No notes, books, electronic devices, or outside materials of any kind.

Read each problem carefully and simplify your answers.

Unless otherwise indicated, supporting work will be required on every problem worth more than 2 points.

You are expected to know the values of all trig functions at multiples of $\pi/4$ and of $\pi/6$.

$$\int \sin^n x\,dx = -\frac{1}{n}\sin^{n-1}x\cos x + \frac{n-1}{n}\int \sin^{n-2}x\,dx$$
$$\int \cos^n x\,dx = \frac{1}{n}\cos^{n-1}x\sin x + \frac{n-1}{n}\int \cos^{n-2}x\,dx$$
$$\int \tan^n x\,dx = \frac{1}{n-1}\tan^{n-1}x - \int \tan^{n-2}x\,dx$$
$$\int \sec^n x\,dx = \frac{1}{n-1}\sec^{n-2}x\tan x + \frac{n-2}{n-1}\int \sec^{n-2}x\,dx \qquad (n \neq 1)$$

1(16 pts). Let T be the interior of the triangle in the xy-plane with vertices $(0,0)$, $(1,-1)$, $(2,1)$. Rewrite the double integral $\iint_T (2x-y)\,dA$ as an iterated integral in the variables $u = x - 2y$ and $v = x + y$, but **do not evaluate**.

2(12 pts). Let $\mathbf{F} = y^2\mathbf{i} + z^2\mathbf{j} + (x^2+z^2)\mathbf{k}$. Find each of the following or state that it does not exist.

a. curl $\mathbf{F}$ b. div $\mathbf{F}$ c. grad(div $\mathbf{F}$)

3(14 pts). Find a potential function or show that none exists.

a. $\langle 2y, 1\rangle$ b. $\langle 2x + 2xy^2, 2x^2y + e^y\rangle$

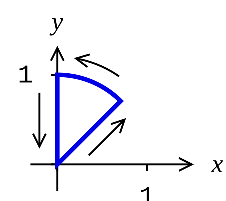
4(32 pts). Evaluate the given line integral.

a. $\int_B 2y\,dx + dy$, where B is the line segment from $(-1,0)$ to $(0,2)$.

b. $\int_C (2x+2xy^2)\,dx + (2x^2y + e^y)\,dy$, where C is parametrized by $\mathbf{r}(t) = \langle t^2+t, 2t^2+t\rangle$ for $0 \le t \le 1$.

c. $\int_B 2y\,ds$, where B is the line segment from $(-1,0)$ to $(0,2)$.

d. $\int_D (2y + e^{x^2})\,dx + (x + \cos\sqrt{y})\,dy$, where D is the closed path consisting of the line segment from $(0,0)$ to $(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}})$, the arc of the unit circle up to $(0,1)$, and the line segment back down to $(0,0)$.

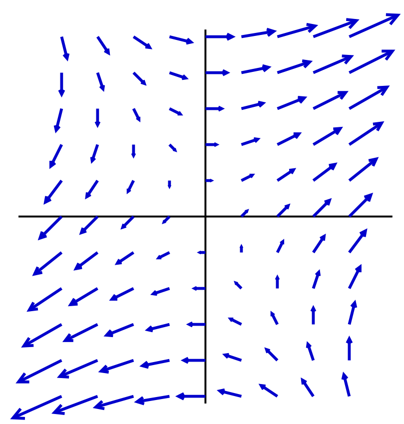


5(12 pts). Let P be the surface parametrized by $\mathbf{r}(u,v) = \langle u+v, uv, u-v\rangle$. Find the plane tangent to P at the point (x,y,z) corresponding to $u=3$, $v=1$. Express the plane **either** parametrically **or** as an equation in x, y, and z.

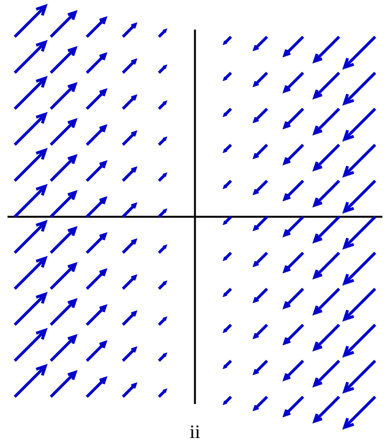
6(10 pts). Find a parametrization of the part of the cone $x^2 = y^2 + z^2$ between $x = 2$ and $x = 3$ for which $y \ge 0$. State the (constant) limits of your parameters necessary to generate this surface exactly one time.

7(4 pts). Find the graph of the given vector field.

