More problems for section 7.7 of *Calculus, Early Transcendentals* by James Stewart, 8e.

$$|E_T| \le \frac{K(b-a)^3}{12n^2} \qquad \text{and} \qquad |E_S| \le \frac{L(b-a)^5}{180n^4}$$

where $|f''(x)| \le K$ and $|f^{(4)}| \le L$ on the interval $[a, b]$.

Example: I plan to approximate $\int_{-1}^{2} f(x)\,dx$ using the Trapezoidal Rule, and $|f''(x)| \le 5$ on $[-1, 2]$.
a. If I use $n = 8$ subintervals, how large might the absolute error be?
b. If I need the absolute error to be less than 10^{-6}, how large an n must I use?
Solution:
a. Using $K = 5$ in the inequality above gives $|E_T| \le \frac{5(2-(-1))^3}{12 \cdot 8^2} = \frac{5 \cdot 3^3}{12 \cdot 8^2} \le 0.176$.
b. To ensure $|E_T| \le 10^{-6}$, chose n so as to make $\frac{K(b-a)^3}{12n^2} = \frac{5 \cdot 3^3}{12 \cdot n^2} \le 10^{-6}$. Solving, $n^2 \ge \frac{5 \cdot 3^2}{4} 10^6$ and so $n \ge \frac{3}{2} 10^3 \sqrt{5} \approx 3354.1$. Since n must be an integer, $n \ge 3355$.

In the exercises below, the same four questions appear with various wording. State each of your answers as an inequality (in the right direction).

1. Suppose $|f^{(2)}(x)| \le 48$ and $|f^{(4)}(x)| \le 120$ on $[1, 3]$.
a. If we approximate $\int_1^3 f(x)\,dx$ using TRAP with $n = 10$, how large could the absolute error be?
b. If we want $|E_T|$ to be less than 10^{-4}, how large must n be?
c. Bound the absolute error that could occur when we approximate $\int_1^3 f(x)\,dx$ using SIMP and $n = 10$.
d. What's the minimum n that will guarantee $|E_S|$ to be less than 10^{-4}?

2. Suppose $|f^{(2)}(x)| \le 3$ and $|f^{(4)}(x)| \le 16$ on $[10, 20]$.
a. If we approximate $\int_{10}^{20} f(x)\,dx$ using TRAP with $n = 20$, find the largest possible $|E_T|$.
b. Find an upper bound for the error that could occur if we approximate $\int_{10}^{20} f(x)\,dx$ using SIMP and $n = 20$.
c. For what range of n can we be certain that $|E_T| < 10^{-8}$?
d. What values of n will ensure $|E_S|$ is no more than 10^{-8}?

3. Suppose $-20 \le f^{(2)}(x) \le 0$ and $0 < f^{(4)}(x) < 10$ on $[2, 7]$.
a. What can be said of the absolute error that occurs when $\int_2^7 f(x)\,dx$ is approximated by SIMP and $n = 100$.
b. Find the maximum possible absolute error when TRAP is used to estimate $\int_2^7 f(x)\,dx$ with $n = 100$.
c. If we require $|E_S|$ to be at most 10^{-12}, what's the smallest acceptable value of n?
d. For what range of n do we know that $|E_T|$ is below 10^{-12}?

4. Suppose that $-6 < f^{(2)}(x) < 5$ and $-100 < f^{(4)}(x) < 25$ for all $0 \le x \le 2\pi$.
a. When TRAP is used to approximate $\int_0^{2\pi} f(x)\,dx$ with $n = 40$, how far from the true integral could the estimate be?
b. If we approximate $\int_0^{2\pi} f(x)\,dx$ using SIMP and 40 subintervals, how large could the resulting error be?
c. If the trapezoidal rule approximation must be within 10^{-8} of the actual integral, what number of subintervals should one use?
d. If we want $|E_S| < 10^{-8}$, how large must n be?

Answers

1a. $|E_T| \le \frac{32}{100}$ 1b. $n \ge 400\sqrt{2}$ 1c. $|E_S| \le \frac{64}{3} 10^{-4}$ 1d. $n \ge 20\sqrt[4]{\frac{4}{3}}$ 2a. $|E_T| \le \frac{3 \cdot 10^3}{12 \cdot 20^2}$ 2b. $|E_S| \le \frac{16 \cdot 10^5}{180 \cdot 20^4}$ 2c. $n \ge \sqrt{\frac{3 \cdot 10^3}{12}} 10^4$ 2d. $n \ge \sqrt[4]{\frac{16 \cdot 10^5}{180}} 10^2$ 3a. $|E_S| \le \frac{10 \cdot 5^5}{180 \cdot 100^4}$ 3b. $|E_T| \le \frac{20 \cdot 5^3}{12 \cdot 100^2}$ 3c. $n \ge \sqrt[4]{\frac{10 \cdot 5^5}{180}} 10^3$ 3d. $n \ge \sqrt{\frac{20 \cdot 5^3}{12}} 10^6$ 4a. $|E_T| \le \frac{100(2\pi)^3}{12 \cdot 40^2}$ 4b. $|E_S| \le \frac{6(2\pi)^5}{180 \cdot 40^4}$ 4c. $n \ge \sqrt{\frac{100(2\pi)^3}{12}} 10^4$ 4d. $n \ge \sqrt[4]{\frac{6(2\pi)^5}{180}} 10^2$