





14.7.re3. Find the absolute extrema of $f(x, y) = (x - y)e^{2xy}$ on the square bounded by the lines $x = 0$, $x = -1$, $y = 0$, $y = 1$.

The solution is to identify all the places in the interior or on the boundary at which the extrema might occur.

First look for interior critical points. When setting partials equal zero, it helps to factor.

$$f_x = e^{2xy} + 2(x - y)ye^{2xy} = e^{2xy}(1 + 2(x - y)y) = 0$$
$$f_y = -e^{2xy} + 2(x - y)xe^{2xy} = e^{2xy}(-1 + 2(x - y)x) = 0$$

Remembering that e^{2xy} cannot equal zero, the system of equations becomes

$$1 + 2(x - y)y = 0$$
$$-1 + 2(x - y)x = 0$$

Notice that $x - y$ can't equal zero, we can divide by it to solve for x and y in each equation.

$$y = \frac{-1}{2(x - y)}$$
$$x = \frac{1}{2(x - y)} = -y$$

Set $x = -y$ and the first equation becomes $y = \frac{-1}{-4y}$ which implies $y = \pm\frac{1}{2}$ and therefore $x = \mp\frac{1}{2}$. Of the critical points we found, only $(-\frac{1}{2}, \frac{1}{2})$ is in the interior of the region in question.

Now we'll find where along the boundary f could take its extrema. Remember that the max of a function of *one* variable on a closed interval can occur only at critical points interior to the interval or at the endpoints of the interval.
Let's analyze each line segment separately.
Segment 1: $x = -1, 0 \le y \le 1$: Along this segment, $f(-1, y) = (-1-y)e^{-2y}$, the derivative of which is (after some calculations, which you should double-check) $e^{-2y}(1 + 2y)$. This equals zero only at $y = -\frac{1}{2}$, which is outside the interval $0 \le y \le 1$. So, the extrema on this segment can only occur at the endpoints $(-1, 0)$ and $(-1, 1)$.
Segment 2: $x = 0$, $0 \le y \le 1$: Along this segment, $f(0, y) = -y$, the derivative of which is never zero. So, the extrema on this segment can only occur at the endpoints $(0, 0)$ and $(0, 1)$.
Segment 3: $y = 1$, $-1 \le x \le 0$: Along this segment, $f(x, 1) = (x - 1)e^{2x}$, the derivative of which is $e^{2x}(2x - 1)$. This equals zero only at $x = \frac{1}{2}$, which is outside the interval $-1 \le x \le 0$. So, the extrema on this segment can only occur at the endpoints $(-1, 1)$ and $(0, 1)$, which we identified earlier.
Segment 4: $y = 0$, $-1 \le x \le 0$: Along this segment, $f(x, 0) = x$, the derivative of which is never zero. So, the extrema on this segment can only occur at the endpoints $(-1, 0)$ and $(0, 0)$.
We now know that the absolute extrema of f can only occur at one of the points of interest identified above. When evaluate f at these five points, its absolute max and min must be in the list of calculated values.

(x, y)	$f(x, y)$
$(-\frac{1}{2}, \frac{1}{2})$	-0.61
$(0, 0)$	0
$(0, 1)$	-1
$(-1, 0)$	-1
$(-1, 1)$	-0.27

Conclusion: the absolute max of f on the square is 0, and its absolute min is -1.

14.7.re4. Find the absolute extrema of the function on the region bounded by the given curves.

a. $2x^2 - 4x + y^2 - 4y + 1$; $x = 0$, $y = 2$, $y = 2x$.
b. $x^2 - xy + y^2 - 3x$; $x = 0$, $x = 3$, $y = 0$, $y = 2$.
c. $-\frac{1}{2}x^2 + \frac{1}{2}y^2 - x$; $x^2 + y^2 = 1$

In addition to the problems in our text, you'll find more of these at
https://kunklet.people.cofc.edu/MATH221/stew1407prob.pdf

Answers

14.7.re2a. Saddle at $(\frac{1}{2}, -\frac{1}{2})$. Loc max at $(-\frac{1}{2}, \frac{1}{2})$. 14.7.re2b. Local max at $(2, -3)$. Saddle point at $(2, -1)$. 14.7.re2c. Local min at $(-1, 1)$. Saddle point at $(0, 0)$. 14.7.re4a. Abs max is 1 at $(0, 0)$. Abs min is -5 at $(1, 2)$. 14.7.re4b. Abs max is 4 at $(0, 2)$. Abs min is -$\frac{9}{4}$ at $(\frac{3}{2}, 0)$, $(3, \frac{3}{2})$, and $(\frac{5}{2}, 2)$. 14.7.re4c. Abs max is $\frac{3}{4}$ at $(-\frac{1}{2}, \pm\frac{\sqrt{3}}{2})$. Abs min is -$\frac{3}{2}$ at $(1, 0)$.

14.8: Lagrange multipliers

In this section we search for the extrema of a function subject to a constraint equation.

The method of Lagrange multipliers with one constraint equation

Fact 14.8.1: If f and g are functions of 2 or 3 variables, then the extrema of f subject to the constraint

$$g = \text{a constant}$$

can only occur at points on the constraint where

$$\nabla f \times \nabla g = \mathbf{0}. \tag{14.8.2}$$

14.8.re1. Find the extreme values of ye^x subject to the constraint $x^2 + 2y^2 = 2$.
Solution: Set $f(x, y) = ye^x$ and $g(x, y) = x^2 + 2y^2$. The extrema of f along $g = 2$ can only occur at those points where

$$\begin{aligned} \nabla f \times \nabla g &= \langle ye^x, e^x, 0\rangle \times \langle 2x, 4y, 0\rangle \\ &= 2e^x \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ y & 1 & 0 \\ x & 2y & 0 \end{vmatrix} = 2e^x\langle 0, 0, 2y^2 - x\rangle \end{aligned}$$

is the zero vector. Therefore, the critical points are the solutions to the system

$$\begin{aligned} 2e^x(2y^2 - x) &= 0 \\ x^2 + 2y^2 &= 2 \end{aligned}$$

Since e^x is never zero, $2y^2 = x$ at the critical points. Substituting this into $g = 2$ gives $x^2 + x = 2$, whose solutions are $x = -2$ and $x = 1$. Since $x = 2y^2$ must be nonnegative, we ignore $x = -2$. Substitute $x = 1$ into $2y^2 = x$ to find $y = \pm\sqrt{\frac{1}{2}}$. Therefore, f can take its max and min over $g = 2$ at the points $x = 1, y = \pm\sqrt{\frac{1}{2}}$.
Evaluate f at these two points.

(x, y)	ye^x	conclusion
$(1, \frac{1}{\sqrt{2}})$	$\frac{e}{\sqrt{2}}$	absolute max
$(1, -\frac{1}{\sqrt{2}})$	$-\frac{e}{\sqrt{2}}$	absolute min

Therefore the absolute max and min of f along the curve $g = 2$ are $\frac{e}{\sqrt{2}}$ and $-\frac{e}{\sqrt{2}}$, respectively.

14.8.re2. Find the points on $x^2 + y^2 + z^2 = 6$ that are nearest to and farthest from the point $(-1, 1, 1)$.

Tip: Distance is maximized (or minimized) exactly when distance-squared is maximized (or minimized).

Solution: We wish to maximize and minimize distance-squared from points (x, y, z) to $(-1, 1, 1)$, so set $f(x, y, z) = (x + 1)^2 + (y - 1)^2 + (z - 1)^2$. Set $g(x, y, z) = x^2 + y^2 + z^2$, so that the constraint is $g = 6$.
f can take its max and min along the constraint only at points at which $\nabla f \times \nabla g = \mathbf{0}$:

$$\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2(x+1) & 2(y-1) & 2(z-1) \\ 2x & 2y & 2z \end{vmatrix} = 4\langle y - z, -x - z, x + y\rangle = \mathbf{0}.$$

Setting these vectors equal means

$$y - z = 0 \qquad -x - z = 0 \qquad x + y = 0$$

which tells us that $x = -y = -z$. Substitute this into the constraint equation

$$x^2 + y^2 + z^2 = 6,$$

which implies $3z^2 = 6$, and $z = \pm\sqrt{2} = y = -x$. Now evaluate f at the two critical points:

critical point	$(x+1)^2 + (y-1)^2 + (z-1)^2$	conclusion
$(\sqrt{2}, -\sqrt{2}, -\sqrt{2})$	$3(\sqrt{2}+1)^2$	absolute maximum
$(-\sqrt{2}, \sqrt{2}, \sqrt{2})$	$3(\sqrt{2}-1)^2$	absolute minimum

To answer the question, the point closest to $(-1, 1, 1)$ is $(-\sqrt{2}, \sqrt{2}, \sqrt{2})$ and the farthest is $(\sqrt{2}, -\sqrt{2}, -\sqrt{2})$. Had we been asked to report those distances, we'd take square root to obtain $\sqrt{3}(\sqrt{2} \pm 1)$.

The method of Lagrange multipliers with two constraint equations

Fact 14.8.3: If f, g, and h are functions of 3 variables, then the extrema of f subject to the constraints

$$g(x, y, z) = \text{a constant}$$
$$h(x, y, z) = \text{a constant}$$

can occur only at points where

$$\nabla f \cdot (\nabla g \times \nabla h) = 0. \tag{14.8.4}$$

14.8.re3. Find the minimum distance between the origin and the intersection of the planes $x + 2y = 12$ and $y + z = 6$.
Solution: The constraints are level surfaces of the functions $x+2y$ and $y+z$, the gradients of which are

$$\nabla(x + 2y) = \langle 1, 2, 0\rangle \qquad \nabla(y + z) = \langle 0, 1, 1\rangle.$$

To minimize distance-squared-to-origin, set $f = x^2 + y^2 + z^2$, calculate its gradient $\nabla f = \langle 2x, 2y, 2z\rangle$, and search for the point(s) on the two constraints at which the triple product

$$\langle 2x, 2y, 2z\rangle \cdot \big(\langle 1, 2, 0\rangle \times \langle 0, 1, 1\rangle\big) = \begin{vmatrix} 2x & 2y & 2z \\ 1 & 2 & 0 \\ 0 & 1 & 1 \end{vmatrix}$$

is zero. You can take out common factors from any row or column. In particular, factor 2 out of the top row:

$$\begin{aligned} &= 2\begin{vmatrix} x & y & z \\ 1 & 2 & 0 \\ 0 & 1 & 1 \end{vmatrix} \\ &= 2\left(x\begin{vmatrix} 2 & 0 \\ 1 & 1 \end{vmatrix} - y\begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} + z\begin{vmatrix} 1 & 2 \\ 0 & 1 \end{vmatrix}\right) \\ &= 2(2x - y + z). \end{aligned}$$

The critical point is the solution to the system of equations

$$\begin{aligned} x + 2y \qquad &= 12 \\ y + z &= 6 \\ 2x - y + z &= 0. \end{aligned}$$

It's not too hard to find that the sole solution to this system is $x = 2$, $y = 5$, $z = 1$, and since we know some point on the constraint line is closest to the origin, this must be that point. Its distance to the origin is $\sqrt{f(2, 5, 1)} = \sqrt{30}$.

14.8.re4. Find the max and min of the function subject to the constraint.

a. $x^2 + 4y^3$; $x^2 + 2y^2 = 1$.
b. $3x^2 + 2y^2 - 4y + 1$; $x^2 + y^2 = 16$
c. $x + 2y - z$; $x^2 + 3y^2 + z^2 = \frac{5}{2}$.

For the record, the critical point condition in Lagrange multipliers is traditionally stated as

$$\nabla f = \lambda \nabla g \text{ (for some } \lambda) \text{ or } \nabla g = 0,$$

which is equivalent to (14.8.2). In case of two constraint equations, you're more likely to see

$$\nabla f = \lambda_1 \nabla g + \lambda_2 \nabla h \text{ (for some } \lambda_1 \text{ and } \lambda_2) \text{ or } \nabla g = 0 \text{ or } \nabla h = 0$$

than the simpler (14.8.4). The method is named for the "multipliers" λ, λ_1, λ_2. Equations (14.8.2) and (14.8.4) are generally simpler to solve than these, since they avoid the unnecessary λs.

In addition to the problems in our text, you'll find more of these at
https://kunklet.people.cofc.edu/MATH221/stew1408prob.pdf

Answers

14.8.re4a. max $= \sqrt{2}$ at $(0, \frac{1}{\sqrt{2}})$. min $= -\sqrt{2}$ at $(0, -\frac{1}{\sqrt{2}})$. 14.8.re4b. max $= 53$ at the two points $(\pm\sqrt{12}, -2)$. min $= 17$ at $(0, 4)$ 14.8.re4c. max is $\frac{5}{3}\sqrt{3}$ at $(\frac{\sqrt{3}}{2}, \frac{\sqrt{3}}{3}, -\frac{\sqrt{3}}{2})$. min is $-\frac{5}{3}\sqrt{3}$ at $(-\frac{\sqrt{3}}{2}, -\frac{\sqrt{3}}{3}, \frac{\sqrt{3}}{2})$.

15.1: Double integrals over rectangles

Review of Riemann sums and the definite integral from calc I

Suppose $f(x)$ is a function defined on the interval $[a, b]$. Divide $[a, b]$ into m subintervals of equal length Δx, and choose "sample points" $x_1^*, x_2^*, \ldots, x_m^*$, one from each subinterval. Then

$$f(x_1^*)\Delta x + f(x_2^*)\Delta x + \cdots + f(x_m^*)\Delta x = \sum_{i=1}^{m} f(x_i^*)\Delta x$$

is a **Riemann sum** for $f(x)$ on $[a, b]$. The **definite integral** of $f(x)$ on $[a, b]$ is defined to be the limit of its Riemann sums:

$$\int_a^b f(x)\,dx = \lim_{m\to\infty} \sum_{i=1}^{m} f(x_i^*)\Delta x$$

15.1.re1. Below, a Riemann sum is calculated for a function $f(x)$ using 6 subintervals and their midpoints.
Because $f(x) > 0$ on $[a, b]$, this Riemann sum is an approximation to the area under the graph of f over this interval on the x-axis.

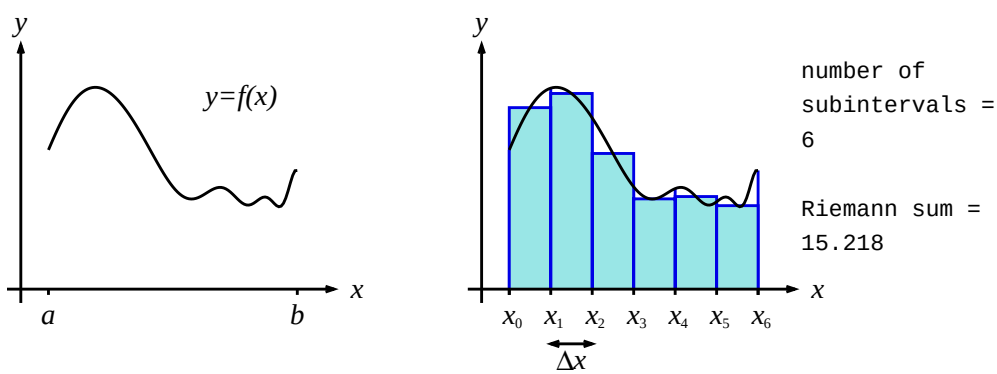


Figure 15.1.re1

Riemann sums in two variables over rectangles

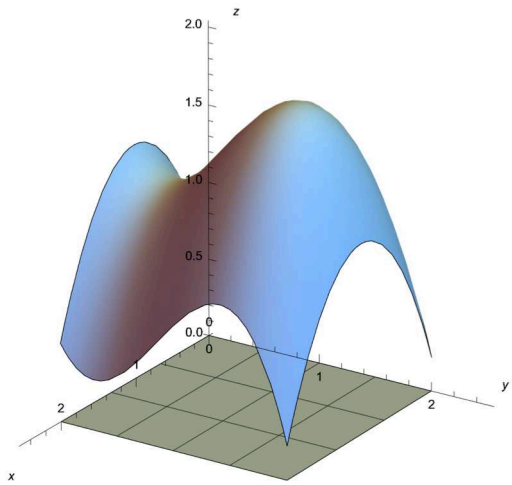
The rectangle

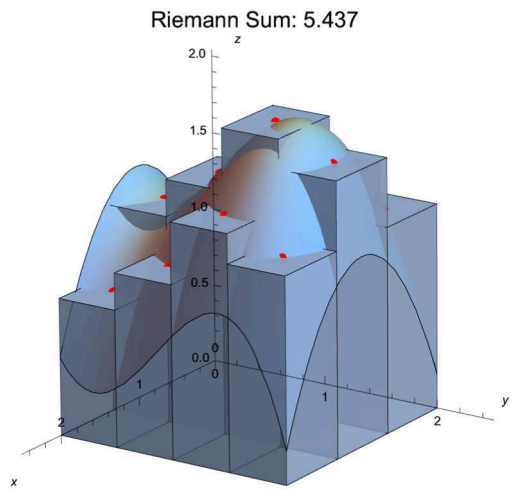
$$\{(x, y) \mid a \le x \le b, c \le y \le d\}$$

in the xy-plane is denoted $[a, b] \times [c, d]$.
If we divide $[a, b]$ into m subintervals and $[c, d]$ into n subintervals, and choose a sample point $(x_{i,j}^*, y_{i,j}^*)$ in each of the resulting $\Delta x \times \Delta y$ subrectangles of $[a, b] \times [c, d]$, then $\displaystyle\sum_{i=1}^{m}\sum_{j=1}^{n} f(x_{i,j}^*, y_{i,j}^*)\,\Delta x\,\Delta y$ is a Riemann sum for $f(x, y)$ over $[a, b] \times [c, d]$.
The **double integral** of $f(x, y)$ over this rectangle is defined as the limit of its Riemann sums as both m and n go to ∞:

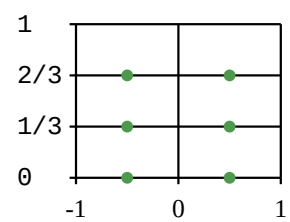
$$\int\int_{[a,b]\times[c,d]} f(x, y)\,dA = \lim_{n,m\to\infty} \sum_{i=1}^{m}\sum_{j=1}^{n} f(x_{i,j}^*, y_{i,j}^*)\,\Delta x\Delta y$$

15.1.re2. Below, a Riemann sum is calculated for a function $f(x, y)$ on the rectangle $[0, 2] \times [0, 2]$ using 3 subintervals in the x-direction and 4 in the y-direction. Sample points are taken to be the midpoints of the subrectangles. Because $f(x, y) > 0$ on $[0, 2] \times [0, 2]$, this Riemann sum is an approximation to the area under the graph of f over this rectangle in the xy-plane.