$\mathbf{F}(x, y) = \langle 1, 1 \rangle$ $\quad \mathbf{G}(x, y) = \langle y, -x \rangle$ $\quad \mathbf{H}(x, y) = \langle x + y, x \rangle$ $\quad \mathbf{K}(x, y) = \langle y, y \rangle$



i



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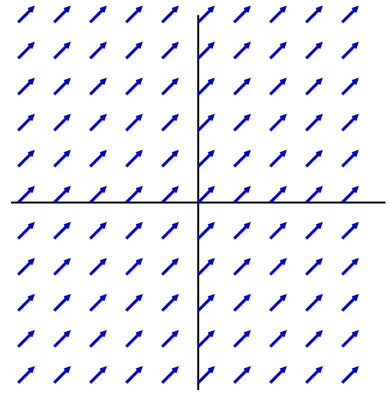
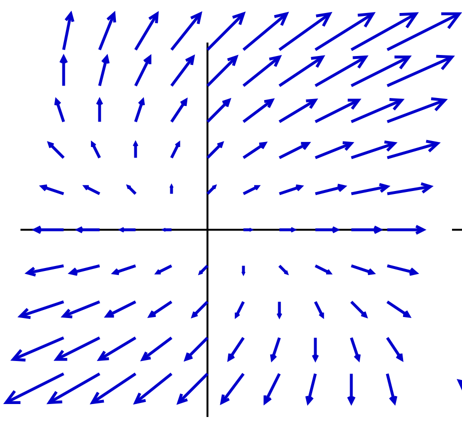
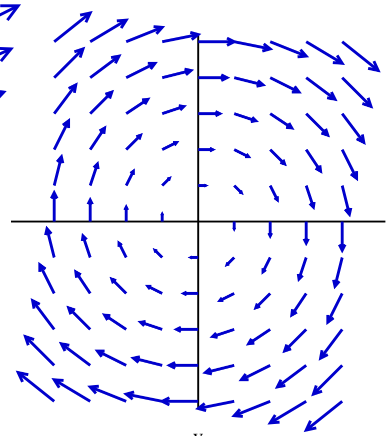


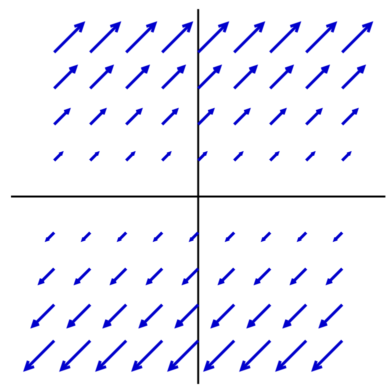
iii



iv



v



vi

1(16 pts).(Source: 15.9.15) Solve for x and y to find

$$x = \frac{1}{3}u + \frac{2}{3}v \qquad y = -\frac{1}{3}u + \frac{1}{3}v$$

Equations of the edges of the triangle are

$$\begin{array}{rrr} x - 2y = 0 & x + y = 0 & 2x - y = 3 \\ u = 0 & v = 0 & u + v = 3 \end{array}$$

The Jacobian is

$$\frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} x_u & x_v \\ y_u & y_v \end{vmatrix} = \begin{vmatrix} 1/3 & 2/3 \\ -1/3 & 1/3 \end{vmatrix} = \frac{1}{3}$$

and the integrand $2x - y = u + v$. In these new variables, the double integral is

$$\int_0^3 \int_0^{3-u} (u+v)\frac{1}{3}\, dv\, du.$$

2.(Source: 16.5.1,12)

grad $= \nabla$	div $= \nabla\cdot$	curl $= \nabla\times$
grad scalar = vector	div vector = scalar	curl vector = vector
grad vector DNE	div scalar DNE	curl scalar DNE

2a.(5 pts) $\operatorname{curl}\mathbf{F} = \langle \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\rangle \times \langle y^2, z^2, x^2+z^2\rangle = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y^2 & z^2 & x^2+z^2 \end{vmatrix} = \langle -2z, -2x, -2y\rangle.$

2b.(4 pts) $\operatorname{div}\mathbf{F} = \langle \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\rangle \cdot \langle y^2, z^2, x^2+z^2\rangle = (y^2)_x + (z^2)_y + (x^2+z^2)_z = 0+0+2z = 2z.$

2c.(3 pts) $\operatorname{grad}(\operatorname{div}\mathbf{F}) = \nabla 2z = \langle (2z)_x, (2z)_y, (2z)_z\rangle = \langle 0, 0, 2\rangle.$

3a(4 pts).(Source: 16.3.3-10) If $f_x = 2y$ and $f_y = 1$, then $f_{xy} = 2$ and $f_{yx} = 0$, a contradiction. Therefore, $\langle 2y, 1\rangle$ has no potential.

3b(10 pts).(Source: 16.3.3-10) $f_x = 2x + 2xy^2 \implies f = x^2 + x^2y^2 + C(y) \implies f_y = 2x^2y + C'(y)$. Set this equal to $2x^2y + e^y$ to obtain $C'(y) = e^y$, and therefore $C = e^y$+any constant. There's a different (correct) answer for every choice of this constant. Choosing the constant 0 gives the potential function $f(x,y) = x^2 + x^2y^2 + e^y$.

