No notes, books, electronic devices, or outside materials of any kind.

Read each problem carefully and simplify your answers.

Supporting work will be required on every problem worth more than 2 points unless instructions say otherwise.

You are expected to know the values of all trig functions at multiples of $\pi/4$ and of $\pi/6$.

1(12 pts). Choose **two** of the three limits below, and then either evaluate the limit or explain why it does not exist. Clearly indicate which two problems you're solving.

a. $\lim_{(x,y)\to(0,0)} \frac{x^4-4y^2}{x^2+2y^2}$ b. $\lim_{(x,y)\to(0,0)} \frac{x^4-9y^4}{x^2+3y^2}$ c. $\lim_{(x,y)\to(0,0)} \frac{x^2y^2}{x^2+y^2}$

2(17 pts). Solve **one** of the following problems. Clearly indicate which one you're solving.

a. The function $\frac{1}{2}x^2 + \frac{1}{2}y^2 - z$ has a minimum but not a maximum on the intersection of the surfaces $x + z = 2$, $x + y = 5$. Find that minimum and the point where it occurs.

b. The function $x-y-z$ attains a maximum but not a minimum on the surface $z = x^2+y^2$. Find that maximum and the point where it occurs.

3(21 pts). Find the points in the xy-plane where $g(x, y) = x^2 + y^3 - 2xy$ has a local max, a local min, or a saddle point (and which occurs at which point).

4(11 pts). Find h_x, h_y, h_{xx}, h_{xy}, and h_{yy} if $h(x, y) = (x^2 + y^2)e^{-x}$.

5. Suppose $p(2, -1) = 5$ and $p_x(2, -1) = 4$ and $p_y(2, -1) = 3$.

a(6 pts). Find the equation of the plane tangent to the graph of $p(x, y)$ at $(x, y) = (2, -1)$.

b(3 pts). Use linear approximation to estimate $p(2.1, -0.99)$.

c(11 pts). If $x = \frac{t+3}{t+1}$ and $y = \cos(\pi t) - \sin(\pi t)$, find $\frac{dp}{dt}$ at $t = 1$.

6. Let $w(x, y, z) = zy + z^3 + 6x$.

a(11 pts). Find the equation of the plane tangent to the surface $w(x, y, z) = 8$ at the point $(3, -9, 2)$.

b(4 pts). Find the derivative of w in the direction $\langle \frac{1}{3}, -\frac{2}{3}, \frac{2}{3} \rangle$ at $(3, -9, 2)$.

c(4 pts). In which (unit vector) direction does w have the greatest directional derivative at $(3, -9, 2)$? What is the value of the derivative in that direction?

1a(6 pts).(Source: 14.2.9) Along the path $y = 0$, $\frac{x^4-4y^2}{x^2+2y^2} = \frac{x^4}{x^2} = x^2 \to 0$ as $(x, y) \to (0, 0)$. But along the path $x = 0$, $\frac{x^4-4y^2}{x^2+2y^2} = \frac{-4y^2}{2y^2} = -2 \to -2$. Since these two limits disagree, $\lim_{(x,y)\to(0,0)} \frac{x^4-4y^2}{x^2+2y^2}$ does not exist.

1b(6 pts).(Source: 14.2.14) The top is a difference of squares, so we can take the limit after canceling the common factor:

$$\lim_{(x,y)\to(0,0)} \frac{x^4 - 9y^4}{x^2 + 3y^2} = \lim_{(x,y)\to(0,0)} \frac{(x^2 - 3y^2)(x^2 + 3y^2)}{x^2 + 3y^2} = \lim_{(x,y)\to(0,0)} (x^2 - 3y^2) = 0.$$

1c(6 pts).(Source: 14.2.16) Observe that $0 \le x^2 \le x^2 + y^2$ for all x and y. Then divide all three sides by $x^2 + y^2$ and multiply by y^2 to obtain

$$0 \le \frac{x^2y^2}{x^2 + y^2} \le y^2.$$

Since 0 and y^2 both $\to 0$ as $(x, y) \to (0, 0)$, the squeeze theorem implies $\lim_{(x,y)\to(0,0)} \frac{x^2y^2}{x^2+y^2}$ must also equal 0.

2a(17 pts).(Source: 14.8.17) $f(x, y, z) = \frac{1}{2}x^2+\frac{1}{2}y^2-z$ can attain its a minimum only at a point on $g(x, y, z) = x+z = 2$, $h(x, y, z) = x+y = 5$ where the triple product $\nabla f\cdot(\nabla g\times\nabla h) = 0$:

$$\begin{vmatrix} x & y & -1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{vmatrix} = x \begin{vmatrix} 0 & 1 \\ 1 & 0 \end{vmatrix} - y \begin{vmatrix} 1 & 1 \\ 1 & 0 \end{vmatrix} + 1 \begin{vmatrix} 1 & 0 \\ 1 & 1 \end{vmatrix} = -x + y - 1 = 0$$

The minimum must occur at a solution to the system

$$\begin{aligned} -x + y &= 1 \\ x + y &= 5 \\ x + z &= 2 \end{aligned} \quad\implies\quad \begin{aligned} x &= 2 \\ y &= 3 \\ z &= 0 \end{aligned}$$

Since this system has only one solution, the minimum of f must occur at the point $(2, 3, 0)$, where $f = \frac{1}{2} \cdot 4 + \frac{1}{2} \cdot 9 - 0 = \frac{13}{2}$.