15.1.re3. Calculate a Riemann sum for the function e^{2x+3y} on $[-1, 1] \times [0, 1]$ using $m = 2$ and $n = 3$ subintervals in the x- and y-directions, respectively. Use the midpoints of the lower side of each subrectangle for sample points



Solution: It helps to picture the subdivision of the rectangle: The sample points (•) are at $x = \pm\frac{1}{2}$ and $y = 0, \frac{1}{3}, \frac{2}{3}$. Dimensions of the subrectangles are $\Delta x = 1$ by $\Delta y = \frac{1}{3}$, and so the Riemann sum is

$$1 \cdot \frac{1}{3} \cdot \left(e^{-2\cdot\frac{1}{2}+3\cdot 0} + e^{-2\cdot\frac{1}{2}+3\cdot\frac{1}{3}} + e^{-2\cdot\frac{1}{2}+3\cdot\frac{2}{3}} + e^{2\cdot\frac{1}{2}+3\cdot 0} + e^{2\cdot\frac{1}{2}+3\cdot\frac{1}{3}} + e^{2\cdot\frac{1}{2}+3\cdot\frac{2}{3}}\right)$$
$$= \frac{1}{3} \cdot \left(e^{-1} + e^{0} + e^{1} + e^{1} + e^{2} + e^{3}\right).$$

15.1.re4. Calculate the Riemann sum for the given function, rectangle, m, n, and sample points.

a. $xy + x - y - 1$; $[0, 2] \times [1, 3]$; $n = 2$; $m = 2$; upper right corners
b. $\cos(x + y)$; $[0, \pi] \times [-\frac{\pi}{2}, \frac{\pi}{2}]$; $m = 3$; $n = 2$; midpoints.
c. $x^2 + y^2$; $[-1, 1] \times [0, 2]$; $m = 4$; $n = 2$; points closest to $(0, 0)$.

Iterated integration

We can calculate double integrals by integrating iteratively: first with respect to one variable, then with respect to the other. **Fubini's Theorem** promises that the value of the integral is independent of the order of integration, provided the integrand is piecewise continuous.

15.1.re5. Demonstrate that the value of the double integral of $x^2 + y$ over the rectangle $[-1, 1] \times [0, 2]$ is the same regardless of the order of integration.

Let's first integrate with respect to x, then y:

$$\int_0^2 \int_{-1}^1 (x^2 + y)\, dx\, dy = \int_0^2 \left((\tfrac{1}{3}x^3 + xy)\Big|_{-1}^1 \right) dy$$
$$= \int_0^2 \left((\tfrac{1}{3} + y) - (-\tfrac{1}{3} - y) \right) dy = \int_0^2 (\tfrac{2}{3} + 2y)\, dy$$
$$= (\tfrac{2}{3}y + y^2)\Big|_0^2 = \tfrac{16}{3}.$$

This time, integrate first with respect to y:

$$\int_{-1}^1 \int_0^2 (x^2 + y)\, dy\, dx = \int_{-1}^1 \left((yx^2 + \tfrac{1}{2}y^2)\Big|_0^2 \right) dx$$
$$= \int_{-1}^1 \left(2x^2 + 2 \right) dx = (\tfrac{2}{3}x^3 + 2x)\Big|_{-1}^1 = \tfrac{16}{3}.$$

15.1.re6. Evaluate the double integral by iterated integration.

a. $\iint_{[1,2]\times[1,3]} (xy + x - y - 1)\, dA$
b. $\iint_{[0,\frac{\pi}{4}]\times[-\frac{\pi}{2},\frac{\pi}{2}]} (\cos(x + y))\, dA$
c. $\iint_{[-2,2]\times[-1,1]} (2x + y^2)\, dA$

Average value

The **average value** of a function $f(x, y)$ over a region R in the xy-plane is defined as

$$f_{\text{ave}} = \frac{1}{\text{area}(R)} \iint_R f(x, y)\, dA.$$

Consequently,

$$\text{area}(R) \cdot f_{\text{ave}} = \iint_R f(x, y)\, dA.$$

That is, f_{ave} is the constant function having the same integral over R as f.

15.1.re7. Calculate the average value over the given rectangle of the given function.

a. $[1, 2] \times [1, 3]$; $xy + x - y - 1$
b. $[0, \frac{\pi}{4}] \times [-\frac{\pi}{2}, \frac{\pi}{2}]$; $\cos(x + y)$
c. $[-2, 2] \times [-1, 1]$; $2x + y^2$

Answers

15.1.re4a. 7. 15.1.re4b. $\frac{\pi^2}{6}\left(\cos(-\frac{\pi}{12}) + \cos(\frac{3\pi}{12}) + \cos(\frac{5\pi}{12}) + \cos(\frac{7\pi}{12}) + \cos(\frac{9\pi}{12}) + \cos(\frac{13\pi}{12})\right)$. 15.1.re4c. 5/2. 15.1.re6a. 3. 15.1.re6b. $\sqrt{2}$. 15.1.re6c. $\frac{8}{3}$. 15.1.re7a. 3/2. 15.1.re7b. $4\sqrt{2}/\pi^2$. 15.1.re7c. $\frac{1}{3}$.

15.2: Double integrals over general regions and properties of the double integral.

Integration over non-rectangular regions

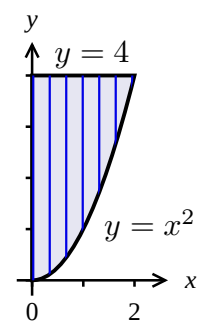
We can integrate iteratively over regions other than rectangles by letting the limits of the inner integral depend on the variable of the outer integrals, e.g.

$$\int_a^b \int_{g(x)}^{h(x)} f(x,y)\,dy\,dx \quad \text{or} \quad \int_a^b \int_{g(y)}^{h(y)} f(x,y)\,dx\,dy.$$

That is, limits on the outer integral must be constant, but limits on the inner integral can depend on the outer variable of integration.

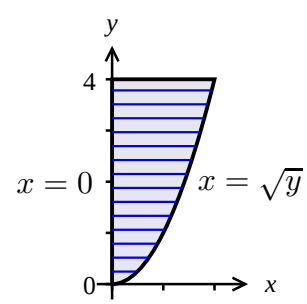
15.2.re1. Calculate $\iint_R x(y-1)\,dA$, where R is the region in quadrant one bounded by $y=4$, $x=0$, and $y=x^2$, twice, using the two different orders of integration.

Evaluation by $\iint\ dy\,dx$:



$$\int_0^2 \int_{x^2}^4 x(y-1)\,dy\,dx = \int_0^2 \left(x(\tfrac{1}{2}y^2 - y)\Big|_{x^2}^4 \right) dx$$
$$= \int_0^2 x\Big((8-4) - (\tfrac{1}{2}x^4 - x^2)\Big)\,dx = \int_0^2 x(4 + x^2 - \tfrac{1}{2}x^4)\,dx$$
$$= \int_0^2 (4x + x^3 - \tfrac{1}{2}x^5)\,dx = (2x^2 + \tfrac{1}{4}x^4 - \tfrac{1}{12}x^6)\Big|_0^2 = \tfrac{20}{3}$$

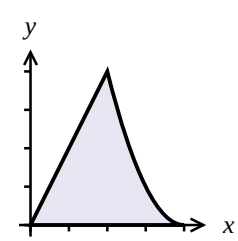
Evaluation by $\iint\ dx\,dy$:



$$\int_0^4 \left(\int_0^{\sqrt{y}} x(y-1)\,dx \right) dy = \int_0^4 \left(\tfrac{1}{2}x^2(y-1)\Big|_0^{\sqrt{y}} \right) dy$$
$$= \int_0^4 \tfrac{1}{2}y(y-1)\,dy = \tfrac{1}{2}\int_0^4 (y^2 - y)\,dy$$
$$= \tfrac{1}{2}(\tfrac{1}{3}y^3 - \tfrac{1}{12}y^2)\Big|_0^4 = \tfrac{20}{3}$$

In some cases, we choose the order of integration to make the problem easier.

15.2.re2. Let D be the region bounded by $y=0$, $y=2x$ and $y=(x-4)^2$ shown here. Write $\iint_D f(x,y)\,dA$ as an iterated integral in both possible orders.



15.2.re3. Evaluate $\displaystyle\int_0^1 \int_{-1}^{-\sqrt{x}} \sqrt{y^3+1}\,dy\,dx$.

15.2.re4. Evaluate the double integral, using whichever order of integration is easiest.

a. $\iint_D (y+1)(x-1)\,dA$, $D = \{(x,y) \mid x^2 - 1 \le y \le 1 - x^2\}$

b. $\iint_E xy\,dA$, $E =$ the triangle with vertices $(0,0), (1,0), (2,2)$.

c. $\iint_F y\,dA$; $F =$ the upper half of the circle centered at $(0,0)$ of radius 2.

15.2.re5. Find the average value of the function on the given region.

a. $(y+1)(x-1)$, on $\{(x,y) \mid x^2 - 1 \le y \le x^2 + 1\}$

b. y, on the upper half of the circle centered at $(0,0)$ of radius 2.

Properties of the double integral

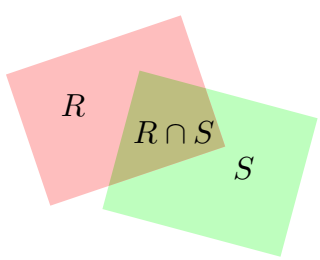
Fact. If b and c are constants, and R and S are regions in the plane on which both $f(x,y)$ and $g(x,y)$ exist, then

1. $\displaystyle\iint_R \big(bf(x,y) + cg(x,y)\big)\,dA = b\iint_R f(x,y)\,dA + c\iint_R g(x,y)\,dA.$

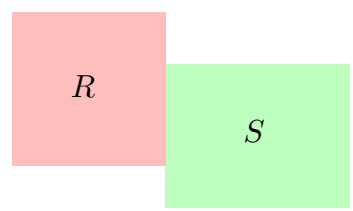
2. $\displaystyle\iint_R c\,dA = c \cdot \text{area}(R).$

3a. $\displaystyle\iint_{R\cup S} f(x,y)\,dA = \iint_R f(x,y)\,dA + \iint_S f(x,y)\,dA - \iint_{R\cap S} f(x,y)\,dA,$ and so

3b. $\displaystyle\iint_{R\cup S} f(x,y)\,dA = \iint_R f(x,y)\,dA + \iint_S f(x,y)\,dA,$ if $R \cap S$ has zero area.

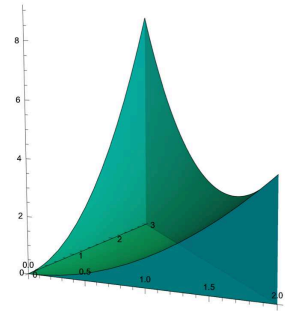


3a.



3b.

15.2.re6. Find the volume of the region in xyz-space bounded by the planes $x = 0$, $y = 0$, $3x + 2y = 6$, $z = 0$, and the surface $z = x^2 + y^2$.



Answers

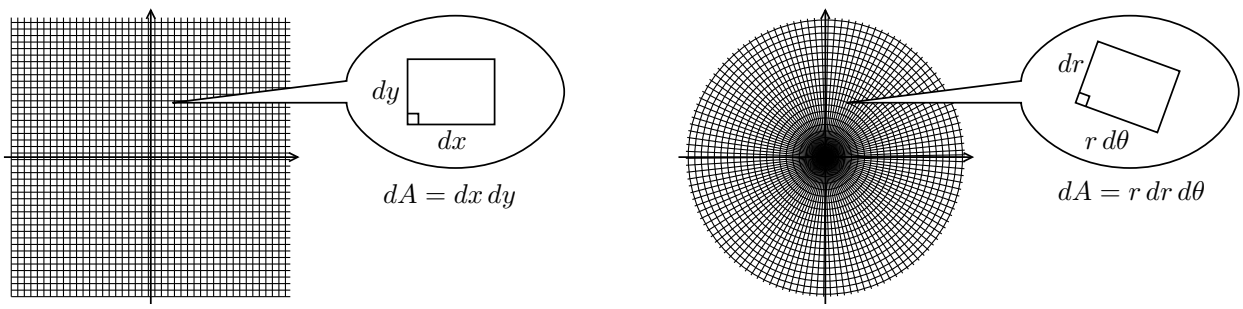
15.2.re2. $\int_0^4 \int_{y/2}^{4-\sqrt{y}} f(x,y)\,dx\,dy$, and $\int_0^2 \int_0^{2x} f(x,y)\,dy\,dx + \int_2^4 \int_0^{(x-4)^2} f(x,y)\,dy\,dx$ 15.2.re3. To avoid integration of $\sqrt{y^3+1}\,dy$, change order of integration to $\int_{-1}^0 \int_0^{y^2} \sqrt{y^3+1}\,dx\,dy$ and evaluate. Result: $\frac{2}{9}$. 15.2.re4a. $-8/3$. 15.2.re4b. 1. 15.2.re4c. $16/3$. 15.2.re5a. -1. 15.2.re5b. $\frac{8}{3\pi}$. 15.2.re6. $\frac{13}{2}$.

15.3: Double integrals in polar coordinates.

The polar coordinate system is coordinate system for the plane. For a review of polar coordinates, see the "Polar Coordinates" section (especially the first page and half) of my MATH 220 review notes at https://kunklet.people.cofc.edu/MATH220/220review.pdf.

Differential area dA in polar coordinates.

All the lines along which x or y are constant divide the plane into infinitesimally small rectangles, whose area is $dA = dx\,dy$. Likewise, the circles along which r is positive and constant and the rays along with θ is constant divide the plane into infinitesimally small rectangles with dimensions dr by $r\,d\theta$, and so $dA = r\,dr\,d\theta$.



Warning: Since dA must be positive, the formula

$$dA = r\,dr\,d\theta$$

requires $r > 0$. In general, $dA = |r|\,dr\,d\theta$.

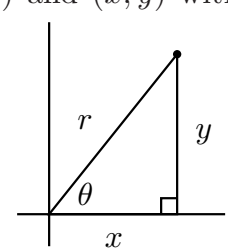
It may be useful to write

$$\iint_D f(x,y)\,dA$$

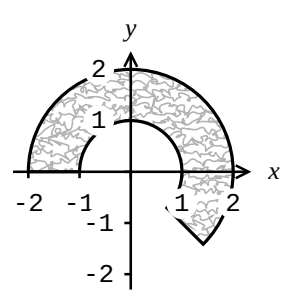
in polar coordinates if either the region D or the integrand f is easier to write in polar coordinates than in rectangular.

Tip: practice writing the following four relationships between (r,θ) and (x,y) with the help of this picture (drawn as if r were positive and θ were acute):

$$x = r\cos\theta \qquad r^2 = x^2 + y^2$$
$$y = r\sin\theta \qquad \tan\theta = y/x$$



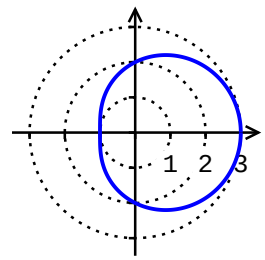
15.3.re1. Integrate $\iint_D x\,dA$, where D is the region pictured here, bounded by the circles centered at $(0,0)$ of radii 1 and 2, the negative x axis, and the line $y = -x$.



Solution: D is easily described in polar coordinates: $-\frac{\pi}{4} \le \theta \le \pi$, $1 \le r \le 2$. In polar coordinates, $x\,dA = (r\cos\theta)r\,dr\,d\theta$, so the integral is

$$\int_{-\pi/4}^{\pi}\int_1^2 r^2\cos\theta\,dr\,d\theta = \int_{-\pi/4}^{\pi}\frac{1}{3}r^3\Big|_1^2 cos\theta\,d\theta = \frac{7}{3}\int_{-\pi/4}^{\pi}\cos\theta\,d\theta$$
$$= \frac{7}{3}\sin\theta\Big|_{-\pi/4}^{\pi} = \frac{7}{3}\big(\sin\pi - \sin(-\pi/4)\big) = \frac{7}{3\sqrt{2}}$$

15.3.re2. Find the area inside the polar curve $r = 2 + \cos\theta$.
To integrate $\cos^2\theta$, rewrite it using the half angle identity:



$$\cos^2\theta = \frac{1}{2}(1+\cos(2\theta)).$$

$\text{area}(D) = \iint_D dA =$

$$\int_0^{2\pi}\int_0^{2+\cos\theta} r\,dr\,d\theta = \frac{1}{2}\int_0^{2\pi}(2+\cos\theta)^2\,d\theta = \frac{1}{2}\int_0^{2\pi}(4+4\cos\theta+\cos^2\theta)\,d\theta$$
$$= \frac{1}{2}\int_0^{2\pi}(4+4\cos\theta+\frac{1}{2}(1+\cos(2\theta)))\,d\theta = \frac{1}{2}\int_0^{2\pi}(\frac{9}{2}+4\cos\theta+\frac{1}{2}\cos(2\theta))\,d\theta$$
$$= \frac{1}{2}(\frac{9}{2}\theta+4\sin\theta+\frac{1}{4}\sin(2\theta))\Big|_0^{2\pi} = \frac{9}{2}\pi$$

15.3.re3. Find the area of one loop of the rose $r = \sin(2\theta)$.

15.3.re4. Evaluate the double integral

a. $\iint_D (x-y)\,dA$, $D =$ the portion of the disk $x^2+y^2 \le 4$ in quadrants I, II, and III.
b. $\iint_E \cos(x^2+y^2)\,dA$, $E =$ the portion of the disk $x^2+y^2 \le 4$ in quadrant 1 below $y = x$.
c. $\iint_F y\,dA$; $F = \{(x,y)\,|\,1 \le x^2+y^2 \le 2, y \ge 0\}$.
d. $\int_0^2\int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}}(x^2+y^2)^{3/2}\,dy\,dx$

15.3.re5. Find the volume of the give solid.

a. Inside $x^2+y^2 = 1$, beween $z = x^2+y^2$ and $z = 0$.
b. Inside both the cylinder $x^2+y^2 = \frac{1}{4}$ and the sphere $x^2+y^2+z^2 = 1$.
c. Under the cone $z = 2-\sqrt{x^2+y^2}$ and above the annulus $1 \le x^2+y^2 \le 4$ in the plane.

Answers

15.3.re3. $\pi/8$ 15.3.re4a. $-\frac{16}{3}$. 15.3.re4b. $\frac{\pi}{8}\sin 4$. 15.3.re4c. $\frac{2}{3}(2^{3/2}-1)$. 15.3.re4d. $32\frac{\pi}{5}$.
15.3.re5a. $\pi/2$. 15.3.re5b. $(\frac{2}{3}-\frac{\sqrt{3}}{4})\pi$. 15.3.re5c. $\frac{4\pi}{3}$.

15.5: Surface area of $z = f(x, y)$.

The area of the surface $z = f(x, y)$ above the region D in the xy-plane is

$$\iint_D \sqrt{1 + f_x^2 + f_y^2}\, dA.$$

Note the similarity of this formula to the *arc length* of the curve $y = f(x)$ over the interval $[a, b]$:

$$\int_a^b \sqrt{1 + f'^2}\, dx$$

that we learned in calculus II.

15.5.re1. Find the area of the surface $z = 5 - x^2 - y^2$ above $z = 1$.
The region of integration in the xy-plane is where $5 - x^2 - y^2 \geq 1$, or $4 \geq x^2 + y^2$, the interior of the circle of radius 2 centered at the origin. Both this and the integrand $\sqrt{1 + z_x^2 + z_y^2} = \sqrt{1 + 4x^2 + 4y^2}$ suggest that we switch to polar coordinates:

$$\int_0^{2\pi} \int_0^2 r\sqrt{1 + 4r^2}\, dr\, d\theta = 2\pi \cdot \frac{\pi}{6}(17^{3/2} - 1).$$

15.5.re2. Find the area of the graph of the given surface.

a. $z = 1 - 2x + 3y$, between $y = x^2$ and $y = 2 - x^2$
b. $z = 3x + y^2$, above the triangle in the plane with vertices $(0, 0)$, $(0, 1)$, and $(2, 1)$.
c. $z = xy$, inside the cylinder $x^2 + y^2 = 3$.
d. the upper half of the sphere $x^2 + y^2 + z^2 = 4$, inside $x^2 + y^2 = 3$.
e. the upper half of the sphere $x^2 + y^2 + z^2 = 4$, inside $x^2 + y^2 = 2x$.
f. the upper half of the cylinder $x^2 + z^2 = 1$, above the rectangle $[0, 1] \times [-1, 1]$.