4a(8 pts).(Source: 16.2.7) Can parametrize the line segment B with $x = t$, $dx = dt$, $y = 2(t+1)$, $dy = 2\,dt$ for $-1 \le t \le 0$ and the integral becomes

$$\int_{-1}^0 4(t+1)\,dt + 2\,dt = 2\int_{-1}^0 (2t+3)\,dt = 2(t^2+3t)\Big|_{-1}^0 = 4.$$

4b(6 pts)(Source: 16.3.13) Use the potential found in 3b. The curve begins and ends at $\mathbf{r}(0) = (0,0)$ and $\mathbf{r}(1) = (2,3)$, and by the Fundamental Theorem of Calculus for line integrals,

$$\int_C (2x + 2xy^2)\,dx + (2x^2y + e^y)\,dy = (x^2 + x^2y^2 + e^y)\Big|_{(0,0)}^{(2,3)} = 39 + e^3.$$

It's not practical to evaluate the integral by using the given parametrization ($x = t^2 + t$, $dx = (2t+1)\,dt$, $y = 2t^2 + t$, $dy = (4t+1)\,dt$) unless you notice that

$$\begin{aligned}\int \big(2(t^2+t)(2t+1) + 2(t^2+t)(2t^2+t)^2(2t+1) \\ +2(t^2+t)^2(2t^2+t)(4t+1) + e^{2t^2+t}(4t+1)\big)\,dt \\ = (t^2+t)^2 + (t^2+t)^2(2t^2+t)^2 + e^{2t^2+t} + C\end{aligned}$$

4c(8 pts).(Source: 16.2.9) Using the parametrization for B found in a,

$$\begin{aligned}\int_B 2y\,ds = \int_{-1}^{0} 4(t+1)\frac{ds}{dt}\,dt = \int_{-1}^{0} 4(t+1)\sqrt{\tfrac{dx}{dt}^2 + \tfrac{dy}{dt}^2}\,dt \\ = 4\sqrt{5}\int_{-1}^{0}(t+1)\,dt = 4\sqrt{5}(\frac{1}{2}t^2 + t)\Big|_{-1}^{0} = 2\sqrt{5}\end{aligned}$$

4d(10 pts).(Source: 16.4.6,7) Since D is a closed path, we can apply Green's Theorem. Let $\mathcal{D}$ denote the interior of D, and rewrite the line integral:

$$\textstyle\int_D (2y + e^{x^2})\,dx + (x + \cos\sqrt{y})\,dy = \iint_{\mathcal{D}} \big((x + \cos\sqrt{y})_x - (2y + e^{x^2})_y\big)\,dA = \iint_{\mathcal{D}} (-1)\,dA.$$

This equals -1 times the area of the eighth-circle $\mathcal{D}$, or $-\frac{\pi}{8}$

5(12 pts).(Source: 16.6.33) The plane passes through the point $\mathbf{r}(3,1) = \langle 4,3,2\rangle$ and is parallel the vectors

$$\mathbf{r}_u = \langle 1, v, 1\rangle = \langle 1,1,1\rangle \text{ and } \mathbf{r}_v = \langle 1, u, -1\rangle = \langle 1,3,-1\rangle.$$

The plane is given in parametric form by the function

$$\mathbf{b}(u,v) = \langle 4,3,2\rangle + u\langle 1,1,1\rangle + v\langle 1,3,-1\rangle.$$

To write its xyz-equation, find the normal vector by crossing:

$$\mathbf{r}_u \times \mathbf{r}_v = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 1 & 1 \\ 1 & 3 & -1 \end{vmatrix} = \langle -4, 2, 2\rangle$$

Then the tangent plane is the graph of the equation $-4(x-4) + 2(y-3) + 2(z-2) = 0$.

6(10 pts).(Source: 16.6.25) For each x between 2 and 3, (y, z) lie on the circle centered at $(0, 0)$ having radius x. We can use $\pm x \sin\theta$ and $\pm x \cos\theta$ for y and z, but since we want $y \geq 0$, it's simplest to use $y = x \sin\theta$ and $z = x \cos\theta$ so that $y \geq 0$ for $0 \leq \theta \leq \pi$. Altogether, the parametrization is

$$\langle x, x\sin\theta, s\cos\theta\rangle \qquad 2 \leq x \leq 3,\ 0 \leq \theta \leq \pi.$$

7(4 pts).(Source: 16.1.29-32) a. iii. b. v. c. i. d. vi.
Notes: **F** is constant. **G** is orthogonal to the position vector $\langle x, y\rangle$. In c, when $x = 0$, **H** is a multiple of **i**, and when $y = 0$, **H** is a multiple of $\mathbf{i} + \mathbf{j}$. Above [below] the x-axis, **K** is a positive [negative] multiple of $\langle 1, 1\rangle$.

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