No notes, books, electronic devices, or outside materials of any kind.

Read each problem carefully and simplify your answers.

Unless otherwise indicated, supporting work will be required on every problem worth more than 2 points.

You are expected to know the values of all trig functions at multiples of $\pi/4$ and of $\pi/6$.

1(19 pts). Find the locations (x, y) of all local extrema and saddle points of $h(x, y) = x^3 - y^3 - 6xy + 6$ and state which is which.

2(17 pts). A particle is at position $\mathbf{r} = \langle e^t, t\sqrt{2}, e^{-t} \rangle$ at time t.

a. Find the distance travelled by the particle between times $t = -1$ and $t = 1$.

b. Find the particle's velocity, acceleration, and normal and tangential components of acceleration at time $t = 0$. Label your answers so I can tell which is which.

3(23 pts). The curve parametrized by $x = t$, $y = 3t - \frac{t^2+1}{2}$, $z = 4t - t^2$ passes through the point $(1, 2, 3)$. Find the following.

a. $\mathbf{T}$, $\mathbf{N}$, and $\mathbf{B}$ at the point $(1, 2, 3)$. Label your answers.

b. The plane parallel both $\mathbf{T}$ and $\mathbf{N}$ and passing through $(1, 2, 3)$.

4(10 pts). Find the linearization of $p(x, y) = (x^2 - y^2)^{3/2}$ at the point $(x, y) = (\sqrt{3}, \sqrt{2})$.

5. Let $g(x, y, z) = xy - yz + xz$.

a(8 pts). Find the derivative of g in the direction $\frac{1}{\sqrt{2}}\langle 1, -1, 0 \rangle$ at the point $(1, 0, 2)$.

b(3 pts). In what direction does the greatest derivative of g at $(1, 0, 2)$ occur? Express your answer as a unit vector.

c(3 pts). Find the equation of the plane tangent to $xy - yz + xz = 2$ at $(1, 0, 2)$.

d(5 pts). Suppose that x and y are functions of t. Find $\frac{dg}{dt}$ at $t = 0$ if, at that time, $x = 1$, $y = 0$, $z = 2$, $\frac{dx}{dt} = -3$, $\frac{dy}{dt} = -4$, and $\frac{dz}{dt} = 5$.

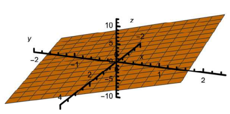
6(8 pts). Choose **one** of the two limits below, and then either evaluate the limit or explain why it does not exist. Clearly indicate which problem you're solving.

a. $\displaystyle \lim_{(x,y)\to(0,0)} \frac{\sqrt{x+y+9} - 3}{x + y}$

b. $\displaystyle \lim_{(x,y)\to(0,0)} \frac{2x^4}{2x^2 + y^2}$

7(4 pts). Find the graph of given function.

a. $\sqrt{4-x^2-y^2}$ b. $\sqrt{x^2+y^2}$ c. $\frac{1}{1+x^2+y^2}$ d. $1-2x+3y$



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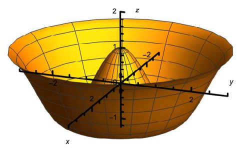

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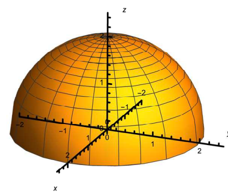

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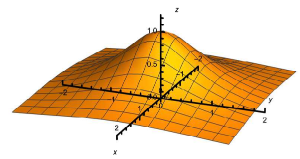

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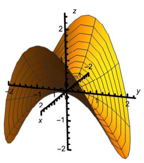

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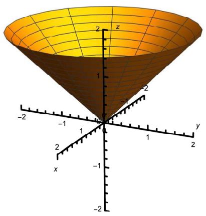

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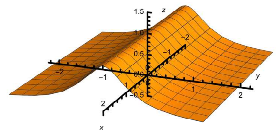

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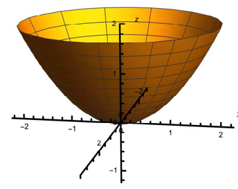

8.