Answers

15.5.re2a. $\frac{8}{3}\sqrt{14}$. 15.5.re2b. $\frac{1}{3}(7\sqrt{14} - 5\sqrt{10})$. 15.5.re2c. $\frac{14\pi}{3}$. 15.5.re2d. 4π. 15.5.re2e. $4\pi - 8$. 15.5.re2f. π.

15.6: Triple integals

Like a double integral, the triple integral of a function $f(x, y, z)$ over a region D in $\mathbb{R}^3$

$$\iiint_D f(x, y, z)\, dV$$

is defined as a limit of Riemann sums and evaluated by iterated integration.

15.6.re1. Evaluate $\int_{-2}^{2} \int_0^{4-x^2} \int_0^z 3y\, dy\, dz\, dx$.

Solution:

$$\int_{-2}^{2} \int_0^{4-x^2} \tfrac{3}{2} y^2 \Big|_0^z dz\, dx = \int_{-2}^{2} \int_0^{4-x^2} \tfrac{3}{2} z^2\, dz\, dx$$
$$= \int_{-2}^{2} \tfrac{1}{2} z^3 \Big|_0^{4-x^2} dx = \int_{-2}^{2} \tfrac{1}{2}(4 - x^2)^3\, dx$$

Since $(4 - x^2)^3$ is even, this equals

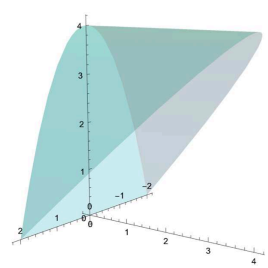
$$\int_0^2 (4 - x^2)^3\, dx = \int_0^2 (4^3 - 3 \cdot 4^2 x^2 + 3 \cdot 4x^4 - x^6)\, dx$$
$$= (64x - 16x^3 + \tfrac{12}{5} x^5 - \tfrac{1}{7} x^7)\Big|_0^2 = \tfrac{12}{5} 2^5 - \tfrac{1}{7} 2^7.$$

We can calculate volumes in three-space using $\iiint_D dV = \text{volume}(D)$:

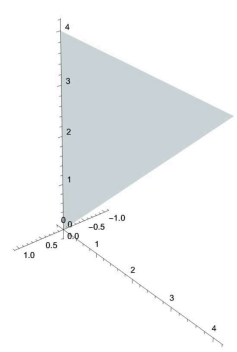
15.6.re2. Find the volume of the region of integration in 15.6.re1. (Hint: replace the integrand in 15.6.re1 with 1 and calculate the triple integral.)

15.6.re3. Rewrite the integral in 15.6.re2 in the order $\iiint dx\, dz\, dy$

To complete this type of exercise, it is essential to make a good sketch of the region in question. Start with a graph of the planes $y = z$ and $y = 0$ and the cylinder $z = 4 - x^2$. See graph right, upper.



The largest and smallest values of y in the entire region are 0 and 4. Now picture the orthogonal projection of the region onto the yz-plane (the triangle pictured right, lower) For each y in $[0, 4]$, the largest and smallest values of z are y and 4. Finally, pass a line parallel the x-axis through a point (y, z) in the shadow of the region in the yz-plane. Observe that it enter and exits the region at $x = \pm\sqrt{4 - z}$.



The required integral is $\displaystyle\int_0^4 \int_y^4 \int_{-\sqrt{4-z}}^{\sqrt{4-z}} dx\, dz\, dy.$

15.6.re4. Evaluate the triple integral.

a. $\int_0^1 \int_0^{1-z} \int_1^{4-3x-3z} e^y \, dy \, dx \, dz$

b. $\int_0^1 \int_0^{2-x} \int_0^{1-x^2} \frac{x}{(3+z)^2} \, dz \, dy \, dx$

c. $\int_0^2 \int_0^{4-x^2} \int_0^{\sqrt{z}} x \, dy \, dz \, dx$

15.6.re5. Rewrite the integrals in the given order order of integration.

a. $\int_0^1 \int_0^{1-z} \int_1^{4-3x-3z} e^y \, dy \, dx \, dz \to \iiint \, dz \, dy \, dx$

b. $\int_0^2 \int_0^{4-x^2} \int_0^{\sqrt{z}} \, dy \, dz \, dx \to \iiint \, dx \, dz \, dy$

c. $\int_{-2}^2 \int_0^{4-x^2} \int_0^z 3y \, dy \, dz \, dx \to \iiint \, dz \, dx \, dy$

15.6.re6. Evaluate the triple integral by first changing the order of integation.

$$\int_{-1}^2 \int_0^{\sqrt{2-z}} \int_y^{\sqrt{2-z}} (1 + \sqrt{1+x^2}) \, dx \, dy \, dz$$

Answers

15.6.re2. 256/15. 15.6.re4a. $-\frac{17}{18}e + \frac{1}{9}e^4$. 15.6.re4b. $-\frac{7}{9} + 2\ln\frac{3}{2}$. 15.6.re4c. $\frac{64}{15}$.
15.6.re5a. $\int_0^1 \int_1^{4-3x} \int_0^{\frac{4}{3}-x-\frac{1}{3}y} e^y \, dz \, dy \, dx$. 15.6.re5b. $\int_0^2 \int_{y^2}^4 \int_0^{\sqrt{4-z}} \, dx \, dz \, dy$.
15.6.re5c. $\int_0^4 \int_{-\sqrt{4-y}}^{\sqrt{4-y}} \int_y^{4-x^2} 3y \, dz \, dx \, dy$. 15.6.re6. $\frac{323}{60}$.

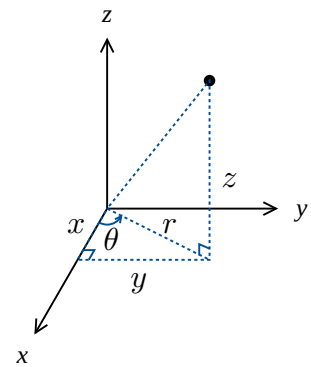
15.7: Cylindrical coordinates

In the cylindrical coordinate system, a point (x, y, z) in space is represented by the variables (r, θ, z), where (r, θ) are polar coordinates for the point (x, y). Since $dx\,dy = r\,dr\,d\theta$,

$$dV = dx\,dy\,dz = r\,dr\,d\theta\,dz.$$

The relations between (r, θ) and (x, y) are the same as seen section 15.3, remembered with the help of this picture (drawn as if r were positive and θ were acute):

$$x = r\cos\theta \qquad r^2 = x^2 + y^2$$
$$y = r\sin\theta \qquad \tan\theta = y/x$$



15.7.re1. Evaluate the integral $\iiint_D (x+y+z)\,dV$ where D is the region $\{(x, y, z) \mid x^2+y^2 \leq 1, 0 \leq z \leq y\}$.

Solution:

$$\begin{aligned}
&\int_0^\pi \int_0^1 \int_0^{r\sin\theta} r(r\cos\theta + r\sin\theta + z)\,dz\,dr\,d\theta \\
&= \int_0^\pi \int_0^1 \int_0^{r\sin\theta} (r^2\cos\theta + r^2\sin\theta + rz)\,dz\,dr\,d\theta \\
&= \int_0^\pi \int_0^1 \left(r^3\cos\theta\sin\theta + r^3\sin^2\theta + \frac{1}{2}r^3\sin^2\theta\right)\,dr\,d\theta \\
&= \int_0^1 r^3\,dr \int_0^\pi \left(\cos\theta\sin\theta + \frac{3}{2}\sin^2\theta\right)\,d\theta \\
&= \frac{1}{4}\int_0^\pi \left(\cos\theta\sin\theta + \frac{3}{4}\big(1 - \cos(2\theta)\big)\right)\,d\theta \\
&= \frac{1}{4}\left(\frac{1}{2}\sin^2\theta + \frac{3}{4}\theta - \frac{3}{8}\sin(2\theta)\right)\Big|_0^\pi = \frac{3\pi}{16}
\end{aligned}$$

15.7.re2. Find the volume of the region below $z = 2x$ and above $z = x^2 + y^2$.

Solution: The surfaces intersect on the circle $2x = x^2 + y^2$, or

$$2r\cos\theta = r^2, \tag{15.7.1}$$

which implies that either $r = 0$ or $2\cos\theta = r$. The only point on the graph of $r = 0$ is the origin, which is also on the graph of $2\cos\theta = r$, so we can safely replace (15.7.1) by

$$r = 2\cos\theta.$$

The circle $r = 2\cos\theta$ is traced once when $-\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$. You can check that the plane is above the paraboloid in the interior of the circle by calculating $2x$ and $x^2 + y^2$ at the circle's center $(1, 0)$. And so the volume equals

$$\int_{-\pi/2}^{\pi/2}\int_0^{2\cos\theta}\int_{r^2}^{2r\cos\theta} 1\,dz\,r\,dr\,d\theta = \int_{-\pi/2}^{\pi/2}\int_0^{2\cos\theta}(2r\cos\theta - r^2)r\,dr\,d\theta$$
$$= \int_{-\pi/2}^{\pi/2}\int_0^{2\cos\theta}(2r^2\cos\theta - r^3)\,dr\,d\theta = \int_{-\pi/2}^{\pi/2}(\frac{2}{3}r^3\cos\theta - \frac{1}{4}r^4)\Big|_0^{2\cos\theta}\,d\theta$$
$$= \int_{-\pi/2}^{\pi/2}(\frac{2}{3}r^3\cos\theta - \frac{1}{4}r^4)\Big|_0^{2\cos\theta}\,d\theta = \int_{-\pi/2}^{\pi/2}\frac{4}{3}\cos^4\theta\,d\theta.$$

This is a challenging integral which is best evaluated using the reduction formula

$$\boxed{\int \cos^n\theta\,d\theta = \frac{1}{n}\cos^{n-1}\theta\sin\theta + \frac{n-1}{n}\int\cos^{n-2}\theta\,d\theta}$$

from calculus II. (See "Integration by parts" in my MATH 220 review notes: https://kunklet.people.cofc.edu/MATH220/220review.pdf)

$$\frac{4}{3}\int_{-\pi/2}^{\pi/2}\cos^4\theta\,d\theta = \frac{4}{3}\left(\frac{1}{4}\cos^3\theta\sin\theta\Big|_{-\pi/2}^{\pi/2} + \frac{3}{4}\int_{-\pi/2}^{\pi/2}\cos^2\theta\,d\theta\right)$$
$$= \frac{4}{3}\left(0 + \frac{3}{4}\left(\frac{1}{2}\cos\theta\sin\theta\Big|_{-\pi/2}^{\pi/2} + \frac{1}{2}\int_{-\pi/2}^{\pi/2}d\theta\right)\right)$$
$$= \frac{4}{3}\left(\frac{3}{4}\left(0 + \frac{1}{2}\pi\right)\right) = \frac{\pi}{2}$$

15.7.re3. Find the volume of the given region.

a. $\{(x, y, z) \mid x^2 + y^2 \le 1, 0 \le z \le y\}$

b. $\{(x, y, z) \mid x^2 + y^2 \le 1, 0 \le y \le z \le 1\}$

c. The region between the paraboloid $z = x^2 + y^2$ and the cone $z = \sqrt{x^2 + y^2}$.

15.7.re4. Evalute the triple integral.

a. $\int_{-1}^{1}\int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}}\int_0^{\sqrt{x^2+y^2}} z^{1/3}\,dz\,dy\,dx$

b. $\iiint_D e^{x^2+y^2-z}\,dV$, where D is the region between the cylinders $x^2 + y^2 = 1$ and $x^2 + y^2 = 4$, above $z = 0$ and below $z = x^2 + y^2$.

c. $\iiint_E \frac{x+y}{\sqrt{x^2+y^2}}\,dV$, where E is the portion of the unit sphere and its interior in the first octant.

Answers

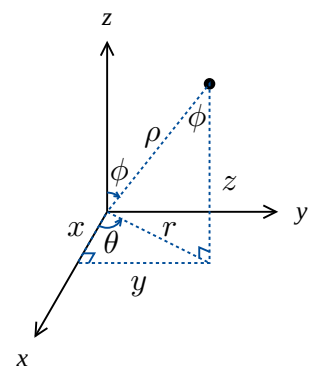
15.7.re3a. $\frac{2}{3}$. 15.7.re3b. $\frac{\pi}{2} - \frac{2}{3}$. 15.7.re3c. $\frac{\pi}{6}$. 15.7.re4a. $\frac{9\pi}{20}$. 15.7.re4b. $\pi(e^4 - e - 3)$. 15.7.re4c. $\frac{2}{3}$.

15.8: Spherical coordinates

In the spherical coordinate system, a point (x, y, z) in space is represented by the variables (ρ, ϕ, θ), where ρ is the distance from the point to the origin, ϕ is the angle from the positive z-axis to the vector $\langle x, y, z\rangle$, and θ is the same as in cylindrical coordinates. According to this definition, $\rho \geq 0$ and $0 \leq \phi \leq \pi$.
In these coordinates,

$$dV = \rho^2 \sin\phi \, d\rho \, d\phi \, d\theta$$

The relations between rectangular, cylindrical, and spherical coordinates can be remembered with the help of this picture:



$$\begin{aligned} x &= r\cos\theta & z &= \rho\cos\phi \\ y &= r\sin\theta & r &= \rho\sin\phi \\ \frac{y}{x} &= \tan\theta & \frac{r}{z} &= \tan\phi \\ r^2 &= x^2 + y^2 & \rho^2 &= r^2 + z^2 \end{aligned}$$

15.8.re1. Identify the graph of $\rho = 2\cos\phi$ by converting to rectangular coordinates.
To convert, multiply both sides by ρ:

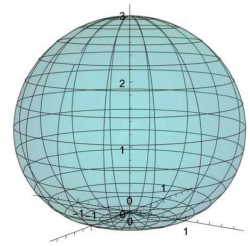
$$\begin{aligned} \rho^2 &= 2\rho\cos\phi & x^2 + y^2 + z^2 - 2z &= 0 \\ x^2 + y^2 + z^2 &= 2z & x^2 + y^2 + z^2 - 2z + 1 &= 1 \\ & & x^2 + y^2 + (z-1)^2 &= 1 \end{aligned}$$

The surface is a sphere or radius 1 centered at the point $(0, 0, 1)$.

15.8.re2. Verify that the volume of the sphere of radius r is $\frac{4}{3}\pi r^3$.
Solution: We'll find the volume of the sphere by using volume$(D) = \iiint_D 1 \, dV =$

$$\begin{aligned} &\int_0^{2\pi}\int_0^{\pi}\int_0^{r} \rho^2 \sin\phi \, d\rho \, d\phi \, d\theta = \int_0^{2\pi}\int_0^{\pi} \sin\phi \frac{1}{3}\rho^3\Big|_0^r \, d\phi \, d\theta \\ &= \frac{1}{3}r^3 \int_0^{2\pi} (-\cos\phi)\Big|_0^{\pi} \, d\theta = \frac{1}{3}r^3 \int_0^{2\pi} (-(-1) - (-1)) \, d\theta \\ &= \frac{1}{3}r^3 \cdot 2\pi \cdot 2 = \frac{4}{3}\pi r^3 \end{aligned}$$

15.8.re3. Find the average value of $\frac{1}{\rho}$ on the set $\{(x, y, z) \mid 1 \leq x^2 + y^2 + z^2 \leq 4\}$. Hint: you can use 15.8.re2 to find the volume of this set.



15.8.re4. Find the volume contained by the surface $\rho = 1 + 2\cos\phi$. Note: by their definition, $\rho \geq 0$ and $0 \leq \phi \leq \pi$. Use these to find the correct interval of ϕ for the integral.

15.8.re5. Find the volume inside the surface $\rho = 1 + 2\cos\phi$ below the cone $\phi = \frac{\pi}{3}$ and above the xy-plane.

15.8.re6. Compute the integral:

$$\int_0^1 \int_{-\sqrt{1-y^2}}^{\sqrt{1-y^2}} \int_{\sqrt{x^2+y^2}}^{1+\sqrt{1-x^2-y^2}} \sqrt{x^2+y^2}\, dz\, dx\, dy$$

Hint: convert to spherical *or* cylindrical coordinates.

Answers

15.8.re3. $\frac{9}{14}$ 15.8.re4. $\frac{27\pi}{4}$ 15.8.re5. $\frac{5\pi}{4}$ 15.8.re6. $\frac{\pi}{2}$

15.9: Change of variables in multiple integrals.

If we wish to write a double integral in the xy-plane in terms of new variables u and v, and if we can express x and y in terms of u and v,

$$x = x(u, v) \quad y = y(u, v),$$

then differential area in the plane can be written

$$dA = \left|\left| \begin{matrix} x_u & x_v \\ y_u & y_v \end{matrix} \right|\right| du\, dv. \tag{15.9.1}$$

Here, the outer $|\ \ |$ denote the absolute value and the inner $|\ \ |$ denote a determinant.
If we wish to write a triple integral in xyz-space in terms of new variables u, v, and w, and

$$x = x(u, v, w) \quad y = y(u, v, w) \quad z = z(u, v, w),$$

then differential volume in space can be written

$$dV = \left|\left| \begin{matrix} x_u & x_v & x_w \\ y_u & y_v & y_w \\ z_u & z_v & z_w \end{matrix} \right|\right| du\, dv\, dw. \tag{15.9.2}$$

The formulas we encountered earlier for dA and dV in polar, spherical, and cylindrical coordinates are all special cases of (15.9.1) and (15.9.2).
These determinants are called **Jacobians** and are denoted

$$\frac{\partial(x, y)}{\partial(u, v)} = \begin{vmatrix} x_u & x_v \\ y_u & y_v \end{vmatrix} \qquad \frac{\partial(x, y, z)}{\partial(u, v, w)} = \begin{vmatrix} x_u & x_v & x_w \\ y_u & y_v & y_w \\ z_u & z_v & z_w \end{vmatrix}.$$

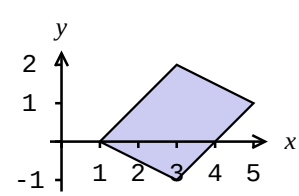
15.9.re1. Find the Jacobian of the given change of variables.

a. $x = (u + v)^2$, $y = u^2 - v^2$

b. $x = u \cos v$, $y = u \sin v$

c. $x = u + vw$, $y = uv + w^2$, $z = \ln w$

15.9.re2. Evaluate the double integral $\iint_D e^{x-4y}\, dA$ where D is the parallelogram with vertices $(1, 0)$, $(3, -1)$, $(3, 2)$, and $(5, 1)$.



