1a (8 pts). Sketch the graph of \( x^2 - y^2 - z^2 = 8 \) on the axes provided. Label the axes \( x, y, z \) however you like, but indicate the positive direction in each variable so that the \( x-, y-, z- \) axes form a right-handed system.

1b (7 pts). How does the graph of \( x^2 - 6x - y^2 - 4y - z^2 = 3 \) compare to the graph above? Specifically, what can you do the surface \( x^2 - y^2 - z^2 = 8 \) to obtain the graph of \( x^2 - 6x - y^2 - 4y - z^2 = 3 \)? Please refer to the positive/negative \( x-, y-, z- \) directions rather than use words like “forward, backward, left, right, . . .”

2. Find the following, if they exist.
When asked for a point, give its coordinates; for line, give a parametrization; for a plane or sphere, give an equation.

   a (5 pts). The sphere centered at \((-2, 1, 3)\) and tangent to the \( yz \)-plane.
   b (5 pts). The line of intersection of the planes \( 2x - y + z = 0, x + y = 1 \).
   c (7 pts). The plane containing the point \((1, 2, 0)\) and the line \((1-t, 1+3t, -2t)\).
   d (9 pts). The points of intersection of the curve \((2t, \sin(\pi t), \cos(\pi t))\) and the surface \( x^2 - y^2 - z^2 = 8 \).
   e (3 pts). The line tangent to the curve parametrized by \( \mathbf{r}(t) \) at the point corresponding to \( t = 1 \), if

\[
\begin{align*}
\mathbf{r}(1) &= \langle 1, 2, 0 \rangle \\
\mathbf{r}'(1) &= \langle 0, 3, 4 \rangle
\end{align*}
\]

3 (29 pts). Find the following at \( t = -1 \) if \( \mathbf{r}(t) = \langle t^2, t^2, \frac{1}{3}t^3 \rangle \).

   a. \( \frac{d\mathbf{r}}{dt} \) and \( \frac{d^2\mathbf{r}}{dt^2} \)  
   b. \( \mathbf{T}, \mathbf{N}, \mathbf{B} \)  
   c. \( \kappa \)  
   d. \( a_T \) and \( a_N \)

4 (16 pts). Let \( \mathbf{u} = \langle 1, -2, 2 \rangle \) and \( \mathbf{v} = \langle 3, 6, 2 \rangle \). Find the following, if they exist.

   a. \( 2\mathbf{u} - \mathbf{v} \)  
   b. \( |\mathbf{u}| \)  
   c. a unit vector parallel to \( \mathbf{u} \)  
   d. the cosine of the angle between \( \mathbf{u} \) and \( \mathbf{v} \).  
   e. the vector projection of \( \mathbf{u} \) onto \( \mathbf{v} \).

5 (11 pts). Find the length of the curve parametrized by \( \mathbf{r}(t) = \langle t^2, t^2, \frac{1}{3}t^3 \rangle \) for \( 0 \leq t \leq 1 \).
1a (8 pts). (Source: 12.6.14,18) It helps to consider the cross-sections of this surface along planes where one variable is constant. When \( y \) or \( z \) is a constant, the cross-section is a hyperbola that opens in the \( \pm x \) direction. When \( x \) is a constant, the cross section is a circle if \( x^2 \geq 8 \) and does not exist otherwise.

The graph is a hyperbola of two sheets. Here’s a nice drawing by Mathematica. The positive \( x \), \( y \), and \( z \) directions in this figure are up, down-and-left, and down-and-right. Because \( x \) can’t equal zero in the equation \( x^2 - y^2 - z^2 = 8 \), the surface misses the \( yz \)-plane.

1b. (Source: 12.6.36,12.1.17-20) Rewrite the equation in 1b by completing the square.

\[
\begin{align*}
x^2 - 6x - y^2 - 4y - z^2 &= 3 \\
x^2 - 6x + 9 - y^2 - 4y - 4 - z^2 &= 3 + 9 - 4 \\
(x - 3)^2 - (y + 2)^2 - z^2 &= 8
\end{align*}
\]

This graph is obtained by shifting the graph in 1a 3 units in the positive \( x \) direction and 2 units in the negative \( y \) direction.

