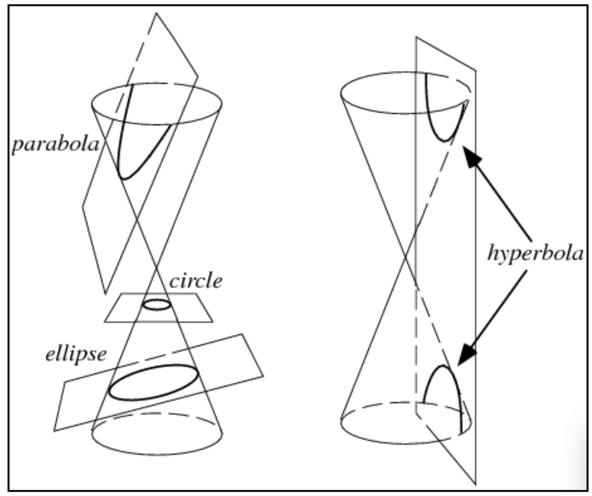
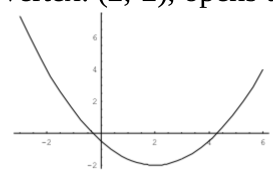
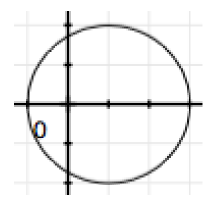
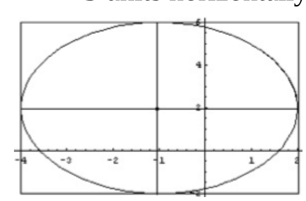
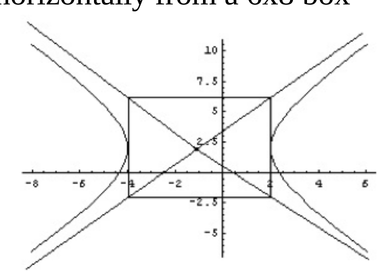
10.5 Conic Sections



- All quadratic equations of the form $Ax^2 + Cy^2 + Dx + Ey + F = 0$ where at least one of A and C are not zero have graphs that are conic sections (parabola, circle, ellipse, hyperbola, or, in degenerate cases, lines or even individual points).

- Generally, if either A or C is zero, you get a parabola; if A and C are the same sign, you get an ellipse; and if A and C are of opposite sign, you get a hyperbola. If A=C, then the ellipse is a circle.

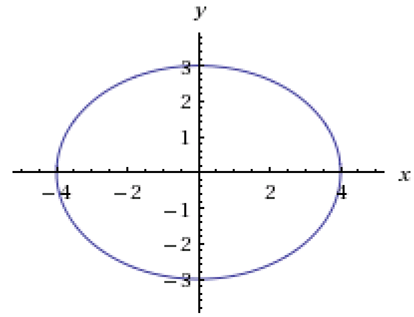
	Typical Equation	Important Information and Examples	
Parabola	$y = \frac{(x-h)^2}{4p} + k$ or $4p(y-k) = (x-h)^2$ or $4p(x-h) = (y-k)^2$	• Vertex: (h,k) • If x term is squared, parabola opens vertically • If y term is squared, parabola opens horizontally • If p>0, parabola opens up/right; p<0 → down/left *(p is in reference to the focal distance)*	$3x^2 - 12x - 8y = 4$ $\rightarrow \frac{8}{3}(y+2) = (x-2)^2$ $\rightarrow$ vertex: (2,-2), opens up 
Circle	$(x-h)^2 + (y-k)^2 = r^2$ or $\frac{(x-h)^2}{r^2} + \frac{(y-k)^2}{r^2} = 1$	• Center: (h,k) • Radius: r **Ex:** $\frac{(x-1)^2}{4} + \frac{y^2}{4} = 1$ Center: (1,0); Radius: 2	
Ellipse	$\frac{(x-h)^2}{a^2} + \frac{(y-k)^2}{b^2} = 1$	• Center: *(h,k)* • The ellipse expands a units from the center in the x direction. • The ellipse expands b units from the center in the y direction. **Ex:** $16x^2 + 9y^2 + 32x - 36y = 92$ $\Rightarrow \frac{(x+1)^2}{3^2} + \frac{(y-2)^2}{4^2} = 1$	Center: $(-1, 2)$ Expands: 4 units vertically, 3 units horizontally. 
Hyperbola	$\frac{(x-h)^2}{a^2} - \frac{(y-k)^2}{b^2} = 1$ or $\frac{(y-k)^2}{b^2} - \frac{(x-h)^2}{a^2} = 1$	• Center: *(h,k)* • If the x term is positive, then the hyperbola opens horizontally, and if the y term is positive, it opens vertically. • If you draw a rectangle $2a$ wide and $2b$ high with *(h, k)* at the center, then the hyperbola will be outside of the rectangle but just touch the sides (or top and bottom) of the rectangle (depending on whether it opens horizontally or vertically). The hyperbola will approach the extended diagonals of the rectangle as asymptotes.	$\frac{(x+1)^2}{3^2} - \frac{(y-2)^2}{4^2} = 1$ $\Rightarrow$ Center: $(-1, 2)$; opens horizontally from a 6x8 box 

