MATH 220–02 (Kunkle), Quiz 6 | Name: ______________________________
10 pts, 10 minutes | Oct 24, 2023

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1 (10 pts). Determine whether the series is convergent or divergent:

$$\sum_{n=2}^{\infty} \frac{1}{n(\ln n)^2}$$

Show all work leading to your conclusion. The correct answer by itself is worth no points.

Solution:

1.(Source: 11.3.21,22) The function $f(x) = \frac{1}{x(\ln x)^2}$ is positive on $[2, \infty)$. It's also decreasing on $[2, \infty)$, since $x(\ln x)^2$ is increasing on that interval. Therefore the Integral Test tells us that the infinite series and the improper integral

$$\sum_{n=2}^{\infty} \frac{1}{n(\ln n)^2} \qquad \text{and} \qquad \int_2^{\infty} \frac{1}{x(\ln x)^2}\, dx$$

must either both converge or both diverge. Calculate the integral directly.

First find the antiderivative: if we let $v = \ln x$, then $dv = \frac{1}{x}\, dx$, and the indefinite integral becomes $\int \frac{1}{x(\ln x)^2}\, dx = \int v^{-2}\, dv = -v^{-1} + C = -(\ln x)^{-1} + C$. Now evaluate the improper integral:

$$\int_2^{\infty} \frac{1}{x(\ln x)^2}\, dx = \lim_{B\to\infty} \int_2^{B} \frac{1}{x(\ln x)^2}\, dx = \lim_{B\to\infty} -(\ln x)^{-1}\Big|_2^B$$
$$= \lim_{B\to\infty} \left(-\frac{1}{\ln B} + \frac{1}{\ln 2}\right) = \frac{1}{\ln 2}.$$

Since the improper integral converges, so does the infinite series. **(done)**

Comments:

a. $\lim_{n\to\infty} \frac{1}{n(\ln n)^2} = 0$, so the nth term test is inconclusive.

b. $\lim_{n\to\infty} \frac{(n+1)(\ln(n+1))^2}{n(\ln n)^2} = 1$, so the ratio test is inconclusive.

c. Comparing the series to $\sum_{n=2}^{\infty} \frac{1}{n^2}$, $\sum_{n=2}^{\infty} \frac{1}{n}$, or $\sum_{n=2}^{\infty} \frac{1}{(\ln n)^2}$ is inconclusive, since the first of these converges, the second and third diverge, and

$$\frac{1}{n^2} < \frac{1}{n(\ln n)^2} < \frac{1}{n} < \frac{1}{(\ln n)^2}$$

for all sufficiently large values of n.

One way to see that these three inequalities are true is to show that each of the limits

$$\lim_{n\to\infty} \frac{\frac{1}{n^2}}{\frac{1}{n(\ln n)^2}} \qquad \lim_{n\to\infty} \frac{\frac{1}{n(\ln n)^2}}{\frac{1}{n}} \qquad \lim_{n\to\infty} \frac{\frac{1}{n}}{\frac{1}{(\ln n)^2}}$$

is zero, so that, in each, the numerator must be less than the denominator for all sufficiently large values of n.

Because these three limits are zero, limit-comparing the series to $\sum_{n=2}^{\infty} \frac{1}{n^2}$, $\sum_{n=2}^{\infty} \frac{1}{n}$, or $\sum_{n=2}^{\infty} \frac{1}{(\ln n)^2}$ is also inconclusive.

d. Although the series converges, it does so very slowly. For instance, if we approximated its sum by the partial sum of its first million terms, by [2], p. 723, our error would be between

$$\int_{10^6+1}^{\infty} \frac{1}{x(\ln x)^2}\,dx = \frac{1}{\ln(10^6+1)} \approx .072382408$$

and

$$\int_{10^6}^{\infty} \frac{1}{x(\ln x)^2}\,dx = \frac{1}{\ln(10^6)} \approx .072382414.$$

That is, the millionth partial sum agrees with the true sum only to about one place after the decimal. (If we would take advantage of the bounds above to get a better estimate of the sum, we'd have an accuracy of about seven places after the decimal.)