2a. (Source: 12.1.23) The radius of the sphere is the distance from the point \((-2, 1, 3)\) to the \( yz \)-plane, or 2, so the equation is \((x + 2)^2 + (y - 1)^2 + (z - 3)^2 = 2^2\).

2b. (Source: 12.5.12) We need a point on the line and vector parallel the line. You can find two points on the line by picking an arbitrary value for \( x \) and solving the two given equations for \( y \) and \( z \). In this way, I found \((0, 1, 1)\) and \((1, 0, -2)\). The vector between these points, \((1, -1, -3)\), is parallel to the line. Using this vector and the first point, parametrize the line by \(\langle t, 1 - t, 1 - 3t \rangle\).

2c. (Source: 12.5.35) We need a point on the plane and vector orthogonal to the plane. To find the vector, observe that the line is parallel \(\langle -1, 3, -2 \rangle\) and passes through the point \((1, 1, 0)\). The plane is parallel both \(\langle -1, 3, -2 \rangle\) and the vector between our two points, \(\langle 0, 1, 0 \rangle\). Therefore, their cross-product

\[
\begin{vmatrix}
i & j & k \\
-1 & 3 & -2 \\
0 & 1 & 0 \\
\end{vmatrix} = i \begin{vmatrix} 3 & -2 \\
1 & 0 \\
\end{vmatrix} - j \begin{vmatrix} -1 & -2 \\
0 & 0 \\
\end{vmatrix} + k \begin{vmatrix} -1 & 3 \\
0 & 1 \\
\end{vmatrix} = \langle 2, 0, -1 \rangle
\]

is orthogonal to the plane. Using this vector and the point \((1, 1, 0)\), obtain the equation \(2(x - 1) - z = 0\).
2d. (Source: 13.1.32) Substitute \( x = 2t, y = \sin(\pi t), \) and \( z = \cos(\pi t) \) into \( x^2 - y^2 - z^2 = 8 \) and solve for \( t \):

\[
4t^2 - \sin^2(\pi t) - \cos^2(\pi t) = 8 \implies 4t^2 - 1 = 8 \implies t^2 = \frac{9}{4} \implies t = \pm \frac{3}{2}
\]

Find the points of intersection by evaluating the parametrization at these \( t \)-values. At \( t = \frac{3}{2} \), the point is \( (2 \cdot \frac{3}{2}, \sin(\frac{3}{2} \pi), \sin(\frac{3}{2} \pi)) = (3, -1, 0) \). At \( t = -\frac{3}{2} \), the point is \( (-3, 1, 0) \).

2e. (Source: 13.2.23-26) The line passing through the point \( (1, 2, 0) \) and parallel to \( (0, 3, 4) \) can be parametrized by \( (1, 2, 0) \).

3a(4 pts). (Source: 13.4.9)

\[
\frac{dr}{dt} = r' = \langle 2t, 2t, t^2 \rangle = \langle -2, -2, 1 \rangle \quad \text{at} \quad t = -1
\]

\[
\frac{d^2r}{dt^2} = r'' = \langle 2, 2, 2t \rangle = \langle 2, 2, -2 \rangle \quad \text{at} \quad t = -1
\]

3b(13 pts). (Source: 13.3.47) \( \mathbf{T} = \frac{1}{|r'|}r' = \frac{1}{3}(-2, -2, 1) \). To find \( \mathbf{B} \), normalize

\[
\mathbf{r}' \times \mathbf{r}'' = \begin{vmatrix}
\mathbf{i} & \mathbf{j} & \mathbf{k} \\
-2 & -2 & 1 \\
2 & 2 & -2
\end{vmatrix} = \mathbf{i} \begin{vmatrix} -2 & 1 \\ -2 & -2 \end{vmatrix} - \mathbf{j} \begin{vmatrix} -2 & 1 \\ 2 & -2 \end{vmatrix} + \mathbf{k} \begin{vmatrix} 2 & 1 \\ 2 & -2 \end{vmatrix} = \langle 2, -2, 0 \rangle,
\]