1(19 pts).(Source: 14.7.more.1l) Search for critical points:

$$h_x(x,y) = 3x^2 - 6y = 0 \Rightarrow y = \tfrac{1}{2}x^2.$$
$$h_y(x,y) - 3y^2 - 6x = 0 \Rightarrow x = -\tfrac{1}{2}y^2 = -\tfrac{1}{2}(\tfrac{1}{2}x^2)^2 = -\tfrac{1}{8}x^4$$

Factoring $0 = x^4 + 8x = x(x^3+8)$ yields $x = 0$ and $x = -2$ Use $y = \frac{1}{2}x^2$ to find the accompanying y-values 0 and 2
Now use the Second Derivative Test at the critical points.

$$D = \begin{vmatrix} h_{xx} & h_{xy} \\ h_{xy} & h_{yy} \end{vmatrix} = \begin{vmatrix} 6x & -6 \\ -6 & -6y \end{vmatrix} = 6 \cdot 6 \begin{vmatrix} x & -1 \\ -1 & -y \end{vmatrix} = 36(-xy-1)$$

critical point	D	h_{xx}	conclusion
$(-2,2)$	$36 \cdot 3$	-6	local maximum
$(0,0)$	-36	irrelevant	saddle point

2a(10 pts).(Source: 13.3.3, 13.4.41,42) Velocity is $\mathbf{v} = \frac{d\mathbf{r}}{dt} = \langle e^t, \sqrt{2}, -e^{-t}\rangle$, and arclength is the integral of speed:

$$s = \int_{-1}^{1} |\langle e^t, \sqrt{2}, -e^{-t}\rangle|\, dt = \int_{-1}^{1} \sqrt{e^{2t} + 2 + e^{-2t}}\, dt = \int_{-1}^{1} \sqrt{(e^t + e^{-t})^2}\, dt$$
$$= \int_{-1}^{1} (e^t + e^{-t})\, dt = (e^t - e^{-t})\Big|_{-1}^{1} = (e - e^{-1}) - (e^{-1} - e) = 2e - 2e^{-1}.$$

2b(7 pts). At time $t = 0$, velocity and acceleration are

$$\mathbf{v} = \frac{d\mathbf{r}}{dt} = \langle e^t, \sqrt{2}, -e^{-t}\rangle = \langle 1, \sqrt{2}, -1\rangle$$
$$\mathbf{a} = \frac{d^2\mathbf{r}}{dt^2} = \langle e^t, 0, e^{-t}\rangle = \langle 1, 0, 1\rangle$$

The normal and tangential components of $\mathbf{a}$ are

$$a_T = \frac{\mathbf{v}\cdot\mathbf{a}}{|\mathbf{v}|} = \frac{\langle 1, \sqrt{2}, -1\rangle \cdot \langle 1, 0, 1\rangle}{|\langle 1, \sqrt{2}, -1\rangle|} = 0 \qquad a_N = \sqrt{|\mathbf{a}|^2 - a_T^2} = |\mathbf{a}| = \sqrt{2}$$

3(23 pts).(Source: 13.3.21-23, 47-50) Note that $\mathbf{r}(t) = \langle t, 3t - \frac{t^2+1}{2}, 4t - t^2\rangle = \langle 1, 2, 3\rangle$ at $t = 1$. Compute $\mathbf{v}$ and $\mathbf{a}$ at that time:

$$\mathbf{v} = \frac{d\mathbf{r}}{dt} = \langle 1, 3 - t, 4 - 2t\rangle = \langle 1, 2, 2\rangle$$
$$\mathbf{a} = \frac{d^2\mathbf{r}}{dt^2} = \langle 0, -1, -2\rangle$$

The cross product of these is

$$\mathbf{v}\times\mathbf{a} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 2 & 2 \\ 0 & -1 & -2 \end{vmatrix} = \begin{vmatrix} 2 & 2 \\ -1 & -2 \end{vmatrix}\mathbf{i} - \begin{vmatrix} 1 & 2 \\ 0 & -2 \end{vmatrix}\mathbf{j} + \begin{vmatrix} 1 & 2 \\ 0 & -1 \end{vmatrix}\mathbf{k} = \langle -2, 2, -1\rangle.$$

Now we can answer part a.:

$$\begin{aligned} \mathbf{T} &= \frac{\mathbf{v}}{|\mathbf{v}|} = \tfrac{1}{3}\langle 1,2,2\rangle \\ \mathbf{B} &= \frac{\mathbf{v}\times\mathbf{a}}{|\mathbf{v}\times\mathbf{a}|} = \tfrac{1}{3}\langle -2,2,-1\rangle \\ \mathbf{N} &= \mathbf{B}\times\mathbf{T} = \tfrac{1}{9}\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 2 & 2 \\ -2 & 2 & -1 \end{vmatrix} = \tfrac{1}{9}\langle -6,-3,6\rangle = \tfrac{1}{3}\langle -2,-1,2\rangle \end{aligned}$$

