No notes, books, electronic devices, or outside materials of any kind.

Read each problem carefully and simplify your answers.

Unless otherwise indicated, supporting work will be required on every problem worth more than 2 points.

You are expected to know the values of all trig functions at multiples of $\pi/4$ and of $\pi/6$.

$$\int \sin^n x\,dx = -\frac{1}{n}\sin^{n-1} x\cos x + \frac{n-1}{n}\int \sin^{n-2} x\,dx$$
$$\int \cos^n x\,dx = \frac{1}{n}\cos^{n-1} x\sin x + \frac{n-1}{n}\int \cos^{n-2} x\,dx$$
$$\int \tan^n x\,dx = \frac{1}{n-1}\tan^{n-1} x - \int \tan^{n-2} x\,dx$$
$$\int \sec^n x\,dx = \frac{1}{n-1}\sec^{n-2} x\tan x + \frac{n-2}{n-1}\int \sec^{n-2} x\,dx \qquad (n \neq 1)$$

1a(4 pts). Find the vector projection (also known as the orthogonal projection) of $\langle 1, 2, 1\rangle$ onto $\langle -1, -1, 1\rangle$.

1b(1 pts). Which of these is the best description of the correct answer to 1a? (circle one)

i. The point on the line $\langle -t, -t, t\rangle$ closest to the point $(1, 2, 1)$.

ii. The point on the line $\langle t, 2t, t\rangle$ closest to the point $(-1, -1, 1)$.

iii. The area of the parallelogram with $\langle 1, 2, 1\rangle$ and $\langle -1, -1, 1\rangle$ for its sides.

2(4 pts). Two students are working on the problem of maximizing and minimizing a function $q(x, y)$ on the disk $D = \{(x, y) \mid x^2 + y^2 \leq 1\}$. They observe that $q(x, y)$ is continuous and differentiable everywhere in the xy plane.
The first student has determined that the maximum and minimum values of q on the circle $x^2 + y^2 = 1$ are 3 and -2.
The second student has determined that the only solutions to $q_x(x, y) = q_y(x, y) = 0$ are $(x, y) = (0, 1 \pm \frac{1}{3})$, and that $q(0, \frac{2}{3}) = 7$ and $q(0, \frac{4}{3}) = -4$.
Assuming their work is correct, can you determine the maximum and minimum values of $q(x, y)$ on D?

3(32 pts). Let $\mathbf{r}(t) = \langle e^{1-t}, t^{-1}, \frac{1}{2}t^2 + 1\rangle$ be the position of a particle at time t.

a. Find the particle's velocity, acceleration, and speed at $t = 1$.

b. Find a parametric equation of the line tangent to the particle's path at the point corresponding to $t = 1$.

c. Find a_T and a_N, the tangential and normal components of the particle's acceleration at $t = 1$.

d. Is the particle is speeding up or slowing down at $t = 1$? Explain briefly.

e. Find $\mathbf{T}$, $\mathbf{N}$, and $\mathbf{B}$ and curvature κ at $t = 1$.

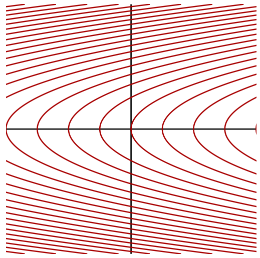
4(4 pts). Find the contour plot of each of the four functions below.

