MATH 221–01 (Kunkle), Exam 4 — Name: ____________________

100 pts, 75 minutes — Apr 18, 2023 —

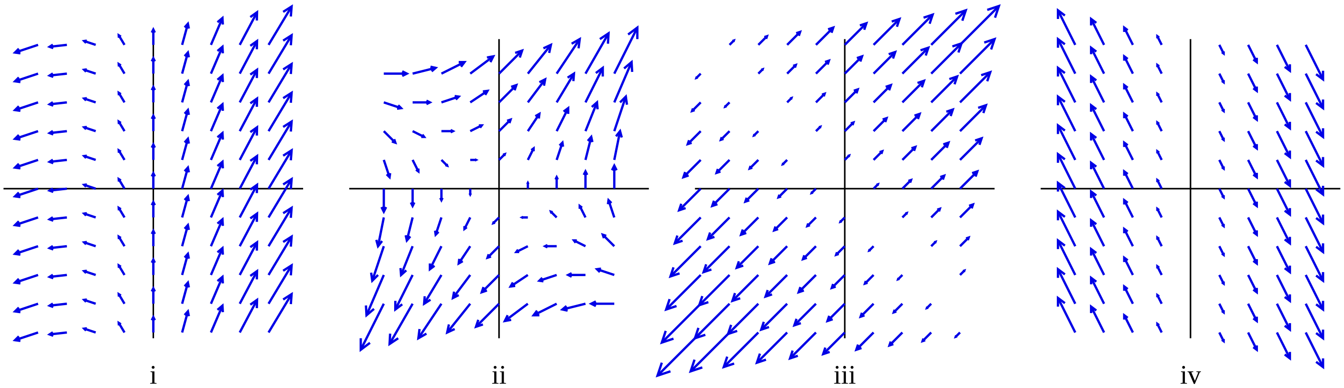
No notes, books, electronic devices, or outside materials of any kind.

Read each problem carefully and simplify your answers.

Supporting work will be required on every problem worth more than 2 points unless instructions say otherwise.

You are expected to know the values of all trig functions at multiples of $\pi/4$ and of $\pi/6$.

1(8 pts). Match each graph of a vector field to its equation.



a. $\langle x+y, x+y\rangle$ b. $\langle x, x+2\rangle$ c. $\langle y, x+y\rangle$ d. $\langle x-y, 0\rangle$ e. $\langle x, -2x\rangle$

2(14 pts). Let C be the curve parametrized by $\langle \sin t, \cos t, 1-t\rangle$ for $0 \le t \le \frac{\pi}{2}$ and valuate the line integral $\int_C xy\, ds$.

3(14 pts). Evaluate the line integral $\int_A (\ln(1+x^2)+y)\,dx + (2x-\ln(2+y))\,dy$ where A is the path from $(1,0)$ to $(-1,0)$ along the parabola $y = x^2-1$ and then from $(-1,0)$ to $(1,0)$ along the x-axis.

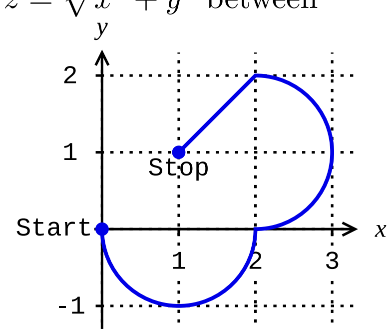
4(12 pts). Let $\mathbf{G} = \langle e^{x+y}, y+z, e^{2x-3z}\rangle$ and find the following, if it exists.

a. div $\mathbf{G}$ b. curl $\mathbf{G}$ c. div (curl $\mathbf{G}$) d. grad(div $\mathbf{G}$)

5(12 pts). Let H be the surface parametrized by $\langle uv, u+v, u-v\rangle$. Find an equation of the plane tangent to H at the point (x,y,z) corresponding to $u=0$ and $v=1$.

6(26 pts). Find the flux of $\mathbf{F} = \langle x, -y, 1\rangle$ across the part of the cone $z = \sqrt{x^2+y^2}$ between the planes $z=1$ and $z=2$, oriented upward.

7(14 pts). Evaluate the line integral $\int_E (e^x + y)\,dx + (x-2y)\,dy$ where E is the path shown in the figure. (Arcs in the figure are semicircles.)