Solution. The equations of the four sides of the parallelogram are

$$x - y = 1 \qquad x + 2y = 1$$
$$x - y = 4 \qquad x + 2y = 7$$

This suggests that we define new variables to be

$$u = x - y$$
$$v = x + 2y$$

Solve for x and y in terms of u and v:

$$\begin{aligned} 2u + v &= 3x \\ v - u &= 3y \end{aligned} \qquad \Longrightarrow \qquad \begin{aligned} x &= \tfrac{2}{3}u + \tfrac{1}{3}v \\ y &= -\tfrac{1}{3}u + \tfrac{1}{3}v \end{aligned}$$

By (15.9.1),

$$dA = \left\| \begin{matrix} x_u & x_v \\ y_u & y_v \end{matrix} \right\| du\,dv = \left\| \begin{matrix} \frac{2}{3} & \frac{1}{3} \\ -\frac{1}{3} & \frac{1}{3} \end{matrix} \right\| du\,dv = \frac{1}{3}\,du\,dv \tag{15.9.3}$$

and the integral equals

$$\int_1^4 \int_1^7 e^{(\frac{2}{3}u+\frac{1}{3}v)-4(-\frac{1}{3}u+\frac{1}{3}v)} \frac{1}{3}\,dv\,du = \frac{1}{3}\int_1^4 \int_1^7 e^{2u-v}\,dv\,du$$
$$= \frac{1}{3}\int_1^4 \int_1^7 e^{2u}e^{-v}\,dv\,du = \frac{1}{3}\int_1^4 e^{2u}\,du \int_1^7 e^{-v}\,dv = \frac{1}{6}(e^8 - e^2)(e^{-1} - e^{-7}).$$

15.9.re3. Compute the integral $\iint_D \frac{x}{y}e^{xy}\,dA$, where D is the region in the first quadrant bounded by

$$xy = 1 \qquad xy = 4 \qquad \frac{x}{y} = 1 \qquad \frac{x}{y} = 3.$$

Choose the variables

$$\begin{aligned} u &= xy \\ v &= \frac{x}{y} \end{aligned} \qquad \Longrightarrow \qquad \begin{aligned} x &= u^{1/2}v^{1/2} \\ y &= u^{1/2}v^{-1/2} \end{aligned}$$

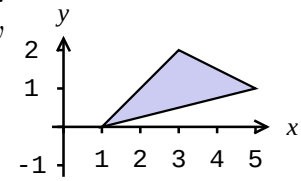
By (15.9.1),

$$\begin{aligned} dA &= \left\| \begin{matrix} x_u & x_v \\ y_u & y_v \end{matrix} \right\| du\,dv \\ &= \left\| \begin{matrix} \frac{1}{2}u^{-1/2}v^{1/2} & \frac{1}{2}u^{1/2}v^{-1/2} \\ \frac{1}{2}u^{-1/2}v^{-1/2} & -\frac{1}{2}u^{1/2}v^{-3/2} \end{matrix} \right\| du\,dv = \left| -\frac{1}{2}v^{-1} \right| du\,dv = \frac{1}{2}v^{-1}\,du\,dv. \end{aligned}$$

(We can use $|v^{-1}| = v^{-1}$, since $v > 0$ in this integral.) The integral equals

$$\int_1^3 \int_1^4 ve^u \frac{1}{2v}\,du\,dv = \frac{1}{2}\int_1^3 \int_1^4 e^u\,du\,dv = e^4 - e.$$

15.9.re4. Let D be the triangle with vertices $(1,0)$, $(3,2)$, and $(5,1)$. Rewrite the integral $\iint_D (x-y)\,dA$ in terms of the variables u and v from 15.9.re2.



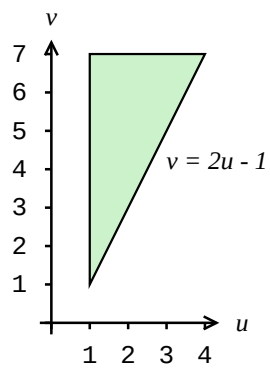
Solution: Using

$$\begin{aligned} u &= x-y & x &= \tfrac{2}{3}u + \tfrac{1}{3}v \\ v &= x+2y & y &= -\tfrac{1}{3}u + \tfrac{1}{3}v \end{aligned}$$

the edges of the triangle become

$$\begin{aligned} x-y &= 1 & x+2y &= 7 & x-4y &= 1 \\ u &= 1 & v &= 7 & (\tfrac{2}{3}u+\tfrac{1}{3}v) - 4(-\tfrac{1}{3}u+\tfrac{1}{3}v) &= 1 \\ & & & & 2u-v &= 1 \end{aligned}$$

which look like this in the uv-plane:



Using this and (15.9.3), the integral can be written either

$$\frac{1}{3}\int_1^4 \int_{2u-1}^7 u\,dv\,du \text{ or } \frac{1}{3}\int_1^7 \int_1^{\frac{1}{2}(1+v)} u\,du\,dv.$$

15.9.re5. Evaluate the integral by making an appropriate change of variables.

a. $\iint_E (x+y)\,dA$, $E =$ the parallelogram with vertices (0,0), (1,2), (4,-1), (5,1).

b. $\iint_C (x+y)\,dA$, $C =$ the triangle with vertices (1,2), (4,-1), (5,1).

c. $\iint_D (x+y)^{-2}\,dA$, D is bounded by $\frac{x}{y}=2$, $\frac{x}{y}=\frac{1}{2}$, $x+y=1$, and $x+y=3$.

d. $\iiint_P (x^2-z^2)\,dV$, where P is given by

$$\begin{aligned} 0 \le x+z \le 1 \\ 0 \le x-z \le 2 \\ -1 \le x+y+z \le 1 \end{aligned}$$

Answers

15.9.re1a. $-4(u+v)^2$ 15.9.re1b. u 15.9.re1c. $w^{-1}u-v$ 15.9.re5a. New variables are $2x-y$ and $x+4y$. Integral $=27$. 15.9.re5b. New variables are $2x-y$ and $x+4y$. Integral $=18$. 15.9.re5c. Let $u=\frac{x}{y}$, $v=x+y$. Then $y=\frac{v}{1+u}$ and $x=\frac{uv}{1+u}$. Jacobian $=\frac{v}{(1+u)^2}$ and integral $=\frac{1}{3}\ln 3$. 15.9.re5d. 1

16.1: Vector fields.

A **vector field** is a function from $\mathbb{R}^2$ into $\mathbb{R}^2$ or from $\mathbb{R}^3$ into $\mathbb{R}^3$. Think of the inputs as points (x, y) in the plane (or points (x, y, z) in space) and the outputs as vectors in $\langle u, v\rangle$ in the plane (or vectors $\langle u, v, w\rangle$ in space).

Examples of vector fields

1. If f is a differentiable function of two or three variables, then its gradient ∇f is a vector field.

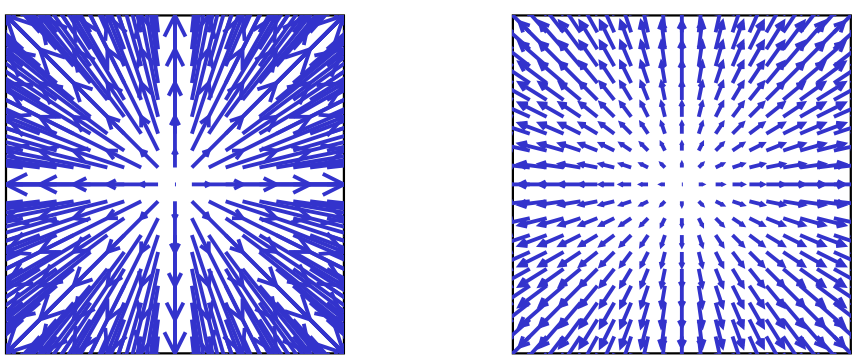
2. If a fluid is flowing across the plane or through space, then its velocity vector $\mathbf{v}$ at at each point (x, y) or (x, y, z) is a vector field.

3. Anything that can be described as a force acting on an object depending on its position is a vector field. For instance, the gravitational force acting on an object of some fixed mass in space due to the presence of nearby masses is a vector field; its direction and magnitude depend on the object's position.

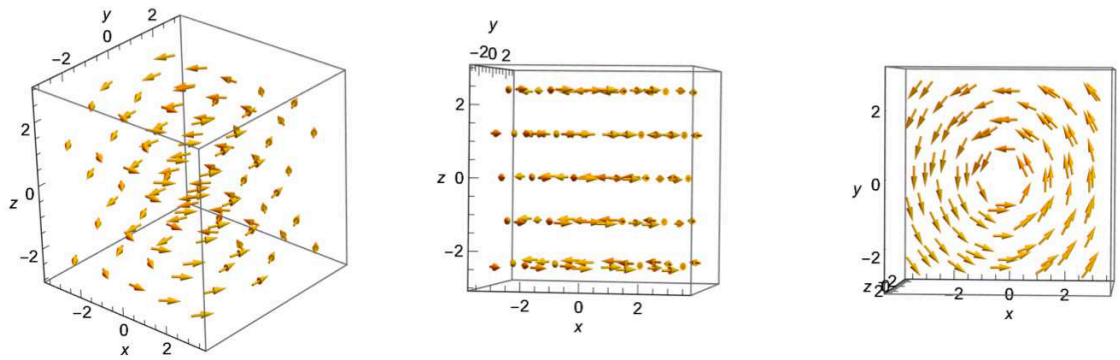
The graph of a vector field

A **graph** of a vector field is a drawing of its values at selected points in the domain, drawn as vectors originating at those points. Graphing a vector field is best done by a computer, although drawing one by hand can be a worthwhile exercise.

16.1.re1. To make the graph of a vector field easier to read, it is common to draw the vectors at a reduced scale. Below are two graphs of the vector field $\mathbf{F}(x, y) = \langle x, y\rangle$. On the left, drawing $\langle x, y\rangle$ at its actual magnitude causes a lot of overlap. On the right, by graphing $c\langle x, y\rangle$ for some small scalar c, we can better see the behavior of $\langle x, y\rangle$.



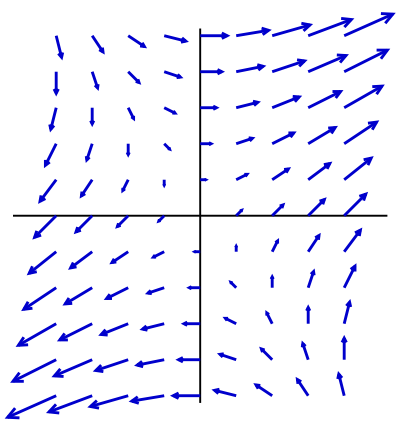
16.1.re2. Even when it is created by on a computer, the graph of a vector field in $\mathbb{R}^3$ can difficult to read without the ability to view it from different angles. Here's the same graph of $\mathbf{G}(x, y, z) = \langle -y, x, 0\rangle$ seen from 3 different points of view.



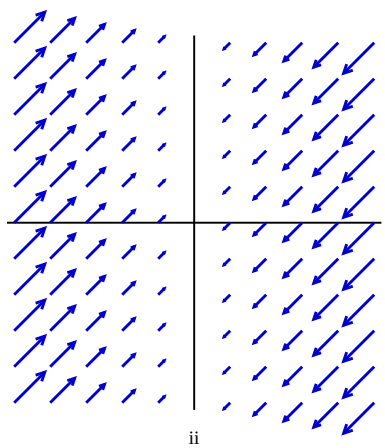
16.1.re3. Find the graph of the given vector field.

a. $\langle 1, 1\rangle$ b. $\langle y, -x\rangle$ c. $\langle x + y, x\rangle$

d. $\langle y, y\rangle$ e. $\langle -x, -x\rangle$ f. $\langle x + y, y\rangle$



i



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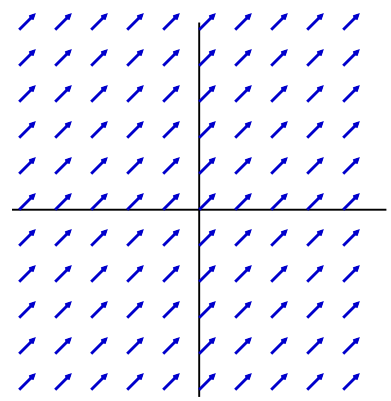
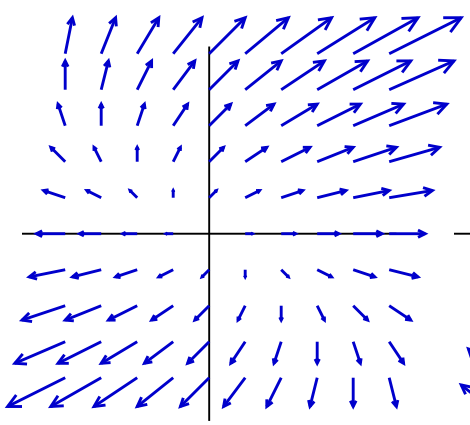
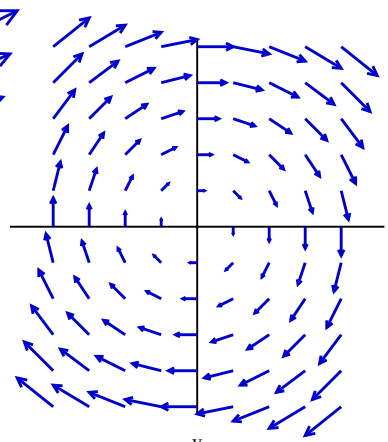


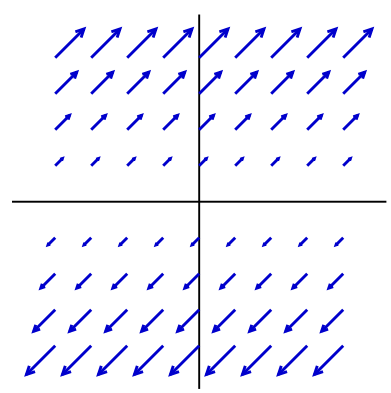
iii



iv

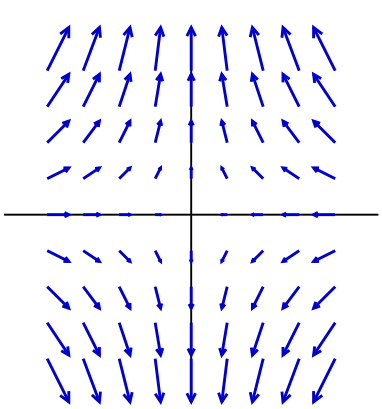


v

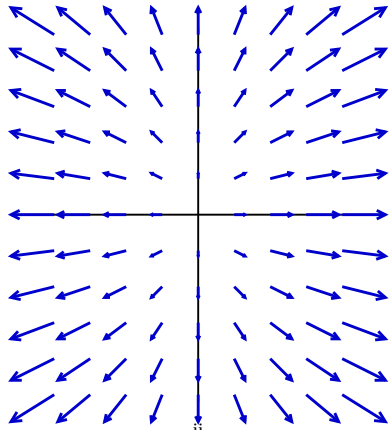


vi

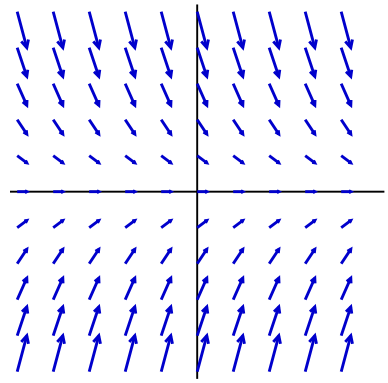
16.1.re4. Find the vector fields graphed below.



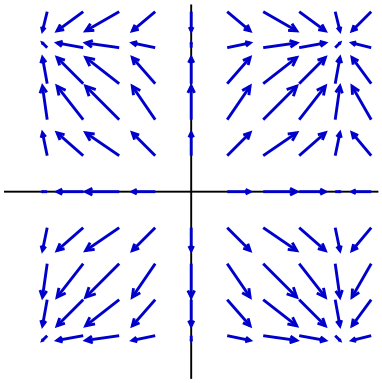
i



ii



iii



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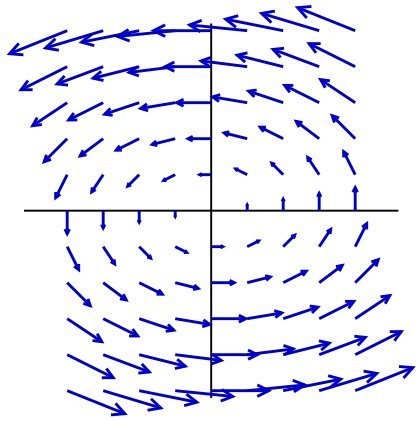

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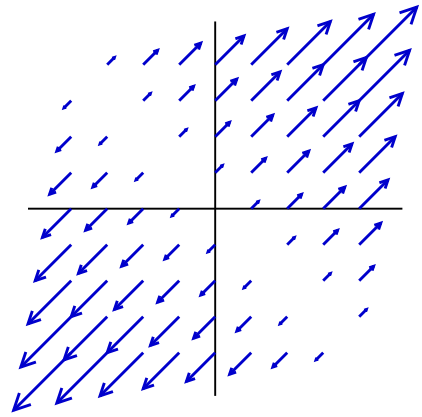


vi

a. $\langle 1, -2x \rangle$

b. $\langle y, 2x \rangle$

c. $\langle 2x, y \rangle$

d. $\langle -2y, x \rangle$

e. $\langle -x, 2y \rangle$

f. $\langle x + y, x + y \rangle$

g. $\langle \sin x, \sin y \rangle$

h. $\langle \cos x, \sin x \rangle$

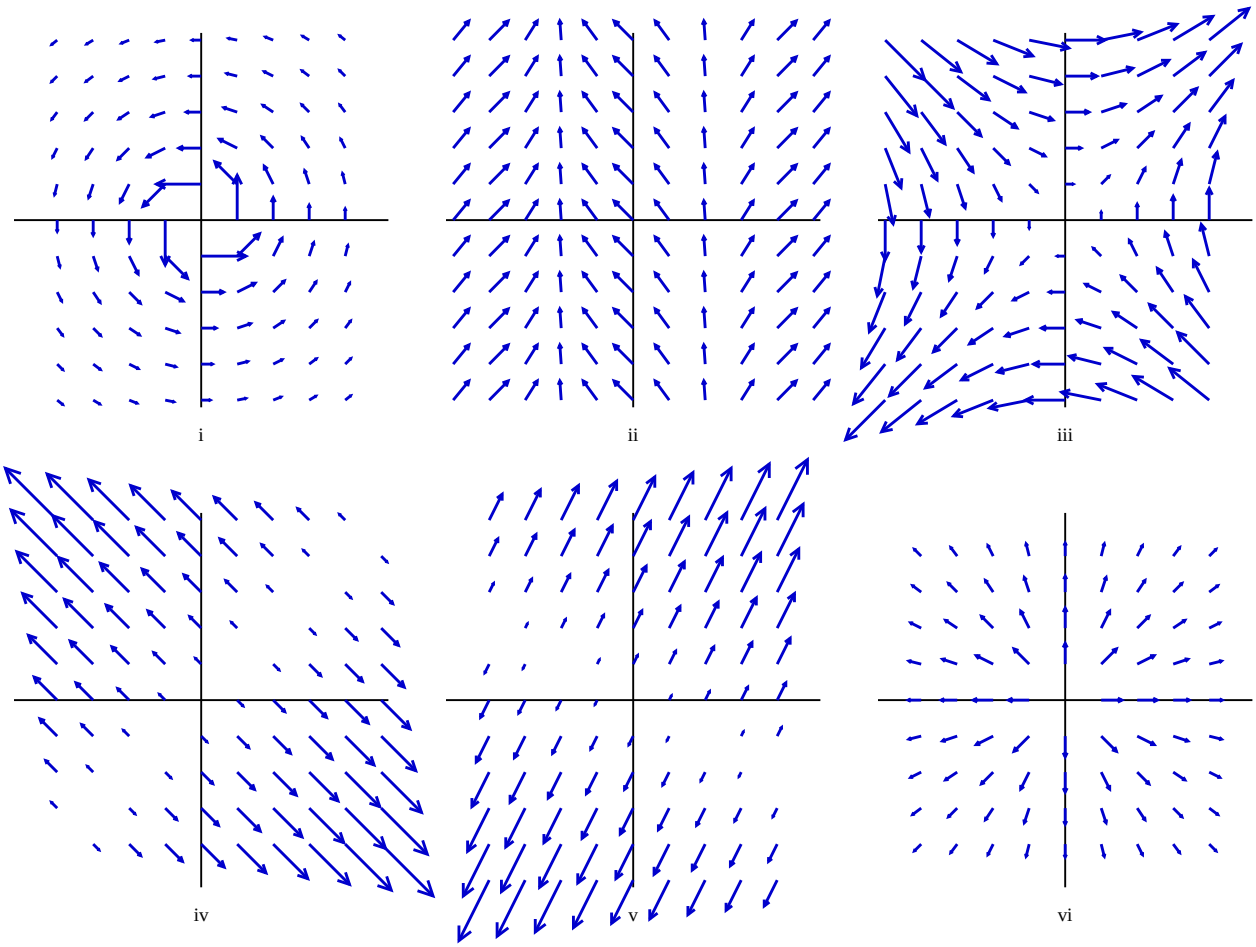
i. $\langle 1, -y \rangle$

As illustrated in example 0, the gradient field of a differentiable function is everywhere orthogonal to the level curves of that function.