2b.(Source: 14.8.7) $f(x, y, z) = x - y - z$ can attain its maximum only at a point on $g(x, y, z) = x^2 + y^2 - z = 0$ where $\nabla f \times \nabla g = \mathbf{0}$:

$$\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & -1 & -1 \\ 2x & 2y & -1 \end{vmatrix} = \mathbf{i} \begin{vmatrix} -1 & -1 \\ 2y & -1 \end{vmatrix} - \mathbf{j} \begin{vmatrix} 1 & -1 \\ 2x & -1 \end{vmatrix} + \mathbf{k} \begin{vmatrix} 1 & -1 \\ 2x & 2y \end{vmatrix} = \langle 1 + 2y, 1 - 2x, 2y + 2x\rangle = \mathbf{0}$$

The minimum must occur at a solution to the system

$$\begin{aligned} 1+2y &= 0 \\ 1-2x &= 0 \\ 2y+2x &= 0 \\ x^2+y^2 &= z \end{aligned} \quad \Longrightarrow \quad \begin{aligned} x &= \tfrac{1}{2} \\ y &= -\tfrac{1}{2} \\ z &= \tfrac{1}{2} \end{aligned}$$

Since this system has only one solution, the maximum of f must occur at the point $(\frac{1}{2}, -\frac{1}{2}, \frac{1}{2})$, where $f = \frac{1}{2} - (-\frac{1}{2}) - \frac{1}{2} = \frac{1}{2}$.

3(21 pts).(Source: 14.7.9) Search for critical points:

$$\begin{aligned} g_x(x,y) &= 2x - 2y = 0 \;\Rightarrow\; y = x. \\ g_y(x,y) &= 3y^2 - 2x = 0 \;\Rightarrow\; 3x^2 - 2x = 0 \end{aligned}$$

Factoring $0 = 3x^2 - 2x = x(3x-2)$ yields $x = 0$ and $x = \frac{2}{3}$. Since $y = x$, the accompanying y-values are 0 and $\frac{2}{3}$
Now use the Second Derivative Test at the critical points.

$$D = \begin{vmatrix} g_{xx} & g_{xy} \\ g_{xy} & g_{yy} \end{vmatrix} = \begin{vmatrix} 2 & -2 \\ -2 & 6y \end{vmatrix} = 4 \begin{vmatrix} 1 & -1 \\ -1 & 3y \end{vmatrix} = 4(3y-1)$$

critical point	D	g_{xx}	conclusion
$(0,0)$	-4	irrelevant	saddle point
$(\frac{2}{3}, \frac{2}{3})$	4	2	local minimum

4(11 pts).(Source: 14.3.17,53-58)

$$h = (x^2+y^2)e^{-x}$$

$$\begin{aligned} h_x &= 2xe^{-x} + (x^2+y^2)e^{-x}(-1) = (2x - x^2 - y^2)e^{-x} \\ h_y &= 2ye^{-x} \end{aligned}$$

$$\begin{aligned} h_{xx} &= (2-2x)e^{-x} + (2x - x^2 - y^2)e^{-x}(-1) = (2 - 4x + x^2 + y^2)e^{-x} \\ h_{xy} &= -2ye^{-x} \\ h_{yy} &= 2e^{-x} \end{aligned}$$

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5a(6 pts).(Source: 14.4.1-6) The linearization of p at $(2,-1)$ is the function $L(x,y) = p(2,-1) + p_x(2,-1)(x-2) + p_y(2,-1)(y+1)$, and the equation of the tangent plane is $z =$ this function, or $z = 5 + 4(x-2) + 3(y+1)$.

5b(3 pts).(Source: 14.4.19) Linear approximation says that $p(x,y) \approx L(x,y)$, for (x,y) near $(2,-1)$. Therefore $p(2.1,-0.99) \approx 5 + 4(2.1-2) + 3(-0.99+1) = 5.43$.

5c(11 pts).(Source: 14.5.13-14, 14.5.2) $\dfrac{dx}{dt} = \dfrac{(t+1)-(t+3)}{(t+1)^2} = \dfrac{-2}{(t+1)^2}$, and $\dfrac{dy}{dt} = -\pi\sin(\pi t) - \pi\cos(\pi t)$. At $t = 1$, $(x,y) = (2,-1)$, $\dfrac{dx}{dt} = -\dfrac{1}{2}$, and $\dfrac{dy}{dt} = \pi$. Therefore, by the chain rule,

$$\frac{dp}{dt} = p_x(2,-1)x_t + p_y(2,-1)y_t = 4 \cdot (\tfrac{-1}{2}) + 3\pi = -2 + 3\pi.$$

6a(11 pts).(Source: 14.6.44) $\nabla w = \langle w_x, w_y, w_z\rangle = \langle 6, z, y+3z^2\rangle$. At $(3,-9,2)$, $\nabla w = \langle 6,2,3\rangle$. The tangent plane passes through the point $(3,-9,2)$ and is perpendicular to $\langle 6,2,3\rangle$, so it's given by the equation

$$6(x-3) + 2(y+9) + 3(z-2) = 0,$$

or, $6x + 2y + 3z = 6$.

6b(4 pts).(Source: 14.6.15) The derivative of w in the direction $\langle \frac{1}{3}, -\frac{2}{3}, \frac{2}{3}\rangle$ at $(3,-9,2)$ is

$$\nabla w(3,-9,2) \cdot \langle \frac{1}{3}, -\frac{2}{3}, \frac{2}{3}\rangle = \langle 6,2,3\rangle \cdot \langle \frac{1}{3}, -\frac{2}{3}, \frac{2}{3}\rangle = 2 - \frac{4}{3} + 2 = \frac{8}{3}.$$

6c(4 pts).(Source: 14.6.24-26) $|\nabla w| = \sqrt{6^2 + 2^2 + 3^2} = \sqrt{49} = 7$. The derivative of w is greatest in the direction $\frac{1}{|\nabla w|}\nabla w = \frac{1}{7}\langle 6,2,3\rangle$. In the direction, the derivative is $|\nabla w| = 7$.