(or, more simply, normalize \( \langle 1, -1, 0 \rangle \)) to obtain \( \mathbf{B} = \frac{1}{\sqrt{2}}(1, -1, 0) \). Finally,

\[
\mathbf{N} = \mathbf{B} \times \mathbf{T} = \frac{1}{3\sqrt{2}} \begin{vmatrix}
\mathbf{i} & \mathbf{j} & \mathbf{k} \\
1 & -1 & 0 \\
-2 & -2 & 1
\end{vmatrix} = \frac{1}{3\sqrt{2}} \begin{vmatrix} -1 & -1 \\ -2 & 1 \\ 1 & 0 \end{vmatrix} = \frac{1}{3\sqrt{2}} \langle -1, -1, -4 \rangle
\]

3c(4 pts). (Source: 13.3.21-23) \( \kappa = \frac{|\mathbf{r}' \times \mathbf{r}''|}{|\mathbf{r}'|^3} = \frac{2\sqrt{2}}{3^3} \)

3d(8 pts). (Source: 13.4.41,42) If \( \theta \) is the angle between \( \mathbf{r}' \) and \( \mathbf{r}'' \), then

\[
a_T = |\mathbf{r}''| \cos \theta = \frac{\mathbf{r}' \cdot \mathbf{r}''}{|\mathbf{r}'|} = \frac{-10}{3} \quad \quad a_N = |\mathbf{r}''| \sin \theta = \frac{|\mathbf{r}' \times \mathbf{r}''|}{|\mathbf{r}'|} = \frac{2\sqrt{2}}{3}
\]

4a(2 pts). (Source: 12.2.22) \( 2\mathbf{u} - \mathbf{v} = \langle 2, -4, 4 \rangle - \langle 3, 6, 2 \rangle = \langle -1, -10, 2 \rangle \).

4b(2 pts). (Source: 12.2.22) \( |\mathbf{u}| = \sqrt{1^2 + (-2)^2 + 2^2} = \sqrt{9} = 3 \).

4c(2 pts). (Source: 12.2.23-25) \( \frac{1}{|\mathbf{u}|}\mathbf{u} = \frac{1}{3}(1, -2, 2), \) or \( \langle \frac{1}{3}, -\frac{2}{3}, \frac{2}{3} \rangle \).

4d(6 pts). (Source: 12.3.17-18) \( |\mathbf{v}| = \sqrt{9 + 36 + 4} = 7 \), and \( \cos \theta = \frac{\mathbf{u} \cdot \mathbf{v}}{|\mathbf{u}||\mathbf{v}|} = \frac{3 - 12 + 4}{3 \cdot 7} = \frac{-5}{21} \).

4e(4 pts). (Source: 12.3.41-42) \( \text{proj}_\mathbf{v} \mathbf{u} = \frac{\mathbf{u} \cdot \mathbf{v}}{|\mathbf{v}|^2} \mathbf{v} = -\frac{5}{19}(3, 6, 2) \).
5(11 pts). (Source: 13.3.5) Arclength $s = \int \frac{ds}{dt} \, dt = \int |\frac{dr}{dt}| \, dt = $

$$\int_0^1 \sqrt{4t^2 + 4t^2 + t^4} \, dt = \int_0^1 \sqrt{8t^2 + t^4} \, dt$$

$$= \int_0^1 \sqrt{(t^2)(8 + t^2)} \, dt = \int_0^1 t\sqrt{(8 + t^2)} \, dt$$

To integrate, substitute $u = 8 + t^2$, so that $du = 2t \, dt$, or $\frac{1}{2} \, du = t \, dt$. New limits are $u = 8$ (when $t = 0$) and $u = 8$ (when $t = 1$). The integral becomes

$$\int_8^9 \frac{1}{2} u^{1/2} \, du = \left. \frac{1}{3} u^{3/2} \right|_8^9 = \frac{1}{3} (9^{3/2} - 8^{3/2}).$$