MATH 221–01 (Kunkle), Exam 1 | Name: ____________________
100 pts, 75 minutes | Feb 7, 2023 |

No notes, books, electronic devices, or outside materials of any kind.

Read each problem carefully and simplify your answers.

Unless otherwise indicated, supporting work will be required on every problem worth more than 2 points.

You are expected to know the values of all trig functions at multiples of $\pi/4$ and of $\pi/6$.

1a(8 pts). Sketch the graph of $x^2 - y^2 - z^2 = 8$ on the axes provided.
Label the axes x, y, z however you like, but indicate the positive direction in each variable so that the x-, y-, and z-axes form a right-handed system.

1b(7 pts). How does the graph of $x^2 - 6x - y^2 - 4y - z^2 = 3$ compare to the graph above? Specifically, what can you do the surface $x^2 - y^2 - z^2 = 8$ to obtain the graph of $x^2 - 6x - y^2 - 4y - z^2 = 3$?
Please refer to the positive/negative x-, y-, or z-directions rather than use words like "forward, backward, left, right, ..."

2. Find the following, if they exist.
When asked for a point, give its coordinates; for line, give a parametrization; for a plane or sphere, give an equation.

a(5 pts). The sphere centered at $(-2, 1, 3)$ and tangent to the yz-plane.

b(5 pts). The line of intersection of the planes $2x - y + z = 0$, $x + y = 1$.

c(7 pts). The plane containing the point $(1, 2, 0)$ and the line $\langle 1-t, 1+3t, -2t\rangle$.

d(9 pts). The points of intersection of the curve $\langle 2t, \sin(\pi t), \cos(\pi t)\rangle$ and the surface $x^2 - y^2 - z^2 = 8$.

e(3 pts). The line tangent to the curve parametrized by $\mathbf{r}$(t) at the point corresponding to $t = 1$, if

$$\mathbf{r}(1) = \langle 1, 2, 0\rangle$$
$$\mathbf{r}'(1) = \langle 0, 3, 4\rangle$$

3(29 pts). Find the following at $t = -1$ if $\mathbf{r}(t) = \langle t^2, t^2, \frac{1}{3}t^3\rangle$.

a. $\frac{d\mathbf{r}}{dt}$ and $\frac{d^2\mathbf{r}}{dt^2}$ b. $\mathbf{T}$, $\mathbf{N}$, $\mathbf{B}$ c. κ d. a_T and a_N

4(16 pts). Let $\mathbf{u} = \langle 1, -2, 2\rangle$ and $\mathbf{v} = \langle 3, 6, 2\rangle$. Find the following, if they exist.

a. $2\mathbf{u} - \mathbf{v}$

b. $|\mathbf{u}|$

c. a unit vector parallel to $\mathbf{u}$

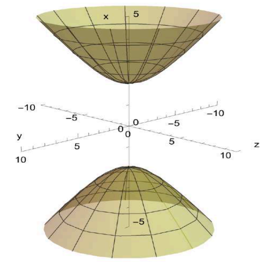
d. the cosine of the angle between $\mathbf{u}$ and $\mathbf{v}$.

e. the vector projection of $\mathbf{u}$ onto $\mathbf{v}$.

5(11 pts). Find the length of the curve parametrized by $\mathbf{r}(t) = \langle t^2, t^2, \frac{1}{3}t^3\rangle$ for $0 \le t \le 1$.

1a(8 pts).(Source: 12.6.14,18) It helps to consider the cross-sections of this surface along planes where one variable is constant. When y or z is a constant, the cross-section is a hyperbola that opens in the $\pm x$ direction. When x is a constant, the cross section is a circle if $x^2 \geq 8$ and does not exist otherwise.

The graph is a hyperbola of two sheets. Here's a nice drawing by Mathematica. The positive x, y, and z directions in this figure are up, down-and-left, and down-and-right. Because x can't equal zero in the equation $x^2 - y^2 - z^2 = 8$, the surface misses the yz-plane.