1(8 pts).(Source: 16.1.11-14) Answers: i.b. ii.c. iii.a. iv.e.
Here's one way to arrive at the answers. Rule out d., since it would be horizontal everywhere. In iii., vectors are parallel; looks like $\langle 1, 1\rangle f(x, y) =$ a. In iv., vectors are parallel and independent of y; looks like $\langle 1, -2\rangle f(x) =$ e. i. is vertical when $x = 0$, as in b. ii. is vertical when $y = 0$, as in c.

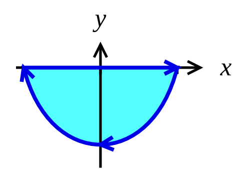
2(14 pts).(Source: 16.2.9) Let $\mathbf{r} = \langle \sin t, \cos t, 1 - t\rangle$. Then

$$ds = \left|\frac{d\mathbf{r}}{dt}\right| dt = |\langle \cos t, -\sin t, -1\rangle|\, dt = \sqrt{\cos^2 t + \sin^2 t + (-1)^2}\, dt = \sqrt{2}\, dt$$

and the integral equals

$$\int_0^{\pi/2} \sin t \cos t \sqrt{2}\, dt = \sqrt{2}\frac{1}{2}\sin^2 t\,\Big|_0^{\pi/2} = \frac{\sqrt{2}}{2}(1-0) = \frac{1}{\sqrt{2}}.$$

3(14 pts).(Source: 16.4.7) By Green's theorem, the line integral around A (in the negative direction) equals the double integral of



$$-(2x - \ln(2+y))_x + (\ln(1+x^2) + y)_y$$

over the region enclosed by A (see figure):

$$\int_{-1}^{1}\int_{x^2-1}^{0} (-2+1)\, dy\, dx = \int_{-1}^{1} (x^2 - 1)\, dx = (\tfrac{1}{3}x^3 - x)\,\Big|_{-1}^{1} = -\tfrac{4}{3}$$

4(12 pts).(Source: 16.5.1-8,12)

grad $= \nabla$	div $= \nabla\cdot$	curl $= \nabla\times$
grad scalar = vector	div scalar DNE	curl scalar DNE
grad vector DNE	div vector = scalar	curl vector = vector

a. $\nabla \cdot \mathbf{G} =$

$$\left\langle \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right\rangle \cdot \langle e^{x+y}, y+z, e^{2x-3z}\rangle = (e^{x+y})_x + (y+z)_y + (e^{2x-3z})_z = e^{x+y} + 1 - 3e^{2x-3z}$$

b. $\nabla \times \mathbf{G} =$

$$\left\langle \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right\rangle \times \langle e^{x+y}, y+z, e^{2x-3z} \rangle = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ e^{x+y} & y+z & e^{2x-3z} \end{vmatrix}$$
$$= \mathbf{i} \begin{vmatrix} \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y+z & e^{2x-3z} \end{vmatrix} - \mathbf{j} \begin{vmatrix} \frac{\partial}{\partial x} & \frac{\partial}{\partial z} \\ e^{x+y} & e^{2x-3z} \end{vmatrix} + \mathbf{k} \begin{vmatrix} \frac{\partial}{\partial x} & \frac{\partial}{\partial y} \\ e^{x+y} & y+z \end{vmatrix}$$
$$= -\mathbf{i} - 2e^{2x-3z}\mathbf{j} - e^{x+y}\mathbf{k}$$
$$= \langle -, -2e^{2x-3z}, -e^{x+y} \rangle$$

c. $\operatorname{div}(\operatorname{curl} \mathbf{G}) = 0$ because $\operatorname{div}(\operatorname{curl} \mathbf{F}) = \nabla \cdot \nabla \times \mathbf{F} = 0$ for *any* twice-continuously differentiable vector field $\mathbf{F}$.

d. $\nabla(e^{x+y} + 1 - 3e^{2x-3z})$

$$= \langle (e^{x+y} + 1 - 3e^{2x-3z})_x, (e^{x+y} + 1 - 3e^{2x-3z})_y, (e^{x+y} + 1 - 3e^{2x-3z})_z \rangle$$
$$= \langle e^{x+y} - 6e^{2x-3z}, e^{x+y}, 9e^{2x-3z} \rangle$$

5(12 pts).(Source: 16.6.33-34) The point of tangency is value of $\mathbf{r} = \langle uv, u+v, u-v \rangle$ at $u = 0$ and $v = 1$, or $\langle 0, 1, -1 \rangle$.