a. $(x+y)^2$

b. $\sqrt{x^2+y^2}$

c. $1-x^2-y^2$

d. $x-y^2$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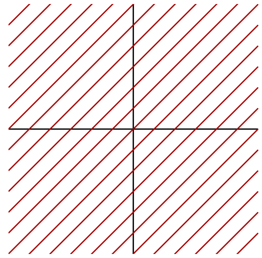
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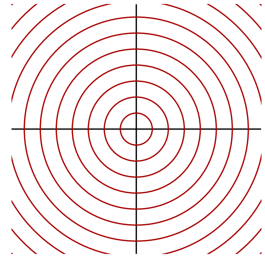
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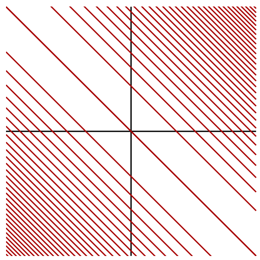
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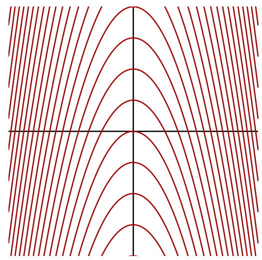
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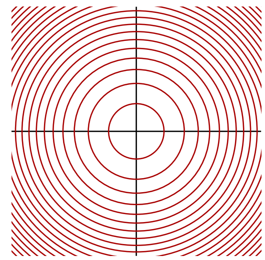
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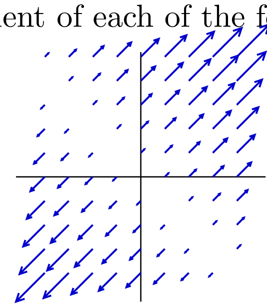
5(8 pts). **Compute** the gradient of each of the four functions **and** find its plot.

a. $\nabla\big((x+y)^2\big) =$

b. $\nabla\big(\sqrt{x^2+y^2}\big) =$

c. $\nabla\big(1-x^2-y^2\big) =$

d. $\nabla\big(x-y^2\big) =$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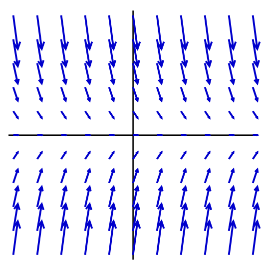
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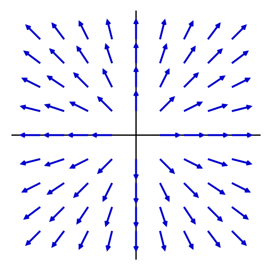
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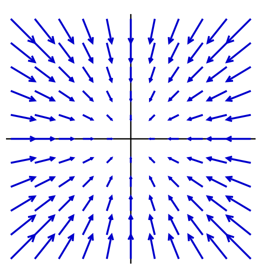
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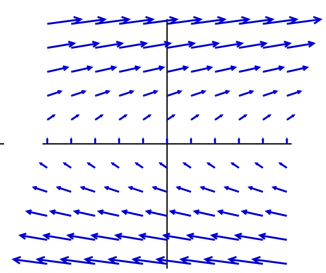
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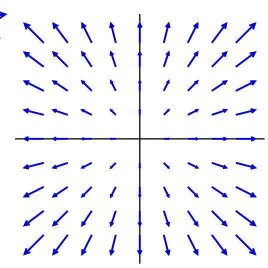
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6a(4 pts). Find the directional derivative of $\ell(x,y) = x - y^2$ in the direction $\langle \frac{3}{5}, \frac{4}{5} \rangle$ at the point $(x,y) = (3,2)$.

6b(4 pts). Suppose that the temperature at every point in the xy-plane is given by $\ell(x,y) = x - y^2$ (degrees), and that an insect traveling in the plane is at position $\langle x(t), y(t) \rangle$ at time t (seconds), and that

$$x(0) = 3 \qquad y(0) = 2 \qquad x_t(0) = 4 \qquad y_t(0) = -5$$

At time 0, at what rate (degrees per second) is the temperature experienced by the insect changing?

7(19 pts). Determine the (x, y) coordinates of all local maxima, local minima, and saddle points of $\phi(x, y) = \frac{1}{3}x^3 - 4x + xy^2$, and state which is which.

8. Find either an xyz-equation or parametric equation(s) of the plane described in each part.

a(5 pts). Passing through the points $(0, 0, 0)$, $(1, 2, 1)$, and $(-1, -1, 1)$.
b(10 pts). Tangent to the surface $xe^{y-z} + yz \ln x = 1$ at the point $(1, 1, 1)$.
c(13 pts). Tangent to the surface parametrized by $\mathbf{r}(u, v) = \langle v^2, uv, \frac{1}{2}u^2 \rangle$ at the point in 3-space corresponding to $(u, v) = (1, 1)$.