The plane described in part b is the osculating plane. We can use $\mathbf{v}\times\mathbf{a} = \langle -2,2,-1\rangle$ as its normal vector, and so its equation is

$$-2(x-1) + 2(y-2) - (z-3) = 0.$$

4(10 pts).(Source: 14.4.11-16)

$$\begin{aligned} p(x,y) &= (x^2-y^2)^{3/2} & p(\sqrt{3},\sqrt{2}) &= (3-2)^{3/2} = 1 \\ p_x(x,y) &= 3x(x^2-y^2)^{1/2} & p_x(\sqrt{3},\sqrt{2}) &= 3\sqrt{3}(3-2)^{1/2} = 3\sqrt{3} \\ p_y(x,y) &= -3y(x^2-y^2)^{1/2} & p_y(\sqrt{3},\sqrt{2}) &= -3\sqrt{2}(3-2)^{1/2} = -3\sqrt{2} \end{aligned}$$

The linearization of p is

$$\begin{aligned} L(x,y) &= p(\sqrt{3},\sqrt{2}) + p_x(\sqrt{3},\sqrt{2})(x-\sqrt{3}) + p_y(\sqrt{3},\sqrt{2})(y-\sqrt{2}) \\ &= 1 + 3\sqrt{3}(x-\sqrt{3}) - 3\sqrt{2}(y-\sqrt{2}) \end{aligned}$$

5a(8 pts).(Source: 14.6.15) $\nabla g = \langle y+z, x-z, x-y\rangle$, which, at the point $(1,0,2)$ equals $\langle 2,-1,1\rangle$. The directional derivative in the direction $\frac{1}{\sqrt{2}}\langle 1,-1,0\rangle$ is

$$\frac{1}{\sqrt{2}}\langle 1,-1,0\rangle\cdot\nabla g = \frac{1}{\sqrt{2}}\langle 1,-1,0\rangle\cdot\langle 2,-1,1\rangle = \frac{3}{\sqrt{2}}.$$

5b(3 pts).(Source: 14.6.21) The greatest directional derivative of g at $(1,0,2)$ occurs in the direction of the gradient: $\frac{1}{\sqrt{6}}\langle 2,-1,1\rangle$.

5c(3 pts).(Source: 14.6.44) The gradient at $(1,0,2)$ is perpendicular to the level surface $xy - yz + xz = 2$, so the plane tangent to the surface there is given by

$$2(x-1) - y + z - 1 = 0, \text{ or } 2x - y + z = 4.$$

5d(5 pts).(Source: 14.5.13,14) By the chain rule,

$$\frac{dg}{dt} = \frac{\partial g}{\partial x}\frac{dx}{dt} + \frac{\partial g}{\partial y}\frac{dy}{dt} + \frac{\partial g}{\partial z}\frac{dz}{dt} = \langle 2, -1, 1\rangle \cdot \langle -3, -4, 5\rangle = -6 + 4 + 5 = 3$$

6a(8 pts).(Source: 14.2.17) Rationalize the numerator:

$$\lim_{(x,y)\to(0,0)} \frac{\sqrt{x+y+9}-3}{x+y} \cdot \frac{\sqrt{x+y+9}+3}{\sqrt{x+y+9}+3} = \lim_{(x,y)\to(0,0)} \frac{x+y+9-9}{(x+y)(\sqrt{x+y+9}+3)}$$
$$= \lim_{(x,y)\to(0,0)} \frac{1}{\sqrt{x+y+9}+3} = \frac{1}{3+3} = \frac{1}{6}.$$

6b(8 pts).(Source: 14.2.16) After taking the limit along a few different paths to the origin and always getting zero, I suspect that the limit is zero. Since there's no common factor I can cancel, I'll try the squeeze theorem.
Observe that

$$0 \le \frac{2x^2}{2x^2+y^2} \le 1.$$

Multiply by the (positive quantity) x^2 to obtain

$$0 \le \frac{2x^4}{2x^2+y^2} \le x^2.$$

Since $\lim_{(x,y)\to(0,0)} 0$ and $\lim_{(x,y)\to(0,0)} x^2$ both equal 0, the squeeze theorem implies that

$$\lim_{(x,y)\to(0,0)} \frac{2x^4}{2x^2+y^2} = 0.$$

7(4 pts).(Source: 14.1.32) a3, b6, c4, d1.