16.1.re5. Find the gradient of the given function and identify its graph below.

a. $y - \sin x$ b. $\ln(1 + x^2 + y^2)$ c. xy

d. $x^2 - 2xy + y^2$ e. $x^2 + 4xy + 4y^2$ f. $\tan^{-1}\left(\frac{y}{x}\right)$



Answers

16.1.re3a. iii 16.1.re3b. v 16.1.re3c. i 16.1.re3d. vi 16.1.re3e. ii 16.1.re3f. iv 16.1.re4a. none. 16.1.re4b. none. 16.1.re4c. ii. 16.1.re4d. v. 16.1.re4e. i. 16.1.re4f. vi. 16.1.re4g. iv. 16.1.re4h. none. 16.1.re4i. iii. 16.1.re5a. $\langle -\cos x, 1\rangle$; ii. 16.1.re5b. $(x^2 + y^2 + 1)^{-1}\langle 2x, 2y\rangle$; vi. 16.1.re5c. $\langle y, x\rangle$; iii. 16.1.re5d. $(2x - 2y)\langle 1, -1\rangle$; iv. 16.1.re5e. $\langle 2x + 4y, 4x + 8y\rangle$; v. 16.1.re5f. $(x^2 + y^2)^{-1}\langle -y, x\rangle$; i.

16.2: Line integrals

If C is a curve parametrized by $\mathbf{r}(t) = \langle x(t), y(t), z(t)\rangle$, $a \le t \le b$, and if $f(x, y, z)$ is a function defined on C, then the **line integral** of f on C

$$\int_C f\,ds$$

can be calculated

$$\int_a^b f\big(\mathbf{r}(t)\big)\frac{ds}{dt}\,dt = \int_a^b f\big(x(t), y(t), z(t)\big)\sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2}\,dt. \tag{16.2.1}$$

If $\mathbf{F} = \langle P, Q, R\rangle$ is a vector field in $\mathbb{R}^3$, then

$$\int_C \mathbf{F}\cdot d\mathbf{r} = \int_C P\,dx + Q\,dy + R\,dz$$

is also a line integral , since it can be written

$$\int_C \left(P\frac{dx}{ds} + Q\frac{dy}{ds} + R\frac{dz}{ds}\right)\,ds = \int_C \mathbf{F}\cdot\mathbf{T}\,ds$$

although it is usually more easily calculated as

$$\int_a^b \left(P\frac{dx}{dt} + Q\frac{dy}{dt} + R\frac{dz}{dt}\right)\,dt. \tag{16.2.2}$$

16.2.re1. If C is parametrized by $\langle x(t), y(t)\rangle = \langle 1 + 2\cos t, -1 + 2\sin t\rangle$, for $0 \le t \le \pi$, calculate the given line integral.

a. $\int_C (xy)\,ds$ b. $\int_C x\,dx + (x + y)\,dy$

a. $ds = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\,dt = \sqrt{(-2\sin t)^2 + (2\cos t)^2}\,dt = 2\,dt$, and so the integral is

$$\begin{aligned}\int_0^\pi 2(1 + 2\cos t)(-1 + 2\sin t)\,dt &= 2\int_0^\pi (4\cos t\sin t - 2\cos t + 2\sin t - 1)\,dt\\ &= 2(2\sin^2 t - 2\sin t - 2\cos t - t)\Big|_0^\pi = 2(4 - \pi).\end{aligned}$$

b. $dx = -2\sin t\,dt$ and $dy = 2\cos t\,dt$, so the integral is

$$\int_0^\pi \big((1 + 2\cos t)(-2\sin t)\,dt + (2\cos t + 2\sin t)(2\cos t)\,dt\big) = \int_0^\pi \big(-2\sin t + 4\cos^2 t\big)\,dt$$
$$= \int_0^\pi \big(-2\sin t + 2(1 + \cos 2t)\big)\,dt = \big(2\cos t + 2t + \sin 2t\big)\Big|_0^\pi = 2\pi - 4.$$

Other parametrizations of C

Fact 16.2.3. If $\mathbf{r}$ and $\tilde{\mathbf{r}}$ are two parametrizations of the curve C, then the value of a line integral

$$\int_C f\,ds$$

is the same for $\mathbf{r}$ and $\tilde{\mathbf{r}}$ (even if they trace out C is opposite directions).

Fact 16.2.4. If $\mathbf{r}$ and $\tilde{\mathbf{r}}$ are two parametrizations that trace out C in the same direction, then

$$\int_C \mathbf{F}\cdot d\mathbf{r} = \int_C \mathbf{F}\cdot d\tilde{\mathbf{r}}.$$

If $\mathbf{r}$ and $\tilde{\mathbf{r}}$ trace out C in opposite directions, then

$$\int_C \mathbf{F}\cdot d\mathbf{r} = -\int_C \mathbf{F}\cdot d\tilde{\mathbf{r}}.$$

16.2.re2. Verify 16.2.3 for $\int_C xy\,ds$, where C is the line segment connecting $(0,0)$ and $(2,4)$, by calculating the integral using these parametrizations.

a. $\mathbf{r}(t) = \langle t, 2t\rangle$ for $0 \le t \le 2$.
b. $\tilde{\mathbf{r}}(t) = \langle 2t^2, 4t^2\rangle$ for $0 \le t \le 1$.
c. $\hat{\mathbf{r}}(t) = \langle 2-t, 4-2t\rangle$ for $0 \le t \le 2$.

16.2.re3. Verify 16.2.4 for $\int_C y\,dx - x\,dy$ by calculating the integral using these parametrizations of C

a. $\mathbf{r}(t) = \langle t, t^2\rangle$ for $0 \le t \le 1$.
b. $\tilde{\mathbf{r}}(t) = \langle \sqrt{1-t}, 1-t\rangle$ for $0 \le t \le 1$.

Work

The **Work** done by a force $\mathbf{F}$ on a particle that moves along a curve C is

$$\int_C \mathbf{F}\cdot d\mathbf{r}.$$

16.2.re4. Find the work done by the force $\langle y, -x\rangle$ in moving a particle along the arc $y = x^2$ from the origin to the point $(1,1)$.

16.2.re5. Calculate the given line integral.

a. $\int_C \frac{x}{1+4y}\,ds$; $\mathbf{r} = \langle t, t^2\rangle$, $0 \le t \le 1$.

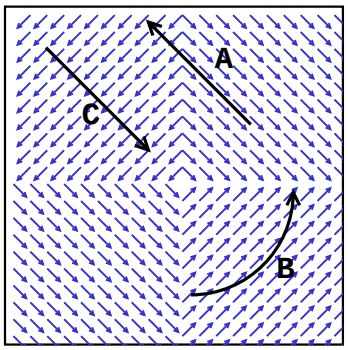
b. $\int_B (x+y)\sin z\,ds$, $\mathbf{r} = \langle \cos t, -\sin t, t\rangle$, $0 \le t \le \frac{\pi}{2}$.

c. $\int_E y^2\,dx + x\,dy$, $E =$ the line segments from $(0,0)$ to $(2,0)$ and from there to $(2,1)$.

d. $\int_H y\,dx - x\,dy$, $H =$ the circular arc $x^2+y^2=1$ in the positive (counterclockwise) direction from $(1,0)$ to $(-1,0)$.

16.2.re6. The figure shows a vector field $\mathbf{F}$ and curves A, B, and C. Determine the sign of the given line integral.

a. $\int_A \mathbf{F}\cdot d\mathbf{r}$ b. $\int_B \mathbf{F}\cdot d\mathbf{r}$ c. $\int_C \mathbf{F}\cdot d\mathbf{r}$



Answers

16.2.re2. All three calculations equal $\frac{16}{3}\sqrt{5}$. 16.2.re3a. $-\frac{1}{3}$ 16.2.re3b. $\frac{1}{3}$ 16.2.re4. Same as 16.2.re3a. 16.2.re5a. $\frac{1}{4}(\sqrt{5}-1)$ 16.2.re5b. $\frac{1}{\sqrt{2}}(1-\frac{\pi}{2})$ 16.2.re5c. 2 16.2.re5d. $-\pi$. 16.2.re6a. negative 16.2.re6b. positive 16.2.re6c. 0

16.3: Fundamental theorem of calculus for line integrals

A summary of the facts and terminology learned in this section.

Definition 16.3.1. The (indefinite) line integral $\int \mathbf{F} \cdot d\mathbf{r}$ is said to be **path-independent** in the domain D if

$$\int_{C_1} \mathbf{F} \cdot d\mathbf{r} = \int_{C_2} \mathbf{F} \cdot d\mathbf{r}$$

for any two curves C_1 and C_2 in D having the same beginning points and ending points.

Theorem 16.3.2. $\int \mathbf{F} \cdot d\mathbf{r}$ is path-independent in D if and only if $\oint_C \mathbf{F} \cdot d\mathbf{r} = 0$ for any closed curve C in D.

Definition 16.3.3. The vector field $\mathbf{F}$ is said to be **conservative**, and $\mathbf{F} \cdot d\mathbf{r}$ to an **exact differential**, and f to be a **potential function** for $\mathbf{F}$ if

$$\nabla f = \mathbf{F}.$$

Fundamental Theorem of Calculus for Line Integrals 16.3.4. If f is a potential function for the vector field $\mathbf{F}$ in the region D, and if C is a path in D from the point $\mathbf{p}$ to the point $\mathbf{q}$, then

$$\int_C \mathbf{F} \cdot d\mathbf{r} = f(\mathbf{q}) - f(\mathbf{p}) \tag{16.3.5}$$

Consequently, if $\mathbf{F}$ is a conservative vector field in D, then $\int \mathbf{F} \cdot d\mathbf{r}$ is path-independent in D, and we can refer to the line integral above simply as

$$\int_{\mathbf{p}}^{\mathbf{q}} \mathbf{F} \cdot d\mathbf{r}.$$

In fact, path-independence and conservativeness are nearly identical:

Theorem 16.3.6. If $\mathbf{F}$ is continuous in D, then $\mathbf{F}$ is conservative in D if and only if $\int \mathbf{F} \cdot d\mathbf{r}$ is path-independent in D.

16.3.re1. Find a potential function for the given vector field, if one exists.

a. $\langle ye^{xy} + \frac{1}{x}, xe^{xy} + \frac{1}{y^2+1} \rangle$ b. $\langle -\cos(x-y) + 2x + y, \cos(x-y) - 3y^2 - x \rangle$

Solution a: A potential function f would satisfy

$$f_x = ye^{xy} + \frac{1}{x} \qquad f_y = xe^{xy} + \frac{1}{y^2+1}.$$

Integrate f_y with respect to y to find

$$f = e^{xy} + \tan^{-1} y + C(x).$$

The "constant" of integration C is only constant with regard to y. That is, C is a function of at most x.
By differentiating this f with respect to x and setting the result equal to the given value of f_x, we learn $C'(x)$:

$$f_x = ye^{xy} + C'(x) = ye^{xy} + \frac{1}{x} \quad \Longrightarrow \quad C'(x) = \frac{1}{x},$$

which implies

$$C(x) = \ln|x| + D,$$

where D is a genuine constant, independent of both x and y. Therefore

$$f(x, y) = e^{xy} + \tan^{-1} y + \ln|x|$$

is a potential function. (In fact, the general potential function is $f(x, y) = e^{xy} + \tan^{-1} y + \ln|x| + D$ for any constant D.)
Solution b: A potential function f would satisfy

$$f_x = -\cos(x - y) + 2x + y \qquad f_y = \cos(x - y) - 3y^2 - x \tag{16.3.7}$$

Integrate f_y with respect to y:

$$f = -\sin(x - y) - y^3 - xy + C(x).$$

Differentiate this respect to x:

$$f_x = -\cos(x - y) - y + C'(x) = -\cos(x - y) + 2x + y \quad \Longrightarrow \quad C'(x) = 2x + 2y,$$

which contradicts the requirement that C be a function of at most x. Therefore no potential function exists.
We could make the same conclusion in b by observing that (16.3.7) implies that

$$\begin{aligned} f_{xy} &= (-\cos(x - y) + 2x + y)_y = -\sin(x - y) + 1 \\ f_{yx} &= (\cos(x - y) - 3y^2 - x)_x = -\sin(x - y) - 1, \end{aligned}$$

violating Clairaut's theorem, which says that f_{xy} must equal f_{yx}.

A similar method works for find potentials of vector fields in $\mathbb{R}^3$:

16.3.re2. Find a potential function for $\langle 1 + y + ye^{xy-z}, x + xe^{xy-z}, -2z - e^{xy-z}\rangle$.
Integrate $g_z = -2z - e^{xy-z}$ with respect to z to obtain $g = -z^2 + e^{xy-z} + C(x, y)$ (where C is a function of as most x and y). Differentiate this with respect to x and y:

$$\begin{aligned} g_x &= ye^{xy-z} + C_x = 1 + y + ye^{xy-z} \\ g_y &= xe^{xy-z} + C_y = x + xe^{xy-z} \end{aligned} \quad \Longrightarrow \quad \begin{aligned} C_x &= 1 + y \\ C_y &= x. \end{aligned}$$

Now solve for C as we solved for f in 16.3.re1a. The rest is left as an exercise.

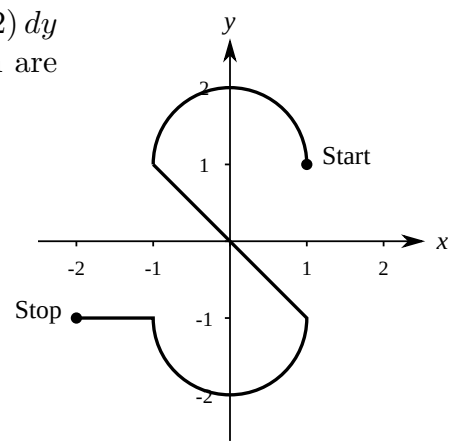
16.3.re3. Evaluate the line integral.

a. $\int_\Omega \frac{x}{\sqrt{x^2+y^2}}\,dx + \frac{y}{\sqrt{x^2+y^2}}\,dy$
$\Omega =$ the graph of $y = x^3$ from $(0,0)$ to $(1,1)$.

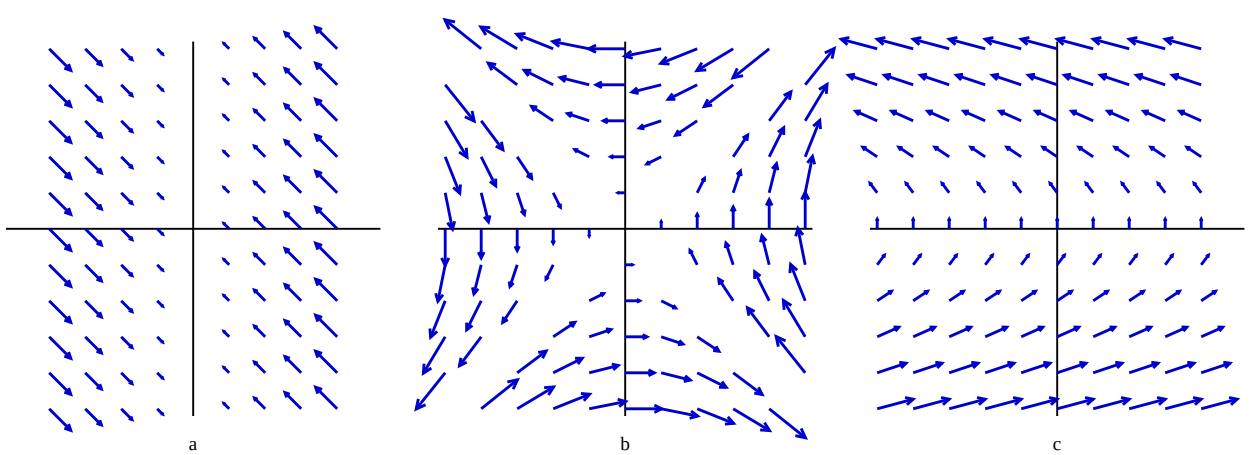
b. $\int_\Gamma (1 + y + ye^{xy-z})\,dx + (x + xe^{xy-z})\,dy - (2z + e^{xy-z})\,dz$
$\Gamma =$ the path of line segments from $(0,0,0)$ to $(0,2,0)$ to $(1,0,-1)$ to $(2,1,1)$.

c. $\int_\Lambda \frac{-y}{\sqrt{x^2+y^2}}\,dx + \frac{x}{\sqrt{x^2+y^2}}\,dy$
$\Lambda =$ the circular arc $x^2 + y^2 = 4$ in the first quadrant from $(2,0)$ to $(0,2)$.

16.3.re4. Evaluate the integral $\int_C (ye^{xy} - 1)\,dx + (xe^{xy} + 2)\,dy$ where C is the path shown at right. (The arcs in the path are semicircles.)

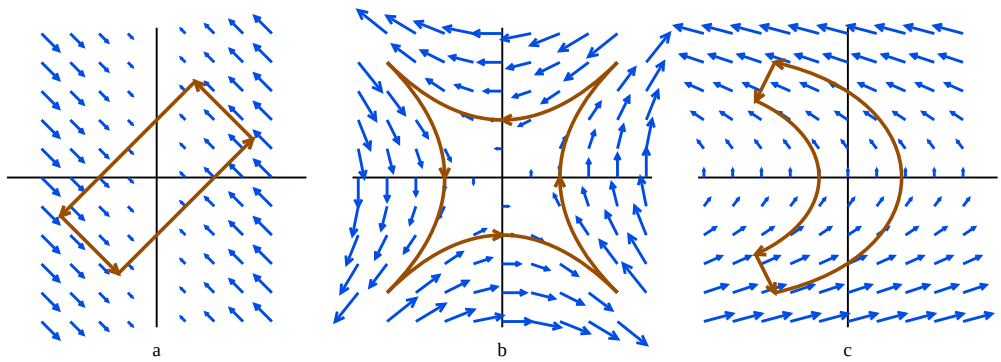


16.3.re5. Explain why the vector fields graphed below are not conservative. (Hint: 16.3.2, 16.3.6.)