1b.(Source: 12.6.36,12.1.17-20) Rewrite the equation in 1b by completing the square.

$$x^2 - 6x - y^2 - 4y - z^2 = 3$$
$$x^2 - 6x + 9 - y^2 - 4y - 4 - z^2 = 3 + 9 - 4$$
$$(x-3)^2 - (y+2)^2 - z^2 = 8$$

This graph is obtained by shifting the graph in 1a 3 units in the positive x direction and 2 units in the negative y direction.

2a.(Source: 12.1.23) The radius of the sphere is the distance from the point $(-2, 1, 3)$ to the yz-plane, or 2, so the equation is $(x+2)^2 + (y-1)^2 + (z-3)^2 = 2^2$.

2b.(Source: 12.5.12) We need a point on the line and vector parallel the line. You can find two points on the line by picking an arbitrary value for x and solving the two given equations for y and z. In this way, I found $(0, 1, 1)$ and $(1, 0, -2)$. The vector between these points, $\langle 1, -1, -3 \rangle$, is parallel to the line. Using this vector and the first point, parametrize the line by $\langle t, 1-t, 1-3t \rangle$.

2c.(Source: 12.5.35) We need a point on the plane and vector orthogonal to the plane. To find the vector, observe that the line is parallel $\langle -1, 3, -2 \rangle$ and passes through the point $(1, 1, 0)$. The plane is parallel both $\langle -1, 3, -2 \rangle$ and the vector between our two points, $\langle 0, 1, 0 \rangle$. Therefore, their cross-product

$$\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -1 & 3 & -2 \\ 0 & 1 & 0 \end{vmatrix} = \mathbf{i} \begin{vmatrix} 3 & -2 \\ 1 & 0 \end{vmatrix} - \mathbf{j} \begin{vmatrix} -1 & -2 \\ 0 & 0 \end{vmatrix} + \mathbf{k} \begin{vmatrix} -1 & 3 \\ 0 & 1 \end{vmatrix} = \langle 2, 0, -1 \rangle$$

is orthogonal to the plane. Using this vector and the point $(1, 1, 0)$, obtain the equation $2(x-1) - z = 0$.

2d.(Source: 13.1.32) Substitute $x = 2t$, $y = \sin(\pi t)$, and $z = \cos(\pi t)$ into $x^2 - y^2 - z^2 = 8$ and solve for t:

$$4t^2 - \sin^2(\pi t) - \cos^2(\pi t) = 8 \implies 4t^2 - 1 = 8 \implies t^2 = \frac{9}{4} \implies t = \pm\frac{3}{2}$$

Find the points of intersection by evaluating the parametrization at these t-values. At $t = \frac{3}{2}$, the point is $\left(2 \cdot \frac{3}{2}, \sin(\frac{3}{2}\pi), \sin(\frac{3}{2}\pi)\right) = (3, -1, 0)$. At $t = -\frac{3}{2}$, the point is $(-3, 1, 0)$.

2e.(Source: 13.2.23-26) The line passing through the point $(1, 2, 0)$ and parallel to $\langle 0, 3, 4\rangle$ can be parametrized by $\langle 1, 2 + 3t, 4t\rangle$.

3a(4 pts).(Source: 13.4.9)

$$\frac{d\mathbf{r}}{dt} = \mathbf{r}' = \langle 2t, 2t, t^2\rangle = \langle -2, -2, 1\rangle \text{ at } t = -1$$
$$\frac{d^2\mathbf{r}}{dt^2} = \mathbf{r}'' = \langle 2, 2, 2t\rangle = \langle 2, 2, -2\rangle \text{ at } t = -1$$

3b(13 pts).(Source: 13.3.47) $\mathbf{T} = \frac{1}{|\mathbf{r}'|}\mathbf{r}' = \frac{1}{3}\langle -2, -2, 1\rangle$. To find $\mathbf{B}$, normalize

$$\mathbf{r}' \times \mathbf{r}'' = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -2 & -2 & 1 \\ 2 & 2 & -2 \end{vmatrix} = \mathbf{i}\begin{vmatrix} -2 & 1 \\ 2 & -2 \end{vmatrix} - \mathbf{j}\begin{vmatrix} -2 & 1 \\ 2 & -2 \end{vmatrix} + \mathbf{k}\begin{vmatrix} -2 & -2 \\ 2 & 2 \end{vmatrix} = \langle 2, -2, 0\rangle,$$