9(16 pts). Find the volume of the solid above the cone $z = \sqrt{3}\sqrt{x^2 + y^2}$ and below the sphere $x^2 + y^2 + z^2 = 16$.

10. A fluid flowing through 3-space has velocity $\langle x^2 - y + z, x + y^2 - z, -x + y + z^2 \rangle$ at the point (x, y, z).

a(14 pts). Find the net rate of flow (a.k.a. "flux") of the fluid out of the rectangular parallelogram with vertices

$$\begin{array}{cccc} (0,0,0) & (0,2,0) & (0,0,3) & (0,2,3) \\ (1,0,0) & (1,2,0) & (1,0,3) & (1,2,3) \end{array}$$

by first writing the flux as a triple integral.

b(22 pts). Find the net rate of flow (a.k.a. "circulation") of the fluid around the triangle

$$\begin{array}{ccccc} & \swarrow & (0,0,2) & \nwarrow & \\ (1,0,0) & & \longrightarrow & & (0,1,0) \end{array}$$

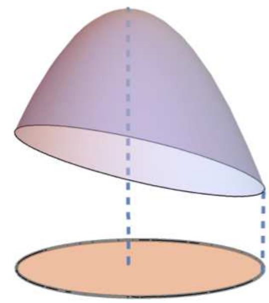
by first writing the circulation as a surface integral.

1a(4 pts).(Source: 12.3.41)

$$\text{proj}_{\langle -1,-1,1\rangle}\langle 1,2,1\rangle = \frac{\langle 1,2,1\rangle\cdot\langle -1,-1,1\rangle}{\langle -1,-1,1\rangle\cdot\langle -1,-1,1\rangle}\langle -1,-1,1\rangle = \frac{-2}{3}\langle -1,-1,1\rangle,$$

or $\langle \frac{2}{3},\frac{2}{3},-\frac{2}{3}\rangle$.

1b(1 pts). Correct response is **i**.



2(4 pts).(Source: 14.8.31-42) The absolute extrema of a function on a region must occur at a boundary point or at a critical point interior to the region. Since the max and min along the boundary of D are 3 and -2 and q equals 7 at the only critical point interior to D, the maximum and minimum of q on D are 7 and -2, respectively.

3a(7 pts). velocity $\mathbf{v} = \frac{d\mathbf{r}}{dt} = \langle -e^{1-t}, -t^{-2}, t\rangle$. acceleration $\mathbf{a} = \frac{d^2\mathbf{r}}{dt^2} = \langle e^{1-t}, 2t^{-3}, 1\rangle$. At $t=1$, $\mathbf{r} = \langle 1,1,\frac{3}{2}\rangle$, $\mathbf{v} = \langle -1,-1,1\rangle$, $\mathbf{a} = \langle 1,2,1\rangle$, and speed $= |\mathbf{v}| = \sqrt{3}$.

3b(3 pts). The tangent line passes through the point $(1,1,\frac{3}{2})$ and is parallel $\langle -1,-1,1\rangle$. The line is parametrized by the vector-valued function

$$\langle 1,1,\frac{3}{2}\rangle + t\langle -1,-1,1\rangle,$$

or $\langle 1-t, 1-t, \frac{3}{2}+t\rangle$.

3c(10 pts).(Source: 13.4.41-42)

$$\mathbf{v}\times\mathbf{a} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -1 & -1 & 1 \\ 1 & 2 & 1 \end{vmatrix} = \mathbf{i}\begin{vmatrix} -1 & 1 \\ 2 & 1 \end{vmatrix} - \mathbf{j}\begin{vmatrix} -1 & 1 \\ 1 & 1 \end{vmatrix} + \mathbf{k}\begin{vmatrix} -1 & -1 \\ 1 & 2 \end{vmatrix} = \langle -3,2,-1\rangle$$

Recall that if θ is the angle between $\mathbf{v}$ and $\mathbf{a}$, then

$$a_T = |\mathbf{a}|\cos\theta = \frac{\mathbf{v}\cdot\mathbf{a}}{|\mathbf{v}|} = \frac{-2}{\sqrt{3}}, \quad \text{and} \quad a_N = |\mathbf{a}|\sin\theta = \frac{|\mathbf{v}\times\mathbf{a}|}{|\mathbf{v}|} = \frac{|\langle -3,2,-1\rangle|}{\sqrt{3}} = \sqrt{\frac{14}{3}}.$$

3d(2 pts).(Source: 13.4. $\boxed{7}$) $\frac{d^2s}{dt^2} = a_T < 0$. The particle is slowing down since its speed is decreasing.

