No notes, books, electronic devices, or outside materials of any kind.

Read each problem carefully and simplify your answers.

Unless otherwise indicated, supporting work will be required on every problem worth more than 2 points. You are expected to know the values of all trigonometric functions at multiples of $\pi/4$ and of $\pi/6$.

You may use any of these formulas that are relevant.

$$\int \sin^n x\,dx = -\frac{1}{n}\sin^{n-1}x\cos x + \frac{n-1}{n}\int \sin^{n-2}x\,dx$$
$$\int \cos^n x\,dx = \frac{1}{n}\cos^{n-1}x\sin x + \frac{n-1}{n}\int \cos^{n-2}x\,dx$$
$$\int \tan^n x\,dx = \frac{1}{n-1}\tan^{n-1}x - \int \tan^{n-2}x\,dx$$
$$\int \sec^n x\,dx = \frac{1}{n-1}\sec^{n-2}x\tan x + \frac{n-2}{n-1}\int \sec^{n-2}x\,dx$$

1(11 pts). Let W be the "triangular" region in the first quadrant bounded by $y = \ln x$, $y = 0$ and $x = e$. Find the volume swept out by W as it is rotated about the line $x = 3$. Express your answer as a definite integral, but **do not evaluate**.

2(37 pts). Evaluate the indefinite integral.

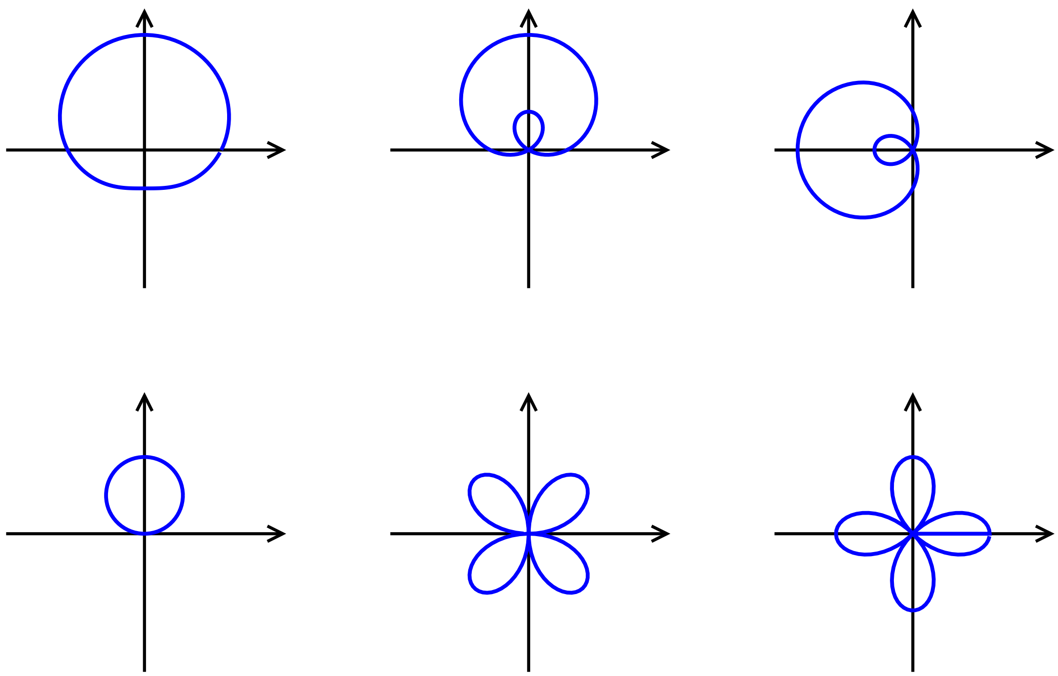
a. $\displaystyle\int \frac{1}{x^4\sqrt{x^2-1}}\,dx$ b. $\displaystyle\int \frac{1}{(x-1)(x+3)}\,dx$ c. $\displaystyle\int x\ln x\,dx$

3(10 pts). Write $\sin(4x)\sin(6x)$ as a sum of sinusoidal functions. (Do not integrate.)

4(7 pts). Find the orthogonal trajectories of the family of curves $y = \dfrac{1}{x+k}$.
Hint: along any member of this family, $\frac{dy}{dx} = -y^2$.

5(12 pts). Six of the nine polar equations below are graphed in the figure. All six graphs are drawn on the same scale. Clearly label each graph with its equation number.

a. $r = 2\cos(2\theta)$	b. $r = 2\sin(2\theta)$	c. $r = 2\cos(4\theta)$
d. $r = 1$	e. $r = 2\sin\theta$	f. $r = 1 + 2\sin\theta$
g. $r = 1 + \sin\theta$	h. $r = 1 - 2\cos\theta$	i. $r = 2 + \sin\theta$



6a(13 pts) Find $\frac{dy}{dx}$ at $\theta = \frac{\pi}{6}$ on the curve $r = \sin(3\theta)$.