(or, more simply, normalize $\langle 1, -1, 0\rangle$) to obtain $\mathbf{B} = \frac{1}{\sqrt{2}}\langle 1, -1, 0\rangle$. Finally,

$$\mathbf{N} = \mathbf{B} \times \mathbf{T} = \frac{1}{3\sqrt{2}}\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & -1 & 0 \\ -2 & -2 & 1 \end{vmatrix} = \frac{1}{3\sqrt{2}}\left(\mathbf{i}\begin{vmatrix} -1 & 0 \\ -2 & 1 \end{vmatrix} - \mathbf{j}\begin{vmatrix} 1 & 0 \\ -2 & 1 \end{vmatrix} + \mathbf{k}\begin{vmatrix} 1 & -1 \\ -2 & -2 \end{vmatrix}\right)$$
$$= \frac{1}{3\sqrt{2}}\langle -1, -1, -4\rangle$$

3c(4 pts).(Source: 13.3.21-23) $\kappa = \dfrac{|\mathbf{r}' \times \mathbf{r}''|}{|\mathbf{r}'|^3} = \dfrac{2\sqrt{2}}{3^3}$

3d(8 pts).(Source: 13.4.41,42) If θ is the angle between $\mathbf{r}'$ and $\mathbf{r}''$, then

$$a_T = |\mathbf{r}''|\cos\theta = \frac{\mathbf{r}' \cdot \mathbf{r}''}{|\mathbf{r}'|} = \frac{-10}{3} \qquad a_N = |\mathbf{r}''|\sin\theta = \frac{|\mathbf{r}' \times \mathbf{r}''|}{|\mathbf{r}'|} = \frac{2\sqrt{2}}{3}$$

4a(2 pts).(Source: 12.2.22) $2\mathbf{u} - \mathbf{v} = \langle 2, -4, 4\rangle - \langle 3, 6, 2\rangle = \langle -1, -10, 2\rangle$.

4b(2 pts).(Source: 12.2.22) $|\mathbf{u}| = \sqrt{1^2 + (-2)^2 + 2^2} = \sqrt{9} = 3$.

4c(2 pts).(Source: 12.2.23-25) $\frac{1}{|\mathbf{u}|}\mathbf{u} = \frac{1}{3}\langle 1, -2, 2\rangle$, or $\langle \frac{1}{3}, -\frac{2}{3}, \frac{2}{3}\rangle$.

4d(6 pts).(Source: 12.3.17-18) $|\mathbf{v}| = \sqrt{9 + 36 + 4} = 7$, and $\cos\theta = \frac{\mathbf{u}\cdot\mathbf{v}}{|\mathbf{u}||\mathbf{v}|} = \frac{3-12+4}{3\cdot 7} = \frac{-5}{21}$.

4e(4 pts).(Source: 12.3.41-42) $\text{proj}_{\mathbf{v}}\, \mathbf{u} = \frac{\mathbf{v}\cdot\mathbf{u}}{\mathbf{v}\cdot\mathbf{v}}\mathbf{v} = \frac{-5}{49}\langle 3, 6, 2\rangle$.

5(11 pts).(Source: 13.3.5) Arclength $s = \int \frac{ds}{dt}\,dt = \int \left|\frac{d\mathbf{r}}{dt}\right|\,dt =$

$$\int_0^1 \sqrt{4t^2 + 4t^2 + t^4}\,dt = \int_0^1 \sqrt{8t^2 + t^4}\,dt$$
$$= \int_0^1 \sqrt{(t^2)(8+t^2)}\,dt = \int_0^1 t\sqrt{(8+t^2)}\,dt$$

To integrate, substitute $u = 8 + t^2$, so that $du = 2t\,dt$, or $\frac{1}{2}\,du = t\,dt$. New limits are $u = 8$ (when $t = 0$) and $u = 8$ (when $t = 1$). The integral becomes

$$\int_8^9 \frac{1}{2}u^{1/2}\,du = \frac{1}{3}u^{3/2}\Big|_8^9 = \frac{1}{3}(9^{3/2} - 8^{3/2}).$$