3e(10 pts).(Source: (Source: 13.3.47-48))

$$\mathbf{T} = \frac{1}{|\mathbf{v}|}\mathbf{v} = \frac{1}{\sqrt{3}}\langle -1,-1,1\rangle \qquad \mathbf{B} = \frac{1}{|\mathbf{v}\times\mathbf{a}|}\mathbf{v}\times\mathbf{a} = \frac{1}{\sqrt{14}}\langle -3,2,-1\rangle$$

$$\mathbf{N} = \mathbf{B}\times\mathbf{T} = \frac{1}{\sqrt{3}\sqrt{14}}\langle -1,-1,1\rangle\times\langle -3,2,-1\rangle = \frac{1}{\sqrt{42}}\langle 1,4,5\rangle$$

$$\kappa = \frac{|\mathbf{v}\times\mathbf{a}|}{|\mathbf{v}|^3} = \sqrt{\frac{14}{27}}$$

4(4 pts).(Source: 14.1.61-66) A contour map of $f(x, y)$ consists of curves $f(x, y) = k$ for some evenly spaced constants k.
a. level curves are $\{(x + y)^2 = k\}$, or $\{x + y = \pm\sqrt{k}\}$, lines of slope -1. graph 4.
b. $\{\sqrt{x^2 + y^2} = k\}$ are circles of radius k. These are evenly spaced when k is. graph 3.
c. $\{1 - x^2 - y^2 = k\}$, or $\{x^2 + y^2 = 1 - k\}$, are circles of radius $\sqrt{1 - k}$. When k is evenly spaced, these are *not*. graph 6.
d. $\{x = y^2 + k\}$ are parabolas opening to the right. graph 1.
5(8 pts).(Source: 14.6.7a,8a,16.1.29-32) The gradient $\nabla f(x, y) = \langle f_x, f_y\rangle$ is perpendicular the level curves of $f(x, y)$.
a. $\nabla((x + y)^2) = \langle 2(x + y), 2(x + y)\rangle$, a scalar multiple of $\langle 1, 1\rangle$ and $\perp$ graph 4. graph 7.
b. $\nabla\sqrt{x^2 + y^2} = \langle \frac{1}{2}(x^2 + y^2)^{-1/2}2x, \frac{1}{2}(x^2 + y^2)^{-1/2}2y\rangle = \frac{1}{\sqrt{x^2+y^2}}\langle x, y\rangle$ = unit vector in the same direction as $\langle x, y\rangle$. Is $\perp$ graph 3. graph 9.
c. $\nabla(1 - x^2 - y^2) = -2\langle x, y\rangle$. Proportional to $\langle x, y\rangle$ in opposite direction. Is $\perp$ graph 6. graph 10.
d. $\nabla(x - y^2) = \langle 1, -2y\rangle$. Points right and down if $y > 0$ and up if $y < 0$. Is $\perp$ graph 1. graph 8.
6a(4 pts).(Source: 14.6.7-8) $\nabla\ell = \langle 1, -2y\rangle = \langle 1, -4\rangle$ at $(3, 2)$. See $\boxed{9}$, p. 950. Derivative in direction $\mathbf{u}$ is $\nabla\ell \cdot \mathbf{u} = \langle 1, -4\rangle \cdot \langle \frac{3}{5}, \frac{4}{5}\rangle = \frac{3}{5} - \frac{16}{5} = -\frac{13}{5}$.
6b(4 pts).(Source: 14.5.13,35) At time 0, $\langle x, y\rangle = \langle 3, 2\rangle$. By the chain rule,

$$\ell_t = \ell_x x_t + \ell_y y_t = 3 \cdot 4 + (-4)(-5) = 32 \text{ deg/sec.}$$