Answers

16.3.re2. $g(x,y,z) = -z^2 + e^{xy-z} + x + xy$. 16.3.re3a. potential $= \sqrt{x^2+y^2}$. 16.3.4 implies line integral $= \sqrt{2} - \sqrt{0} = \sqrt{2}$. 16.3.re3b. Use answer to 16.3.re2. Line integral $= (-z^2 + e^{xy-z} + x + xy)\Big|_{(0,0,0)}^{(2,1,1)} = 2 + e$. 16.3.re3c. Not conservative. Parametrize arc as $x = 2\cos\theta$, $y = 2\sin\theta$; rewrite line integral in terms of θ and calculate to equal π. 16.3.re4. Integral is path-independent. By FTC for line integrals, integral $= (e^{xy} + 2y - x)\Big|_{(1,1)}^{(-2,-1)} = e^2 - e - 1$. 16.3.re5. $\oint_C \mathbf{F} \cdot d\mathbf{r}$ is positive on each of the closed paths C below.

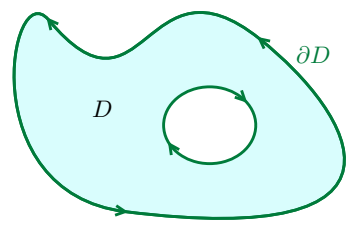


16.4: Green's theorem

Green's theorem tells us that a line integral $\oint_C P\,dx + Q\,dy$ along a closed curve C in the plane can be rewritten as a double integral over the region bounded by that curve.

Green's theorem. If D is a region in the plane and ∂D is its boundary curve, and if P_x and Q_y are continuous on D, then

$$\iint_D (Q_x - P_y)\,dA = \oint_{\partial D} P\,dx + Q\,dy \tag{16.4.1}$$



The line integral in (16.4.1) is taken in the **positive** direction, meaning the direction that keeps D always on the left.

Important: Green's theorem applies only to the line integral of $\mathbf{F}\cdot d\mathbf{r}$ along a *closed, planar* curve, that is, a curve in $\mathbb{R}^2$ that begins and ends at the same point.

16.4.re1. Let C be the triangular path from $(0,0)$ to $(3,0)$ to $(2,1)$ to $(0,0)$. Calculate the line integral

$$\oint_C (y + e^{x-y})\,dx - (x + e^{x-y})\,dy$$

directly, and again using Green's theorem.

Solution: From $(0,0)$ to $(3,0)$:

$$\begin{aligned} x &= t & y &= 0 & 0 \le t \le 3 \\ dx &= dt & dy &= 0 \end{aligned} \qquad \int_0^3 (0 + e^t)\,dt = e^3 - e^0$$

From $(3,0)$ to $(2,1)$, where $y = 3 - x$:

$$\begin{aligned} x &= 3 - t & y &= t \\ dx &= -dt & dy &= dt \\ 0 &\le t \le 1 \end{aligned} \qquad \begin{aligned} &\int_0^1 (t + e^{3-2t})\,(-dt) + (-(3-t) - e^{3-2t})\,dt \\ &= \int_0^1 (-3 - 2e^{3-2t})\,dt = -3t + e^{3-2t}\Big|_0^1 = -3 + e - e^3 \end{aligned}$$

From $(2,1)$ to $(0,0)$, where $y = \frac{1}{2}x$:

$$\begin{aligned} x &= 2 - t & y &= 1 - \tfrac{1}{2}t \\ dx &= -dt & dy &= -\tfrac{1}{2}\,dt \\ 0 &\le t \le 2 \end{aligned} \qquad \begin{aligned} &\int_0^2 (1 - \frac{1}{2}t + e^{1-\frac{1}{2}t})(-dt) + (t - 2 - e^{1-\frac{1}{2}t})(-\tfrac{1}{2}\,dt) \\ &= \int_0^2 -\tfrac{1}{2}e^{1-\frac{1}{2}t}\,dt = e^{1-\frac{1}{2}t}\Big|_0^2 = e^0 - e \end{aligned}$$

Adding these, we find the line integral to be

$$(e^3 - e^0) + (-3 + e - e^3) + (e^0 - e) = -3.$$

Green's theorem says that we'll get the same answer when we replace the integral with the double integral of $\big((-x - e^{x-y})_x - (y + e^{x-y})_y\big) = -2$ over the interior D of the triangle:

$$\iint_D (-2)\, dA = -2\,\mathrm{area}(D) = -2\left(\tfrac{3}{2}\right) = -3.$$

(done)

16.4.re2. Calculate the line directly, and again using Green's theorem. All curves are traversed in the positive direction.

a $\int_B (\cos x - \cos y)\, dx + y\, dy$, where B is the perimeter of the rectangle with vertices $(-2, 0)$, $(2, 0)$, $(-2, \pi)$, $(2, \pi)$.

b $\int_C xe^y\, dx + e^x\, dy$, where C is the perimeter of the triangle with vertices $(0, 0)$, $(2, 0)$, $(2, 2)$.

16.4.re3. Calculate the line integral the positive direction. Hint: you may wish to rewrite the given integral using Green's theorem.

a$\int_D (xy + \sin x + ye^{xy})\, dx + (3x^2 + xe^{xy})\, dy$, with D the line segment from $(0, 0)$ to $(1, 0)$ followed by the three-quarter-circle $x^2 + y^2 = 1$ to $(0, -1)$ and then the line segment back to $(0, 0)$.

b $\int_E -\frac{1}{2}y\, dx + \frac{1}{2}x\, dy$, with E the line segments from $(0, 1)$ to $(0, -1)$ to $(1, 0)$ to $(0, 1)$.

Area as a line integral

Because $(\frac{1}{2}x)_x - (-\frac{1}{2}y)_y = 1$, the area of a region is the line integral of $-\frac{1}{2}y\, dx + \frac{1}{2}x\, dy$ around its perimeter:

$$\mathrm{area}(D) = \int_{\partial D} -\frac{1}{2}y\, dx + \frac{1}{2}x\, dy \tag{16.4.2}$$

16.4.re4. Find the area of the loop that occurs in the graph of $x = t^2 - 1$, $y = t^3 - t$, visible at https://www.desmos.com/calculator/7w95ye8x4i.

Calculating a line integral

Every line integral is ultimately of the form $\int_C f(\mathbf{r})\, ds$ and, in theory, can be calculated by parametrizing the curve C and using (16.2.1). A line integral of the special form $\int_C \mathbf{F} \cdot d\mathbf{r} = \int_C P\, dx + Q\, dy + R\, dz$ can calculated directly from the parametrization (16.2.2), but other, possibly easier, methods for this type of line integral exist in two cases.

1. If $\mathbf{F}$ is conservative (16.3.3), $\int_C \mathbf{F} \cdot d\mathbf{r}$ can be evaluated using (16.3.5).
2. If C is a closed path in $\mathbb{R}^2$, $\int_C \mathbf{F} \cdot d\mathbf{r}$ can be rewritten as a double integral (16.4.1).

Generally, $\int_C P\, dx + Q\, dy + R\, dz$ **cannot** be evaluated by integrating P with respect to x while holding y and z constant, etc., since, as one variable changes along C, the others might also change.

Equivalent descriptions of a conservative vector field

Here's a summary of what we've learned in the last section and in this section about a conservative vector field $\mathbf{F} = \langle P, Q\rangle$ or $\langle P, Q, R\rangle$ on a region D in $\mathbb{R}^2$ or $\mathbb{R}^3$. (This assumes that P, Q, and R have continuous first partial derivatives.)

$\displaystyle\int \mathbf{F}\cdot d\mathbf{r}$ is path-independent in D.

$$\Updownarrow$$

$\displaystyle\oint_C \mathbf{F}\cdot d\mathbf{r} = 0$ for every closed path C in D.

$$\Updownarrow$$

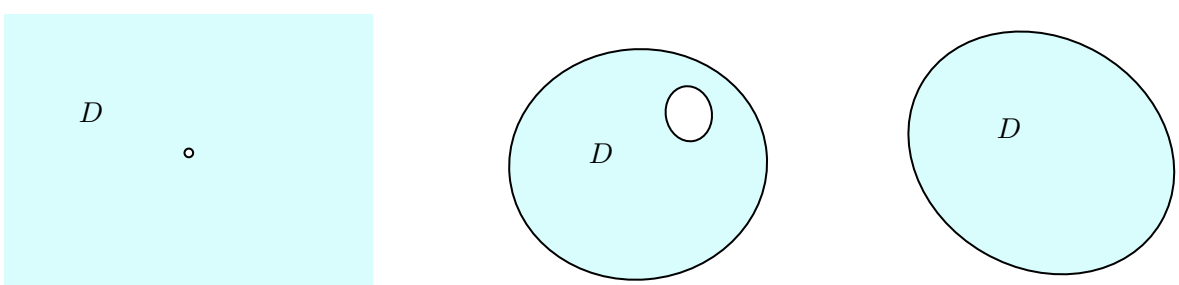
$\mathbf{F} = \nabla f$ in D for some scalar-valued function f.

$$\Updownarrow$$

$\mathbf{F}$ is **conservative** in D $\implies$ $Q_x = P_y$ (and $Q_z = R_y$ and $P_z = R_x$ in $\mathbb{R}^3$).

$\mathbf{F}$ is **conservative** in D $\impliedby$ $Q_x = P_y$, if $D \subset \mathbb{R}^2$ is simply connected.

The set $D \subset \mathbb{R}^2$ is **simply connected** if has no "holes:"



not simply connected | not simply connected | simply connected

Answers

16.4.re2a. -8 16.4.re2b. 2 16.4.re3a. $-\frac{5}{3}$ 16.4.re3b. 1 16.4.re4. $\frac{8}{15}$

16.5: div, grad, and curl

The symbol $\frac{d}{dx}$ from Calculus 1 can be thought of as an operator that acts on functions to produce their derivatives. For instance

$$\frac{d}{dx}x^2 = 2x \qquad \frac{d}{dx}\sin x = \cos x \qquad \frac{d}{dx}f(x) = f'(x).$$

Operators like $\frac{d}{dx}$ that involve differentiation are sometimes called differential operators. Div, grad, and curl are three differential operators that act on either vector-valued functions or scalar-valued functions. The symbol ∇ stands for the operator $\left\langle \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right\rangle$.

grad $= \nabla$	div $= \nabla\cdot$	curl $= \nabla\times$
grad scalar = vector	div scalar DNE	curl scalar DNE
grad vector DNE	div vector = scalar	curl vector = vector

Grad, div, and curl are calculated according to the definition of scalar multiplication, dot product, and cross product. When one of $\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}$ "multiplies" a function, that operator is applied to the function.

16.5.re1. Find the following.

a. $\text{grad}(x + e^{yz})$ b. $\text{div}\,\langle x + e^{yz}, yz, x + \sec z\rangle$ c. $\text{curl}\langle x + e^{yz}, yz, x + \sec z\rangle$

Solution:

a. $\text{grad}(x+e^{yz}) = \nabla(x+e^{yz}) = \left\langle \frac{\partial}{\partial x}(x+e^{yz}), \frac{\partial}{\partial y}(x+e^{yz}), \frac{\partial}{\partial z}(x+e^{yz}) \right\rangle = \langle 1, ze^{yz}, ye^{yz}\rangle.$

b. $\left\langle \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right\rangle \cdot \langle x + e^{yz}, yz, x + \sec z\rangle = \frac{\partial}{\partial x}(x + e^{yz}) + \frac{\partial}{\partial y}(yz) + \frac{\partial}{\partial z}(x + \sec z)$

$$= 1 + z + \sec z \tan z.$$

c. $\left\langle \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right\rangle \times \langle x + e^{yz}, yz, x + \sec z\rangle =$

$$\begin{aligned}
&\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x + e^{yz} & yz & x + \sec z \end{vmatrix} \\
&= \mathbf{i}\begin{vmatrix} \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ yz & x + \sec z \end{vmatrix} - \mathbf{j}\begin{vmatrix} \frac{\partial}{\partial x} & \frac{\partial}{\partial z} \\ x + e^{yz} & x + \sec z \end{vmatrix} + \mathbf{k}\begin{vmatrix} \frac{\partial}{\partial x} & \frac{\partial}{\partial y} \\ x + e^{yz} & yz \end{vmatrix} \\
&= \langle (x + \sec z)_y - (yz)_z,\ (x + e^{yz})_z - (x + \sec z)_x,\ (yz)_x - (x + e^{yz})_y\rangle \\
&= \langle -y,\ ye^{yz} - 1,\ -ze^{yz}\rangle
\end{aligned}$$

Interpreting div and curl

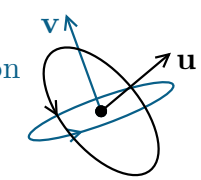
Case 1: $\mathbf{F} = \langle P, Q \rangle$ is the velocity field of a fluid in the plane.

> The value of $\operatorname{div} \mathbf{F} = P_x + Q_y$ at the point (x, y) is the net rate at which fluid is flowing out of (x, y).
>
> The value of $\operatorname{curl} \mathbf{F} \cdot \mathbf{k} = Q_x - P_y$ at (x, y) is the net rate of circulation of the fluid about the point (x, y). The net circulation is in the positive [negative] direction (as seen from above) if $Q_x - P_y$ is positive [negative].

Case 2: $\mathbf{F} = \langle P, Q, R \rangle$ is the velocity field of a fluid in the plane.

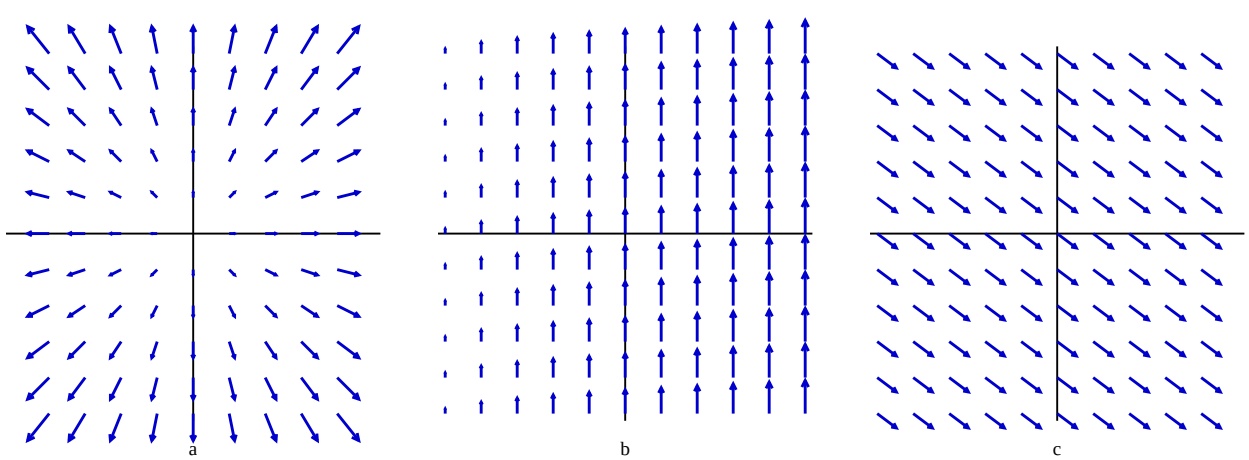
> The value of $\operatorname{div} \mathbf{F} = P_x + Q_y + R_z$ at the point (x, y, z) is the net rate at which fluid is flowing out of (x, y, z).
>
> If $\mathbf{u}$ is a unit vector, then the value of $\mathbf{u} \cdot \operatorname{curl} \mathbf{F}$ at (x, y, z) is the net rate of circulation of the fluid at the point (x, y, z) about an axis parallel to $\mathbf{u}$. The net circulation is in the positive [negative] direction (from the point of view of $\mathbf{u}$) if $\mathbf{u} \cdot \operatorname{curl} \mathbf{F}$ is positive [negative].

Rate of rotation of $\mathbf{F}$ about $\mathbf{v}$ $= \mathbf{v} \cdot \operatorname{curl} \mathbf{F}$



Rate of rotation of $\mathbf{F}$ about $\mathbf{u}$ $= \mathbf{u} \cdot \operatorname{curl} \mathbf{F}$

16.5.re2. Both $\operatorname{div} \mathbf{F}$ and $\mathbf{k} \cdot \operatorname{curl} \mathbf{F}$ have constant sign (positive, negative, or zero) for each of the vector fields shown below. Determine their sign and explain your answer.



16.5.re3. Let $\mathbf{F}(x,y,z) = \langle xy, xz, -yz\rangle$, $\mathbf{G}(x,y,z) = \langle z, y, -x\rangle$, $k(x,y,z) = \tan(xyz)$. Find the following or state that they do not exist. Clearly distinguish between the scalar 0 and the vector $\mathbf{0}$ should these occur in your answers.

a. $\operatorname{curl}\mathbf{F}$	b. $\operatorname{grad}\mathbf{F}$	c. $\operatorname{div}\mathbf{F}$	d. $\operatorname{div}(\mathbf{F}\cdot\mathbf{G})$	e. $\operatorname{div}(\operatorname{curl}\mathbf{G})$
f. $\operatorname{curl} k$	g. $\operatorname{grad} k$	h. $\operatorname{div} k$	i. $\operatorname{grad}(\mathbf{F}\cdot\mathbf{G})$	j. $\operatorname{curl}(\operatorname{grad} k)$

Fact If the second partial derivatives of f and $\mathbf{F}$ are continuous, then

$$\operatorname{curl}(\operatorname{grad} f) = \nabla\times\nabla f = \mathbf{0}$$
$$\operatorname{div}(\operatorname{curl}\mathbf{F}) = \nabla\cdot(\nabla\times\mathbf{F}) = 0$$

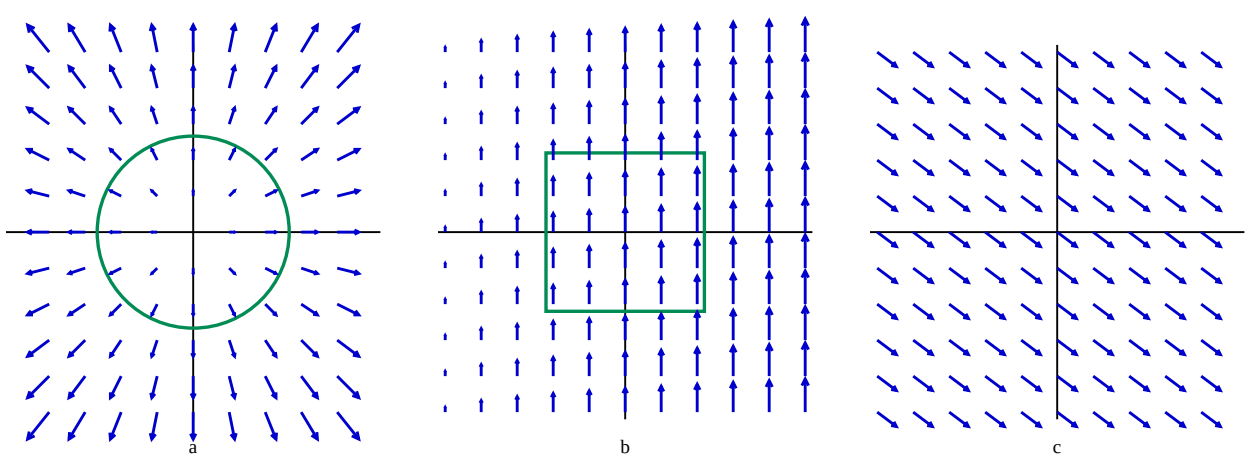
You can remember these by observing their similarity to the more familiar identities

$$\mathbf{u}\times\mathbf{u} = \mathbf{0}$$
$$\mathbf{u}\cdot(\mathbf{u}\times\mathbf{v}) = 0$$

The first of these says that the curl of a conservative vector field is $\mathbf{0}$. In fact, as we'll see in 16.8, the converse is true on sufficiently nice sets D.