EXAMPLE 1: Find the standard form for $9x^2 + 16y^2 = 144$, identify the conic section represented, and sketch the curve.

SOLUTION:

$$9x^2 + 16y^2 = 144 \Rightarrow \frac{9x^2}{144} + \frac{16y^2}{144} = 1 \Rightarrow \frac{x^2}{16} + \frac{y^2}{9} = 1.$$

This is an ellipse with center at $(0,0)$ that expands 4 units horizontally from the center (in both directions) and 3 units vertically from the center (in both directions). The graph looks like:

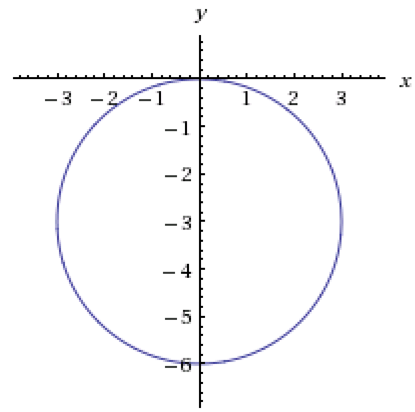


EXAMPLE 2: Convert the polar equation $r^2 = -6r\sin\theta$ to a Cartesian equation in standard form, identify the conic section represented, and graph the curve.

SOLUTION:

$$r^2 = -6r\sin\theta$$
$$\Rightarrow x^2 + y^2 = -6y$$
$$\Rightarrow x^2 + y^2 + 6y = 0$$
$$\Rightarrow x^2 + (y^2 + 6y + 9) = 9$$
$$\Rightarrow x^2 + (y+3)^2 = 9$$

This is a circle with center $(0,-3)$ and radius 3. The graph looks like



Exercise 1: Sketch the conic given by the equation $9x^2 - 4y^2 - 72x + 8y + 176 = 0$ by finding the standard form of the equation. What type of conic is represented?

Solution:

$$9x^2 - 4y^2 - 72x + 8y + 176$$
$$\Rightarrow 9(x^2 - 8x) - 4(y^2 + 2y) = -176$$
$$\Rightarrow 9(x^2 - 8x + 16) - 4(y^2 + 2y + 1) = -176 + 144 - 4$$
$$\Rightarrow 9(x-4)^2 - 4(y+1)^2 = -36$$
$$\Rightarrow \frac{9(x-4)^2}{-36} - \frac{4(y+1)^2}{-36} = 1$$
$$\Rightarrow \frac{(y+1)^2}{9} - \frac{(x-4)^2}{4} = 1$$

which is a hyperbola with center $(-1,4)$. The box drawn around the center extends horizontally 2 units (in both directions) and vertically 3 units (in both directions). The graph looks like