7(19 pts).(Source: 14.7.5-13) First, find ϕ's critical points: $\phi_y(x, y) = 2xy = 0$ implies either $x = 0$ or $y = 0$. Substituting these into $\phi_x = x^2 - 4 + y^2 = 0$ gives four critical points: $(0, 2)$, $(0, -2)$, $(2, 0)$, and $(-2, 0)$.
Now apply the Second Derivative Test at these points.

$$D = \begin{vmatrix} \phi_{xx} & \phi_{xy} \\ \phi_{xy} & \phi_{yy} \end{vmatrix} = \begin{vmatrix} 2x & 2y \\ 2y & 2x \end{vmatrix} = 4(x^2 - y^2).$$

critical point	D	ϕ_{xx}	conclusion
$(2, 0)$	16	4	local minimum
$(-2, 0)$	16	-4	local maximum
$(0, 2)$	-16	irrelevant	saddle point
$(0, -2)$	-16	irrelevant	saddle point

8. See $\boxed{7}$, p. 827. To find the equation of a plane, we need a point on the plane and a vector normal to the plane.
8a(5 pts).(Source: 12.5.31-33) This plane is parallel the vectors $\langle -1, -1, 1\rangle$ and $\langle 1, 2, 1\rangle$, so their cross product $\langle -3, 2, -1\rangle$ is normal to the plane. Since it passes through $(0, 0, 0)$, the plane is given implicitly by the equation $-3x + 2y - z = 0$.
8b(10 pts).(Source: 14.6.41-46) The gradient of a function is perpendicular to its level curves/surfaces. $\nabla(xe^{y-z} + yz\ln x) = \langle e^{y-z} + yz\frac{1}{x}, xe^{y-z} + z\ln x, -xe^{y-z} + y\ln x\rangle$, which, at

$(1,1,1)$, equals $\langle 2,1,-1\rangle$. The plane is given by the equation $2(x-1)+(y-1)-(z-1)=0$, or $2x+y-z=2$.

8c(13 pts).(Source: 16.6.33-36) The point corresponding to $(u,v)=(1,1)$ is given by $\mathbf{r}(1,1)=\langle 1,1,\frac{1}{2}\rangle$. The vector $\mathbf{r}_u\times\mathbf{r}_v=$

$$\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & v & u \\ 2v & u & 0 \end{vmatrix} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & 1 & 1 \\ 2 & 1 & 0 \end{vmatrix} = \langle -1,2,-2\rangle$$

is normal to the plane, which is given implicitly by $-(x-1)+2(y-1)-2(z-\frac{1}{2})=0$, or $-x+2y-2z=0$.

See review notes example 16.6.re.5. Since it is parallel to both $\mathbf{r}_u(1,1)=\langle 0,1,1\rangle$ and $\mathbf{r}_v=\langle 2,1,0\rangle$ and passes through $(1,1,\frac{1}{2})$, the plane can be expressed parametrically by

$$\rho(s,t)=\langle 1,1,\tfrac{1}{2}\rangle+s\langle 0,1,1\rangle+t\langle 2,1,0\rangle,$$

or $\langle 1+2t, 1+s+t, \frac{1}{2}+s\rangle$.

9(16 pts).(Source: 15.3.25,15.7.23,15.8.30) Solution one (cylindrical coordinates):

The cone and sphere intersect when $x^2+y^2+3(x^2+y^2)=16$, or $x^2+y^2=4$. Limits are $0\le\theta\le 2\pi$, $0\le r\le 2$. Using $\sqrt{x^2+y^2}=r$, the volume can be written

$$\begin{aligned}\int_0^{2\pi}\int_0^2\int_{r\sqrt{3}}^{\sqrt{16-r^2}} dz\, r\, dr\, d\theta &= \int_0^{2\pi}\int_0^2 (\sqrt{16-r^2}-r\sqrt{3})r\, dr\, d\theta \\ &= \int_0^{2\pi} (-\frac{1}{3}(16-r^2)^{3/2}-\frac{\sqrt{3}}{3}r^3)\Big|_0^2 d\theta \\ &= 2\pi(-\frac{1}{3}12^{3/2}+\frac{1}{3}16^{3/2}-\frac{\sqrt{3}}{8}),\end{aligned}$$

