MATH 221–02 (Kunkle), Quiz 8 — Name: ______________________________

10 pts, 10 minutes — April 4, 2024

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1 (10 pts). Evaluate the given line integral.

a. $\int_D (2x+y)\,ds$, where D is the line segment from $(0,1)$ to $(1,-1)$.

b. $\int_C \frac{y}{z}\,dx + \frac{x}{z}\,dy - \frac{xy}{z^2}\,dz$ where C is the line segment from $(1,0,3)$ to $(2,-1,2)$.

1a.(Source: 16.2.4) The simplest solution is to notice that the equation of the line from $(0,1)$ to $(1,-1)$ is $2x+y=1$, so the integral is $\int_D 1\,ds$, which simply equals the length of the line segment: $\sqrt{1^2+2^2}=\sqrt{5}$.

But if you didn't see that, you could always parametrize the line segment as

$$x=t \qquad y=1-2t \qquad ds=\frac{ds}{dt}\,dt=\sqrt{1^2+(-2)^2}\,dt=\sqrt{5}\,dt$$

and write the integral as

$$\int_0^1 \big(2t+(1-2t)\big)\sqrt{5}\,dt=\int_0^1\sqrt{5}\,dt=\sqrt{5}.$$

1b.(Source: 16.3.10,15) This time, parametrizing the curve leads to a difficult integral:

$$\int_0^1\left(\frac{-t}{3-t}-\frac{1+t}{3-t}+\frac{t(1+t)}{(3-t)^2}\right)dt=\left(\frac{-t^2-t}{3-t}\right)\Big|_0^1=-1$$

It's far easier to look for a potential function for $\langle \frac{y}{z},\frac{x}{z},-\frac{xy}{z^2}\rangle$, that is, a function $f(x,y,z)$ with the property that

$$f_x=\frac{y}{z} \qquad f_y=\frac{x}{z} \qquad f_z=-\frac{xy}{z^2} \tag{1}$$

Here's a complete solution, but at any point, you might find such a function. As long as you demonstrate that the function you find satisfies (1), you don't need to complete all the following steps.

$$f_x=\frac{y}{z}\Rightarrow f=\frac{xy}{z}+C(y,z)\Rightarrow f_y=\frac{x}{z}+C_y(y,z)=\frac{x}{z}\Rightarrow C_y(y,z)=0\Rightarrow C(y,z)=C(z)$$

Now differentiate f with respect to z:

$$f_z=-\frac{xy}{z^2}+C_z(z)=-\frac{xy}{z^2}\Rightarrow C_z(z)=0\Rightarrow C=\text{constant}$$

We needed only *one* potential function, but we found them all: the general potential function for $\langle \frac{y}{z},\frac{x}{z},-\frac{xy}{z^2}\rangle$ is $f=\frac{xy}{z}+$ any constant.

We can choose the constant to be zero and, by the fundamental theorem of calculus for line integrals, the line integral is

$$\int_C \nabla f\cdot d\mathbf{r}=f(x,y,z)\Big|_{(1,0,3)}^{(2,-1,2)}=f(2,-1,2)-f(1,0,3)=-1-0=-1.$$