Answers

16.5.re2. a. $\operatorname{div}\mathbf{F} > 0$, $\mathbf{k}\cdot\operatorname{curl}\mathbf{F} = 0$. For instance, fluid is flowing out the circle shown below but has zero circulation about the circle. b. $\operatorname{div}\mathbf{F} = 0$, $\mathbf{k}\cdot\operatorname{curl}\mathbf{F} > 0$. Net flow out of the square shown is zero, but net circulation about the square is positive, since the rate of flow is greater on the right side.
c. $\mathbf{F}$ is constant, so its div and curl both $= 0$.



16.5.re3a. $\langle -z-x, 0, z-x\rangle$. 16.5.re3b. DNE. 16.5.re3c. the scalar 0. 16.5.re3d. $=\operatorname{div}(\text{scalar})$: DNE. 16.5.re3e. the scalar 0. 16.5.re3f. DNE. 16.5.re3g. $\sec^2(xyz)\langle yz, xz, xy\rangle$. 16.5.re3h. DNE. 16.5.re3i. $3\langle yz, xz, xy\rangle$. 16.5.re3j. the vector $\mathbf{0}$.

16.6: Parametrized surfaces

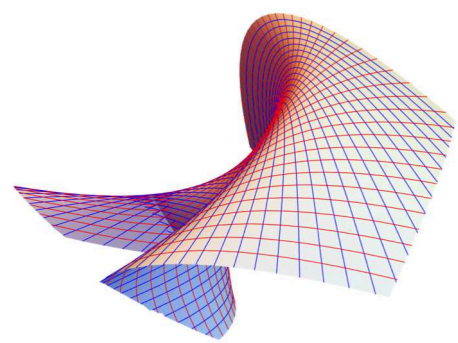
Like curves, surfaces can be represented by either parametric or implicit equations.

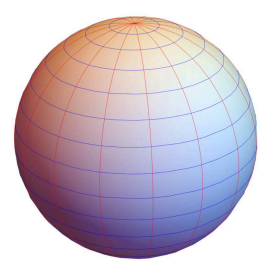
OBJECT	PARAMETRIC EQUATIONS	IMPLICIT EQUATIONS
curve in $\mathbb{R}^2$	$x = x(t)$ $y = y(t)$	$f(x, y) = 0$
curve in $\mathbb{R}^3$	$x = x(t)$ $y = y(t)$ $z = z(t)$	$f(x, y, z) = 0$ $g(x, y, z) = 0$
surface in $\mathbb{R}^3$	$x = x(u, v)$ $y = y(u, v)$ $z = z(u, v)$	$f(x, y, z) = 0$

The parametrization of a surface in $\mathbb{R}^3$ is a map from the uv-plane to xyz-space, often expressed as a vector-valued function

$$\mathbf{r}(u, v) = \langle x(u, v), y(u, v), z(u, v)\rangle$$

of two variables, say u and v. The surface can considered as the collection of all the curves traced by $\mathbf{r}(u, v)$ when one of u or v is fixed and the other is allowed to vary.





16.6.re1. The unit sphere in $\mathbb{R}^3$ can be expressed implicitly by $x^2 + y^2 + z^2 = 1$ and parametrically (by setting $\rho = 1$ and using spherical coordinates) by $x = \sin\phi\cos\theta$, $y = \sin\phi\sin\theta$, $z = \cos\phi$, or as the vector-valued function of two-variables

$$\mathbf{r}(\phi, \theta) = \langle \sin\phi\cos\theta, \sin\phi\sin\theta, \cos\phi\rangle, \quad 0 \le \phi \le \pi \quad 0 \le \theta \le 2\pi$$

Curves traced by $\mathbf{r}$ when θ is fixed are circles of longitude: circles on the sphere that pass through the north and south poles $(0, 0, \pm 1)$. Curves traced by $\mathbf{r}$ when ϕ is fixed on circles of latitude: circles centered at $(0, 0, z)$ of radius $\sqrt{1 - z^2}$.

Generally, the parametrization of a surface is written with some bounds on the variables, as in the last example. When none are given, assume that the variables are meant to take all values in the domain of the function—that is, everywhere the function is defined.

16.6.re2. We can recognize the surface parametrized by

$$\mathbf{r}(\theta, z) = \langle \sqrt{1 + z^2}\sin\theta, \sqrt{1 + z^2}\cos\theta, z\rangle \tag{16.6.1}$$

once we express it implicitly, that is, in a single xyz-equation. Square x and y and add:

$$(1+z^2)\sin^2\theta + (1+z^2)\cos^2\theta = 1+z^2$$
$$x^2+y^2 = 1+z^2,$$

which is the equation of the hyperboloid of one sheet symmetric about the z-axis.
The parametrization (16.6.1) is defined for all θ and z but redraws the surface infinitely many times as θ goes from $-\infty$ to ∞. In some calculations, it would be important to use an interval of θ that draws the surface once, for example, $-\pi \le \theta \le \pi$.

16.6.re3. The graph of the function $z = f(x,y)$ can always be parametrized by using x and y as parameters:

$$\mathbf{r}(x,y) = \langle x, y, f(x,y)\rangle$$

16.6.re4. Parametrize the plane with the given implicit equation.

a. $2x+y+3z=0$

b. $x+y=1$

We can also parametrize a plane by finding using point and two vectors parallel the plane, as in the next example.

16.6.re5. Parametrize the plane passing through the points $(1,2,3)$, $(1,0,2)$, $(3,2,0)$.

Solution: both vectors $\langle 1,0,2\rangle - \langle 1,2,3\rangle = \langle 0,-2,-1\rangle$ and $\langle 3,2,0\rangle - \langle 1,2,3\rangle = \langle 2,2,-2\rangle$ lie in the plane, and since these are non-parallel, every point in the plane can be reached from $(1,2,3)$ by moving in the directions of these two vectors. That is, every point in the plane has the form

$$\begin{aligned}\langle x,y,z\rangle &= \langle 1,2,3\rangle + u\langle 0,-2,-1\rangle + v\langle 2,2,-2\rangle\\ &= \langle 1+2v, 2-2u+2v, 3-u-2v\rangle,\end{aligned}$$

and so this parametrizes the plane.

16.6.re6. Find a parametrization for the surface with the given implicit equation.

a. $x^2-y^2+z^2=0$

b. The plane through $(0,0,0)$, $(1,1,0)$, $(1,-1,1)$.

c. $y^2+4z^2=1$

d. $x=4-y^2$, $x\ge 0$

e. $x=4-y^2-z^2$, $x\ge 0$

f. The part of the plane $x=1$ that lies inside the sphere $x^2+y^2+z^2=4$

g. The part of the sphere $x^2+y^2+z^2=4$ that lies outside the cylinder $x^2+y^2=1$.

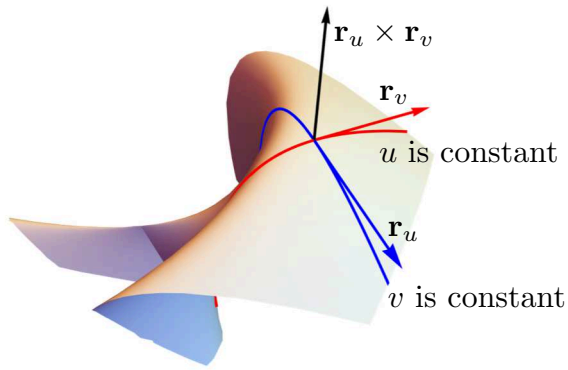
Tangent planes of parametrized surfaces

Every point on the graph of $\mathbf{r}(u,v)$ is the intersection of a curve along which u is constant and one along which v is constant. Since the derivatives

$$\mathbf{r}_u = \langle x_u(u,v), y_u(u,v), z_u(u,v)\rangle$$

and

$$\mathbf{r}_v = \langle x_v(u,v), y_v(u,v), z_v(u,v)\rangle$$



are tangent vectors to those curves, their cross product $\mathbf{r}_u \times \mathbf{r}_v$ must be a normal to the surface and the tangent plane.

16.6.re7. Find an equation of the plane tangent to the surface parametrized by $\langle 2\cos\theta, t+\sqrt{3}\sin\theta, \frac{1}{2}t\rangle$ at the point $(1, \frac{5}{2}, 2)$.

The surface passes through the given point when when $t = 4$ and $\theta = \frac{\pi}{3}$. To find the normal vector, calculate

$$\mathbf{r}_\theta \times \mathbf{r}_t = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -2\sin\theta & \sqrt{3}\cos\theta & 0 \\ 0 & 1 & \frac{1}{2} \end{vmatrix}$$

at $t = 4$, $\theta = \frac{\pi}{3}$:

$$\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -\sqrt{3} & \frac{\sqrt{3}}{2} & 0 \\ 0 & 1 & \frac{1}{2} \end{vmatrix}.$$

Since any vector parallel to this will do, multiply the third row by 2 and divide the second by $\frac{\sqrt{3}}{2}$:

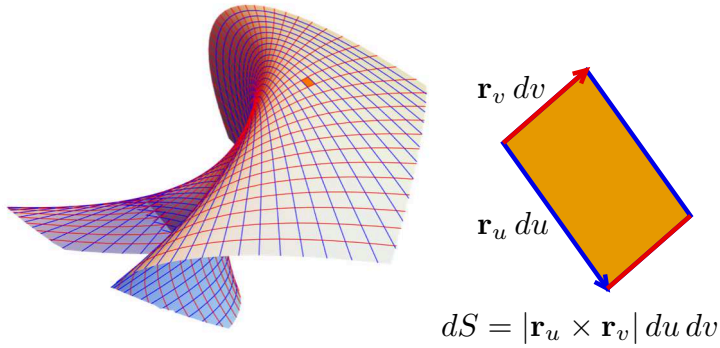
$$\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -2 & 1 & 0 \\ 0 & 2 & 1 \end{vmatrix} = \langle 1, 2, -4\rangle$$

The equation of the tangent plane is $(x-1) + 2(y - \frac{5}{2}) - 4(z-2) = 0$.

16.6.re8. Find the plane tangent to the surface at the given point.

a. $\langle u^2, uv, u+v\rangle$ at $u = 1$, $v = -1$. b. $\langle e^{uv}, u+v, u-v\rangle$ at $x = e^2$, $y = 3$, $z = -1$.

Surface area of parametrized surfaces



The curves along which u or v are constant divide the graph of $\mathbf{r}(u,v)$ into infinitesimal parallelograms with sides $\mathbf{r}_u\,du$ and $\mathbf{r}_v\,dv$. The area of such a parallelogram is the magnitude of the cross product of its sides. We call this area differential surface area dS and integrate dS to find area S of the given surface.

16.6.re9. To find the area of the that part of the surface $z = 4 - x^2 - y^2$ on which $z \geq 0$, parametrize the the surface by

$$\langle r\cos\theta, r\sin\theta, 4 - r^2\rangle, \quad 0 \leq r \leq 2, \quad 0 \leq \theta \leq 2\pi.$$

In contrast to example 16.6.re7, dS must be the magnitude of $\mathbf{r}_\theta \times \mathbf{r}_r$, not just some vector parallel this one.

$$\mathbf{r}_\theta \times \mathbf{r}_r = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -r\sin\theta & r\cos\theta & 0 \\ \cos\theta & \sin\theta & 2r \end{vmatrix} = \langle -2r^2\cos\theta, -2r^2\sin\theta, -r\rangle,$$

and so $dS = |\langle -2r^2\cos\theta, -2r^2\sin\theta, -r\rangle|\,dr\,d\theta = r\sqrt{1+4r^2}\,dr\,d\theta$. Integrate this to find surface area:

$$S = \int_0^{2\pi}\int_0^2 r\sqrt{1+4r^2}\,dr\,d\theta = 2\pi \cdot \frac{1}{12}(1+4r^2)^{3/2}\Big|_0^2 = \frac{\pi}{6}(17^{3/2} - 1).$$

16.6.re10. Find the area of the given surface.

a. The sphere centered at the origin of radius R.
b. $\langle u+v, uv, u-v\rangle$ for $u^2 + v^2 \leq 1$.
c. $z = \sqrt{x^2+y^2}$ that lies inside the sphere $x^2+y^2+z^2 = z$.
d. $x^2 + z^2 = 1$ above the triangle with vertices $(0,0,0)$, $(1,0,0)$, $(1,1,0)$.
e. $y + z = 2\sqrt{x}$ for $0 \leq x \leq 1$ and $0 \leq z \leq \sqrt{x}$

Answers

16.6.re4a. $\langle x, -2x-3z, z\rangle$. 16.6.re4b. $\langle 1-y, y, z\rangle$. 16.6.re6a. $\langle y\cos\theta, y, y\sin\theta\rangle$, $y \in (-\infty,\infty)$, $\theta \in (0, 2\pi)$. 16.6.re6b. $\langle u+v, u-v, v\rangle$. 16.6.re6c. Tip: first parametrize the cylinder $y^2+z^2 = 1$, and then replace z in your answer with $2z$. Result: $\langle x, \cos\theta, \frac{1}{2}\sin\theta\rangle$. 16.6.re6d. $\langle 4-y^2, y, z\rangle$, $-2 \leq y \leq 2$. 16.6.re6e. $\langle 4 - r^2, r\cos\theta, r\sin\theta\rangle$. $0 \leq r \leq 2$. 16.6.re6f. $\langle 1, \sqrt{3}\sin\theta, \sqrt{3}\cos\theta\rangle$. 16.6.re6g. $\langle 2\sin\phi\cos\theta, 2\sin\phi\sin\theta, 2\sin\phi\rangle$, $0 \leq \theta \leq 2\pi$, $\frac{\pi}{6} \leq \phi \leq \frac{5\pi}{6}$. 16.6.re8a. $x+y-z = 0$. 16.6.re8b. $-2x+3e^2y+e^2z = 6e^2$. 16.6.re10a. $4\pi R^2$. 16.6.re10b. $\frac{\pi}{3}(6^{3/2}-8)$. 16.6.re10c. $\frac{\pi}{2}$. 16.6.re10d. 1. 16.6.re10e. $3^{1/2} - 3^{-1}$.

16.7: Surfaces integrals

If E is a surface parametrized by $\mathbf{r}(u,v) = \langle x(u,v), y(u,v), z(u,v)\rangle$, for (u,v) in some domain D in the uv-plane, and if $f(x,y,z)$ is a function defined on E, then the **surface integral** of f on E

$$\iint_E f\,dS$$

can be calculated as a double integral in the variables u and v, using $dS = |\mathbf{r}_u \times \mathbf{r}_v|\,du\,dv$

$$\iint_D f\big(\mathbf{r}(u,v)\big)|\mathbf{r}_u \times \mathbf{r}_v|\,du\,dv. \tag{16.7.1}$$

The **unit normal vector n** of E is defined as either

$$\mathbf{n} = \pm(\mathbf{r}_u \times \mathbf{r}_v)/|\mathbf{r}_u \times \mathbf{r}_v|, \tag{16.7.2}$$

provided the cross product is not zero. If $\mathbf{F} = \langle P, Q, R\rangle$ is a vector field in $\mathbb{R}^3$, then the surface integral

$$\iint_E \mathbf{F}\cdot\mathbf{n}\,dS$$

is called the **flux** of $\mathbf{F}$ across E. (Sometimes the expression $\mathbf{n}\,dS$ is abbreviated $d\mathbf{S}$.) Because

$$\mathbf{n}\,dS = \pm\frac{\mathbf{r}_u \times \mathbf{r}_v}{|\mathbf{r}_u \times \mathbf{r}_v|}|\mathbf{r}_u \times \mathbf{r}_v|\,du\,dv,$$

depending on the choice of $\pm$ in (16.7.2), the flux is can be calculated

$$\pm\iint_D \mathbf{F}\cdot(\mathbf{r}_u \times \mathbf{r}_v)\,du\,dv. \tag{16.7.3}$$

16.7.re1. Calculate the surface integral.

a. $\iint_H (x+y)\,dS$, H given by $\mathbf{r}(r,\theta) = \langle r\cos\theta, r\sin\theta, \theta\rangle$, $0 \le r \le 2$, $0 \le \theta \le \frac{\pi}{2}$.

b. The flux of $\langle y, x, 1\rangle$ across the part of $x^2+y^2 = 1+z^2$ in first octant , $z \le 1$, oriented away from the z-axis.

c. $\iint_J \langle -y, x, x^2+y^2\rangle \cdot \mathbf{n}\,dS$, where J is $z = 1-x^2-y^2 \ge 0$, oriented upward.