Solution two (spherical coordinates):

$\frac{r}{z}=\tan\theta$ and $z=r\sqrt{3}$ together tell us that $\tan\phi=\frac{1}{\sqrt{3}}$. Therefore $\phi=\frac{\pi}{6}$ along the cone. The volume can be written

$$\int_0^{2\pi}\int_0^{\pi/6}\int_0^4 \rho^2\sin\phi\, d\rho\, d\phi\, d\theta = \int_0^{2\pi} d\theta \int_0^{\pi/6}\sin\phi\, d\phi \int_0^4 \rho^2\, d\rho$$

$$2\pi(-\cos(\pi/6)+\cos 0)\frac{1}{3}4^3 = 2\pi(1-\frac{\sqrt{3}}{2})\frac{1}{3}4^3$$

With some effort, you can show that the answers in solutions one and two are the same. In rectangular coordinates, the volume is

$$\int_{-2}^{2}\int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}}\int_{\sqrt{3}\sqrt{x^2+y^2}}^{\sqrt{16-x^2-y^2}} dz\, dy\, dx,$$

but evaluating this would be a challenge.

10a(14 pts).(Source: 16.9.5-6, 15.6.2) The divergence of $\mathbf{F} = \langle x^2 - y + z, x + y^2 - z, -x + y + z^2 \rangle$ is $\nabla \cdot \mathbf{F} = (x^2 - y + z)_x + (x + y^2 - z)_y + (-x + y + z^2)_z = 2x + 2y + 2z$. By the divergence theorem, the flux of $\mathbf{F}$ out of the parallelogram is the triple integral of $\operatorname{div} \mathbf{F}$:

$$\int_0^1 \int_0^2 \int_0^3 (2x + 2y + 2z)\, dz\, dy\, dx.$$

Remembering that the integral of a constant is that constant times the length of the integral, the triple integral is

$$\int_0^1 \int_0^2 (6x + 6y + 9)\, dy\, dx = \int_0^1 (12x + 12 + 18)\, dx = (6 + 12 + 18) = 36.$$

10b(22 pts).(Source: 16.8.7, 16.7.11, 15.2.19,22) Stokes's theorem says that the circulation around the triangular path is the same as the integral of $\operatorname{curl} \mathbf{F} \cdot \mathbf{n}\, dS$ over any surface whose boundary is that triangle. The simplest surface to use is the interior of the triangle. We need an equation of the plane passing through the three points. You can either use the method of problem 8a, or simply observe that $2x + 2y + z = 2$ is satisfied by each of the three given points, and so this linear equation must be the equation of the plane.
Since z is a function of x and y along this plane, we can parametrize the plane using $\mathbf{r}(x, y) = \langle x, y, 2 - 2x - 2y \rangle$. Choosing limits in x and y that describe the shadow of the triangle in the xy-plane gives $0 \le x \le 1$, $0 \le y \le 1 - x$.

$$\mathbf{n}\, dS = \pm\, \mathbf{r}_x \times \mathbf{r}_y\, dx\, dy = \pm \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 0 & -2 \\ 0 & 1 & -2 \end{vmatrix} dx\, dy = \pm \langle 2, 2, 1 \rangle\, dx\, dy.$$

In order for the positive direction around the triangle to be the one indicated in the statement of the problem, choose $\mathbf{n}\, dS = +\langle 2, 2, 1 \rangle\, dx\, dy$.
The curl of $\mathbf{F}$ is

$$\begin{aligned} \nabla \times \mathbf{F} &= \langle \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \rangle \times \langle x^2 - y + z, x + y^2 - z, -x + y + z^2 \rangle \\ &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x^2 - y + z & x + y^2 - z & -x + y + z^2 \end{vmatrix} = \langle 2, 2, 2 \rangle \end{aligned}$$

.
The circulation equals the surface integral

$$\int_0^1 \int_0^{1-x} \langle 2, 2, 2 \rangle \cdot \langle 2, 2, 1 \rangle\, dy\, dx = \int_0^1 \int_0^{1-x} 9\, dy\, dx = \int_0^1 9(1 - x)\, dx = \frac{9}{2}.$$