The normal vector is $\mathbf{r}_u \times \mathbf{r}_v = \langle v, 1, 1 \rangle \times \langle u, 1 - 1 \rangle$ which equals, at $u = 0$ and $v = 1$,

$$\langle 1, 1, 1 \rangle \times \langle 0, 1 - 1 \rangle = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 1 & 1 \\ 0 & 1 & -1 \end{vmatrix}$$
$$= \mathbf{i} \begin{vmatrix} 1 & 1 \\ 1 & -1 \end{vmatrix} - \mathbf{j} \begin{vmatrix} 1 & 1 \\ 0 & -1 \end{vmatrix} + \mathbf{k} \begin{vmatrix} 1 & 1 \\ 0 & 1 \end{vmatrix}$$
$$= -2\mathbf{i} + 1\mathbf{j} + 1\mathbf{k}.$$

The equation of the plane is $-2(x - 0) + (y - 1) + (z + 1) = 0$, or $-2x + y + z = 0$.

Alternate solution. Since the plane passes through $(0, 1, -1)$ and is parallel to $\langle 1, 1, 1 \rangle$ and $\langle 0, 1 - 1 \rangle$, you could instead represent the plane parametrically by

$$\rho(s, t) = \langle 0, 1, -1 \rangle + s\langle 1, 1, 1 \rangle + t\langle 0, 1 - 1 \rangle$$
$$= \langle s, 1 + s + t, -1 + s - t \rangle$$

6(26 pts).(Source: 16.7.24) Parametrize the cone by $\mathbf{r} = \langle r\cos\theta, r\sin\theta, r \rangle$ with $0 \leq \theta \leq 2\pi$ and $1 \leq r \leq 2$. Using this, $\mathbf{n}\, dS = \pm(\mathbf{r}_r \times \mathbf{r}_\theta)\, dr\, d\theta$

$$= \pm \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \cos\theta & \sin\theta & 1 \\ -r\sin\theta & r\cos\theta & 0 \end{vmatrix} dr\, d\theta$$
$$= \pm \langle -r\cos\theta, -r\sin\theta, r \rangle\, dr\, d\theta$$

Since the third component r is positive in our parametrization, use $+$ above so that the $\mathbf{n}$ is oriented upward. Then the flux is

$$\begin{aligned}
&\int_1^2 \int_0^{2\pi} \langle r\cos\theta, -r\sin\theta, 1\rangle \cdot \langle -r\cos\theta, -r\sin\theta, r\rangle \, d\theta\, dr \\
&= \int_1^2 \int_0^{2\pi} (-r^2\cos^2\theta + r^2\sin^2\theta + r)\, d\theta\, dr \\
&= \int_1^2 \int_0^{2\pi} \big(r - r^2\cos(2\theta)\big)\, d\theta\, dr \\
&= \int_1^2 \left(r\theta - r^2\frac{1}{2}\sin(2\theta)\right)\Big|_0^{2\pi} \, dr \\
&= \int_1^2 2\pi r\, dr = \pi r^2 \Big|_1^2 = 3\pi
\end{aligned}$$

7(14 pts).(Source: 16.3.19-20) $\langle e^x + y, x - 2y\rangle$ is conservative on $\mathbb{R}^2$, since $(x-2y)_x = 1 = (e^x + y)_y$, so we can use the Fundamental Theorem to evaluate the line integral. To find a potential for this vector field, begin by integrating f_x.

$$f_x = e^x + y \quad \Longrightarrow \quad f = e^x + xy + c(y).$$

Then differentiate with respect to y, and set the result equal to $x - 2y$:

$$f_y = x + c'(y) = x - 2y \quad \Longrightarrow \quad c'(y) = -2y$$

Therefore, $c(y) = -y^2 + K$ for some constant K. Since the value of this constant won't affect the value of the integral, we can take $K = 0$ and use the potential function

$$f = e^x + xy - y^2.$$

Now evaluate the integral with the Fundamental Theorem for line integrals:

$$\int_E \nabla f \cdot d\mathbf{r} = f\Big|_{(0,0)}^{(1,1)} = \big(e^x + xy - y^2\big)\Big|_{(0,0)}^{(1,1)} = e - 1.$$

Alternate solution. Because $(x-2y)_x = 1 = (e^x+y)_y$, this integral is path-independent, so we are free to choose the simpler straight-line path $\langle t, t\rangle$ for $0 \le t \le 1$. The integral along this path is

$$\int_0^1 (e^t + t)\, dt + (t - 2t)\, dt = \int_0^1 e^t\, dt = e^t \Big|_0^1 = e - 1.$$