Solution: a.

$$\mathbf{r}_\theta \times \mathbf{r}_r = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -r\sin\theta & r\cos\theta & 1 \\ \cos\theta & \sin\theta & 0 \end{vmatrix} = \langle -\sin\theta, \cos\theta, -r\rangle$$

and so $dS = \sqrt{1+r^2}\,dr\,d\theta$, and the surface integral can be calculated as the double integral

$$\begin{aligned}\int_0^{\frac{\pi}{2}}\int_0^2 (r\cos\theta + r\sin\theta)\sqrt{1+r^2}\,dr\,d\theta &= \int_0^{\frac{\pi}{2}}(\cos\theta+\sin\theta)\,d\theta\int_0^2 r\sqrt{1+r^2}\,dr \\ &= (\sin\theta-\cos\theta)\Big|_0^{\frac{\pi}{2}}\frac{1}{3}(1+r^2)^{3/2}\Big|_0^2 = \frac{2}{3}(5^{3/2}-1).\end{aligned}$$

b. Using the parametrization

$$\langle \sqrt{1+z^2}\sin\theta, \sqrt{1+z^2}\cos\theta, z\rangle$$

from 16.6.re2,

$$\begin{aligned}\mathbf{n}\,dS = \pm\mathbf{r}_\theta\times\mathbf{r}_z\,dz\,d\theta &= \pm\begin{vmatrix}\mathbf{i} & \mathbf{j} & \mathbf{k}\\ \sqrt{1+z^2}\cos\theta & -\sqrt{1+z^2}\sin\theta & 0\\ \frac{z}{\sqrt{1+z^2}}\sin\theta & \frac{z}{\sqrt{1+z^2}}\cos\theta & 1\end{vmatrix}\,dz\,d\theta\\ &= \pm\langle -\sqrt{1+z^2}\sin\theta, -\sqrt{1+z^2}\cos\theta, z\rangle\,dz\,d\theta.\end{aligned} \tag{16.7.4}$$

To decide between $\pm$, evaluate this normal at a point and decide whether it points away from the z-axis as required. Observe that $\mathbf{r}=(0,1,0)$ at $z=\theta=0$, and that $\mathbf{r}_\theta\times\mathbf{r}_z = \langle 0,-1,0\rangle$, which points *toward* the z-axis from $(0,1,0)$. So that $\mathbf{n}$ will point away from the z-axis, choose $-$ in (16.7.4). The surface integral is

$$\begin{aligned}&\iint_E \langle\sqrt{1+z^2}\cos\theta, \sqrt{1+z^2}\sin\theta, 1\rangle\cdot\langle\sqrt{1+z^2}\sin\theta, \sqrt{1+z^2}\cos\theta, -z\rangle\,dz\,d\theta\\ &= \int_0^{\pi/2}\int_0^1 (2(z^2+1)\sin\theta\cos\theta - z)\,dz\,d\theta\\ &= \int_0^{\pi/2}(2(\frac{1}{3}z^3+z)\sin\theta\cos\theta - \frac{1}{2}z^2)\Big|_0^1\,d\theta\\ &= \int_0^{\pi/2}(\frac{8}{3}\sin\theta\cos\theta - \frac{1}{2})\,d\theta\\ &= (\frac{4}{3}\sin^2\theta - \frac{1}{2}\theta)\Big|_0^{\pi/2} = \frac{4}{3}-\frac{\pi}{4}.\end{aligned}$$

c. Note that the integral in question is a flux. Use polar coordinates to parametrize the surface:

$$\mathbf{r}(r,\theta) = \langle r\cos\theta, r\sin\theta, 1-r^2\rangle, \quad 0\le r\le 1, \quad 0\le\theta\le 2\pi$$

Then

$$\begin{aligned}\mathbf{n}\,dS = \pm\mathbf{r}_r\times\mathbf{r}_\theta\,dr\,d\theta &= \pm\begin{vmatrix}\mathbf{i} & \mathbf{j} & \mathbf{k}\\ \cos\theta & \sin\theta & -2r\\ -r\sin\theta & r\cos\theta & 0\end{vmatrix}\,dr\,d\theta\\ &= \pm\langle 2r^2\cos\theta, 2r^2\sin\theta, r\rangle\,dr\,d\theta\end{aligned} \tag{16.7.5}$$

For this to point upward requires that its $\mathbf{k}$-coefficient be positive. Since $r\ge 0$, choose the $+$ in (16.7.5). With this choice, $\mathbf{F}\cdot\mathbf{n}\,dS =$

$$\langle -r\sin\theta, r\cos\theta, r^2\rangle\cdot\langle 2r^2\cos\theta, 2r^2\sin\theta, r\rangle\,dr\,d\theta = r^3\,dr\,d\theta$$

and so the flux is

$$\int_0^{2\pi}\int_0^1 r^3\,dr\,d\theta = \frac{\pi}{2}.$$

16.7.re2. Calculate the surface integral.

a. the flux of $\langle 1, 1, y\rangle$ across hemisphere $y = \sqrt{1 - x^2 - z^2}$, oriented away from $(0, 0, 0)$.
b. $\iint_G \langle x, 0, z\rangle \cdot \mathbf{n}\, dS$, G is $z = 1 - y^2 \geq 0$ for $0 \leq x \leq 1$, oriented upward.
c. the flux of $\langle x, x, y\rangle$ across the plane $x + y + 2z = 2$ in 1st octant, oriented upward.
d. $\iint_R x\, dS$, R is $x^2 + z^2 = 1$ above the triangle with vertices $(0, 0, 0)$, $(1, 0, 0)$, $(1, 1, 0)$.
e. $\iint_K e^z\, dS$, K is $z = \sqrt{x^2 + y^2} \leq 3$.
f. $\iint_M x^2 + y^2\, dS$, M given by $\langle u - v, u + v, uv\rangle$, for $u^2 + v^2 \leq 1$.

Answers

16.7.re2a. $\frac{\pi}{2}$. 16.7.re2b. $\frac{4}{3}$. 16.7.re2c. $\frac{8}{3}$. 16.7.re2d. $\frac{\pi}{4}$. 16.7.re2e. $2\pi\sqrt{2}(2e^3 + 1)$.
16.7.re2f. $\frac{\pi}{4}(\frac{2}{5}6^{5/2} - \frac{8}{3}6^{3/2} - \frac{64}{5} + \frac{64}{3})$.

16.8: Stokes's Theorem

Two interpretations of Green's theorem

Suppose $\mathbf{F} = \langle P, Q, 0\rangle$ and D is a region in the plane bounded by the closed curve ∂D. Let $\mathbf{T} = \langle \frac{dx}{ds}, \frac{dy}{ds}\rangle$, the unit tangent vector to ∂D when the path is traveled in the positive direction, i.e., with D always on the left. Then $\mathbf{n} = \langle \frac{dy}{ds}, -\frac{dx}{ds}\rangle$ is the outward-bound unit normal to the curve ∂D. Green's theorem (16.4.1) states that

$$\iint_D (Q_x - P_y)\, dA = \oint_{\partial D} P\, dx + Q\, dy,$$

or

$$\iint_D (\nabla \times \mathbf{F}) \cdot \mathbf{k}\, dA = \oint_{\partial D} \mathbf{F} \cdot \mathbf{T}\, ds. \tag{16.8.1}$$

That is, the integral over D of $\operatorname{curl} \mathbf{F}$ dotted with the unit normal to D equals the circulation of $\mathbf{F}$ around the boundary of D.

If we interchange P and Q and then change the sign of Q, (16.4.1) becomes

$$\iint_D (P_x + Q_y)\, dA = \oint_{\partial D} P\, dy - Q\, dx,$$

or

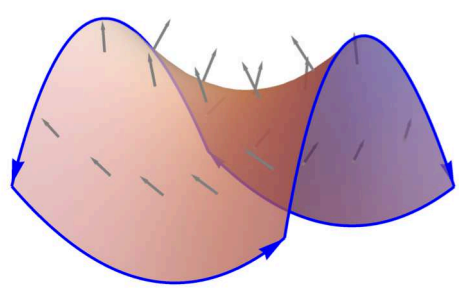
$$\iint_D (\nabla \cdot \mathbf{F})\, dA = \oint_{\partial D} \mathbf{F} \cdot \mathbf{n}\, ds. \tag{16.8.2}$$

That is, the integral over D of $\operatorname{div} \mathbf{F}$ equals the flux of $\mathbf{F}$ out of D across its boundary.

Stokes's theorem is the generalization of (16.8.1) in which the planar surface D is replaced by a surface in $\mathbb{R}^3$. We'll see a generalization of (16.8.2) in the next section.

Stokes's theorem. If E is an orientable surface in $\mathbb{R}^3$ and ∂E is its boundary, and if the derivatives of the components of $\mathbf{F}$ are continuous, then

$$\iint_E \operatorname{curl} F \cdot \mathbf{n}\, dS = \oint_{\partial E} \mathbf{F} \cdot d\mathbf{r}. \tag{16.8.3}$$



The line integral in (16.8.3) is taken in the positive direction from the point of view of the normal vector $\mathbf{n}$ to E, that is, the direction that keeps E always to $\mathbf{n}$'s left.

16.8.re1. Compute both sides of (16.8.3) for $\mathbf{F} = \langle -\frac{1}{2}y, \frac{1}{2}x, z\rangle$ and E equal to the hemisphere $z = \sqrt{1 - x^2 - y^2}$, oriented upward.

It is not difficult to show that $\operatorname{curl} F = \langle 0, 0, 1\rangle$.
By using spherical coordinates with $\rho = 1$, we obtain the parametrization of the sphere

$$\mathbf{r} = \langle \sin\phi\cos\phi, \sin\phi\sin\theta, \cos\theta\rangle, \qquad 0 \le \theta \le 2\pi, \quad 0 \le \phi \le \frac{\pi}{2}. \tag{16.8.4}$$

In example 16.6.10 of our text, it is shown that

$$\mathbf{r}_\phi \times \mathbf{r}_\theta = \langle \sin^2\phi\cos\theta,\ \sin^2\phi\sin\theta,\ \sin\phi\cos\phi\rangle.$$

Since $0 \le \phi \le \frac{\pi}{2}$, the third coordinate of this vector is always positive, so

$$\mathbf{n}\, dS = \mathbf{r}_\phi \times \mathbf{r}_\theta\, d\phi\, d\theta.$$

The left side of (16.8.3) is

$$\int_0^{2\pi}\int_0^{\pi/2} \sin\phi\cos\phi\, d\phi\, d\theta = \int_0^{2\pi} \frac{1}{2}\sin^2\phi\Big|_0^{\pi/2} d\theta = \int_0^{2\pi} \frac{1}{2}\, d\theta = \pi.$$

The right side of (16.8.3) is the line integral

$$\oint_{\partial E} -\frac{1}{2}y\, dx + \frac{1}{2}x\, dy$$

around the unit circle. Since the normal to E points upward, the positive direction around ∂E from the point of view of $\mathbf{n}$ is the same as the positive direction around the circle as viewed from above.
When we set $\phi = \frac{\pi}{2}$ in (16.8.4), we obtain the standard parametrization of the unit circle

$$x = \cos\theta \quad y = \sin\theta$$

Using this, and letting θ go from 0 to 2π, we evaluate the line integral as

$$\int_0^{2\pi} -\frac{1}{2}\sin\theta\, d(\cos\theta) + \frac{1}{2}\cos\theta\, d(\sin\theta) \int_0^{2\pi} \frac{1}{2}\sin^2\theta\, d\theta + \frac{1}{2}\cos^2\theta\, d\theta = \int_0^{2\pi} \frac{1}{2}\, d\theta = \pi,$$

the same result as the double integral.

16.8.re2. Use Stokes's theorem to calculate the given integral.

a. $\iint_B \operatorname{curl}\langle z, xy, xz\rangle \cdot \mathbf{n}\, dS$, $B =$ the five sides of the cube formed by the planes $x = 0$ or 1, $y = 0$ or 1, $z = 0$, between $z = 0$ and $z = 1$, oriented outward.

b. The circulation of $\langle ye^x, xe^y, z\rangle$ around the parallelogram with vertices $(0, 0, 0)$, $(-1, 0, 1)$, $(0, 2, 1)$ and $(-1, 2, 2)$ in the positive direction seen from above.
Hint: parametrize the surface with $\mathbf{r} = u\langle -1, 0, 1\rangle + v\langle 0, 2, 1\rangle$.

c. The circulation of $\langle z, xy, xz\rangle$ around $x^2 + z^2 = 1$ in the plane $y = 0$, traversed in the positive direction from the point of view of the vector $\mathbf{j}$.

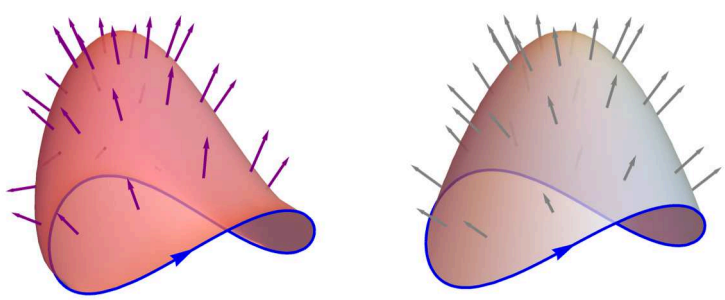
d. $\iint_C \operatorname{curl}\langle y, yz, xz\rangle \cdot \mathbf{n}\, dS$, G is $z = 1 - r^2 \geq 0$, oriented upward.

Surfaces with the same boundary

If two surfaces have the same boundary and the same orientation, meaning that the positive direction around the boundary is the same from the point of view of either of their normals, then the flux of the curl of $\mathbf{F}$

$$\iint_E \operatorname{curl} \mathbf{F} \cdot \mathbf{n}\, dS$$

must be same across either surface.



16.9.re1, continued. Letting C denote the interior of the unit circle in the xy-plane, the line integral

$$\iint_E \langle 0, 0, 1\rangle \cdot \mathbf{n}\, dS = \oint_{\partial E} -\frac{1}{2} y\, dx + \frac{1}{2} x\, dy$$

can be rewritten via Green's theorem as a double integral over C::

$$\oint_{\partial C} -\frac{1}{2} y\, dx + \frac{1}{2} x\, dy = \iint_C 1\, dA$$

(which equals the area of the circle, as seen in (16.4.2)). That is,

$$\iint_E \langle 0, 0, 1\rangle \cdot \mathbf{n}\, dS = \iint_C \langle 0, 0, 1\rangle \cdot \mathbf{k}\, dA.$$

Conservative vector fields and their curl

Another consequence of Stokes's theorem is that, on simply connected regions in $\mathbb{R}^3$, a vector field is conservative if and only if its curl is zero. Here's an update of the table appearing 16.4. This assumes $\mathbf{F} = \langle P, Q\rangle$ or $\langle P, Q, R\rangle$ on a region D in $\mathbb{R}^2$ or $\mathbb{R}^3$, and that P, Q, and R have continuous first partial derivatives.)

$\displaystyle\int \mathbf{F}\cdot d\mathbf{r}$ is path-independent in D.

$$\Updownarrow$$

$\displaystyle\oint_C \mathbf{F}\cdot d\mathbf{r} = 0$ for every closed path C in D.

$$\Updownarrow$$

$\mathbf{F} = \nabla f$ in D for some scalar-valued function f.

$$\Updownarrow$$

$\mathbf{F}$ is **conservative** in D $\implies$ $\operatorname{curl}\mathbf{F} = \mathbf{0}$.

$\mathbf{F}$ is **conservative** in D $\impliedby$ $\operatorname{curl}\mathbf{F} = \mathbf{0}$, if D is simply connected.

The set D in $\mathbb{R}^2$ or $\mathbb{R}^3$ is **simply connected** if every continuous path in D can be continuously contracted to a point. $\mathbb{R}^2$ and $\mathbb{R}^3$ are themselves simply connected.

16.8.re3. Find a potential function for the given vector field or determine that none exists. (See also 16.3.re2.)

a. $\mathbf{F} = \langle z, y^2, xz\rangle$

b. $\mathbf{G} = \langle 1 + e^{-y}, -xe^{-y}, -e^z\rangle$

c. $\mathbf{R} = \langle \frac{1}{2}z^2, y^2, xz\rangle$

d. $\mathbf{H} = \langle y + \frac{1}{2}\sqrt{\frac{z}{x}}, x - 3z, \frac{1}{2}\sqrt{\frac{x}{z}} - 3y\rangle$

Answers

16.8.re2a. $-\frac{1}{2}$. 16.8.re2b. $2e^2 + 4e^{-1} - 6$. 16.8.re2c. π. 16.8.re2d. $-\pi$. 16.8.re3a. $\operatorname{curl}\mathbf{F} = \langle 0, 1 - z, 0\rangle$, so $\mathbf{F}$ is not conservative. 16.8.re3b. $xe^{-y} + x - e^z$. 16.8.re3c. $\frac{1}{2}xz^2 + \frac{1}{3}y^3$. 16.8.re3d. $xy - 3yz + \sqrt{xz}$.

16.9: The divergence theorem

The divergence theorem is the generalization of (16.8.2) in which the planar surface D is replaced by a solid E in $\mathbb{R}^3$.

Divergence theorem. If E is a solid in $\mathbb{R}^3$ and ∂E is its boundary surface and $\mathbf{n}$ is ∂E's unit normal vector in the direction out of E, and if the derivatives of the components of $\mathbf{F}$ are continuous, then

$$\iiint_E \operatorname{div} \mathbf{F}\, dV = \iint_{\partial E} \mathbf{F} \cdot \mathbf{n}\, dS. \tag{16.9.1}$$

That is, the integral over E of the divergence of $\mathbf{F}$ equals the flux of $\mathbf{F}$ out of E.

16.9.re1. Compute the flux of $\mathbf{F} = \langle xz, y, 0\rangle$ out of the cube with vertices $(\pm 1, \pm 1, \pm 1)$ both as a surface integral and, by the divergence theorem, as a triple integral.

First the surface integral.
On the top and bottom faces of the cube, the normals are $\pm\mathbf{k}$. Since $\mathbf{F} \cdot \mathbf{k} = 0$, the flux out of E across these surfaces is zero.
On the faces given by $y = \pm 1$, the normals are $\pm\mathbf{j}$. When $y = 1$, the flux is

$$\int_{-1}^{1}\int_{-1}^{1} \langle xz, 1, 0\rangle \cdot \mathbf{j}\, dx\, dz = \int_{-1}^{1}\int_{-1}^{1} 1\, dx\, dz = 4.$$

When $y = -1$, the flux is the same:

$$\int_{-1}^{1}\int_{-1}^{1} \langle xz, -1, 0\rangle \cdot (-\mathbf{j})\, dx\, dz = \int_{-1}^{1}\int_{-1}^{1} 1\, dx\, dz = 4.$$

On the faces where $x = \pm 1$, the normals are $\pm\mathbf{i}$. When $x = 1$, the flux is

$$\int_{-1}^{1}\int_{-1}^{1} \langle z, y, 0\rangle \cdot \mathbf{i}\, dz\, dy = \int_{-1}^{1}\int_{-1}^{1} z\, dz\, dy = 0,$$

and where $x = -1$, the flux is

$$\int_{-1}^{1}\int_{-1}^{1} \langle -z, y, 0\rangle \cdot (-\mathbf{i})\, dz\, dy = \int_{-1}^{1}\int_{-1}^{1} z\, dz\, dy = 0.$$

Therefore the total flux, as calculated as a surface integral, is 8.
The divergence theorem promises that this same result can be gotten by the triple integral

$$\begin{aligned}\int_{-1}^{1}\int_{-1}^{1}\int_{-1}^{1} \operatorname{div} \langle xz, y, 0\rangle\, dx\, dy\, dz &= \int_{-1}^{1}\int_{-1}^{1}\int_{-1}^{1} (xz)_x + (y)_y + (0)_z\, dx\, dy\, dz \\ &= \int_{-1}^{1}\int_{-1}^{1}\int_{-1}^{1} (x+1)\, dx\, dy\, dz.\end{aligned}$$

Since $\int_{-1}^{1} x\, dx = 0$, the triple integral is the same as $\int_{-1}^{1}\int_{-1}^{1}\int_{-1}^{1} dx\, dy\, dz$, which equals the volume of the cube, 8.

16.9.re2. Compute the flux of the given vector field out of the solid E.

a. $\langle e^x, ye^x, -z\rangle$, $E =$ the cube $[0,1]\times[0,1]\times[0,1]$
b. $\langle 2x-z, 2x+3y, 2y-z\rangle$, $E =$ the solid inside $x^2+y^2=1$ between $z=0$ and $z=2-y$.
c. $\operatorname{grad}(x^4+y^4+z^4-2xy-3z+4y)$, $E =$ the hemisphere $x^2+y^2+z^2\le 1$ $z\ge 0$
d. $\operatorname{curl}\langle e^x, ye^x, -z\rangle$, $E =$ the tetrahedron in the first octant bounded by $x+y+z=4$.

The flux of the curl over a closed surface

As seen in 16.9.re2d, the flux of the curl of $\mathbf{F}$ over a closed surface H in $\mathbb{R}^3$ must be zero:

$$\iint_H \operatorname{curl}\mathbf{F}\cdot\mathbf{n}\,dS = 0,$$

since this equals the triple integral of div $\operatorname{curl}\mathbf{F} = 0$ over the interior of H.

Answers

16.9.re2a. $2e-3$. 16.9.re2b. 8π. 16.9.re2c. The divergence of the gradient is $12(x^2+y^2+z^2)$; flux is $24\frac{\pi}{5}$. 16.9.re2d. The divergence of the curl is 0; flux is 0.