6b(16 pts) Find the area inside one loop of $r = \sin(3\theta)$.

6c(7 pts) Find the length of one loop of the curve $r = \sin(3\theta)$.
In 6c, express your answer as a definite integral, but **do not evaluate**.

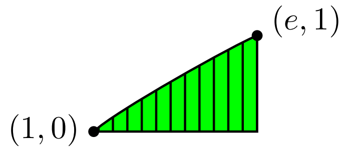
7(13 pts). Find the radius of convergence of the power series $\displaystyle\sum_{n=1}^{\infty} \frac{(x+5)^n}{3^n n(n+1)}$

8(20 pts) Determine whether the series $\displaystyle\sum_{n=2}^{\infty} (-1)^n \frac{1}{n \ln n}$ converges absolutely, converges conditionally, or diverges.

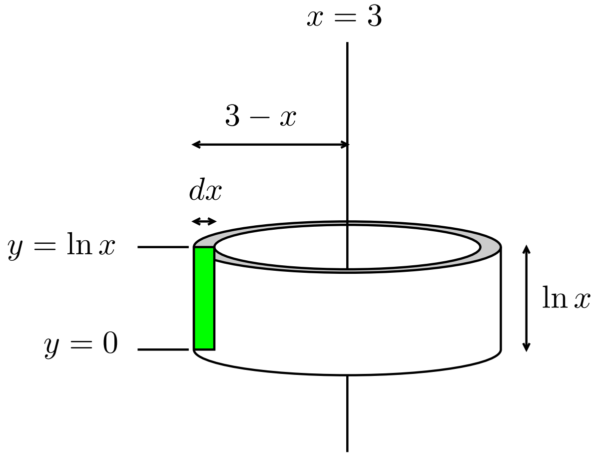
9(14 pts). Find the Taylor series for $\frac{1}{x}$ centered at $a = 2$.

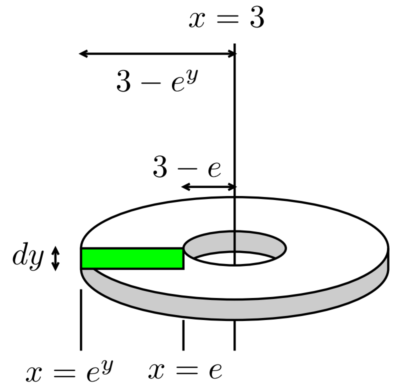
1(11 pts).(Source: 6.2.15,17)

Here's a sketch of R, sliced vertically:



a. When one of these rectangles is rotated about $x = 3$, the result is a cylindrical shell. If we slice R horizontally and rotate, the result is a washer:





There are two correct answers:

$$dV = 2\pi(3 - x)\ln x\,dx \qquad\qquad dV = \pi\big((3 - e^y)^2 - (3 - e)^2\big)\,dy$$

$$V = \int_1^e 2\pi(3 - x)\ln x\,dx \qquad\qquad V = \int_0^1 \pi\big((3 - e^y)^2 - (3 - e)^2\big)\,dy$$

2a(16 pts).(Source: 7.3.13, 7.2.32) Use trig substitution. Choose $x = \sec\theta$ so that $\sqrt{x^2 - 1} = \sqrt{\sec^2\theta - 1} = \tan\theta$. Then $dx = \sec\theta\tan\theta\,d\theta$ and the integral becomes

$$\int \frac{1}{\sec^4\theta\tan\theta}\sec\theta\tan\theta\,d\theta = \int \frac{1}{\sec^3\theta}\,d\theta = \int \cos^3\theta\,d\theta = \int (1 - \sin^2\theta)\cos\theta\,d\theta$$

Now, either use the reduction formula or substitute $s = \sin\theta$, $ds = \cos\theta\,d\theta$:

$$\int (1 - s^2)\,ds = s - \tfrac{1}{3}s^3 + C = \sin\theta - \tfrac{1}{3}\sin^3\theta + C$$

2b(10 pts).(Source: 7.4.9,10) Find the partial fraction decomposition of the integrand. The degree of the numerator is strictly less than that of the denominator, so long division is not necessary. Look for constants A and B for which

$$\frac{1}{(x-1)(x+3)} = \frac{A}{x+3} + \frac{B}{x-1} \quad\Longrightarrow\quad 1 = A(x-1) + B(x+3).$$

Evaluate at $x = -3$ and $x = 1$:

$$\begin{aligned} x = -3: &\qquad 1 = A\cdot(-4) &&\Longrightarrow\quad A = -\tfrac{1}{4} \\ x = 1: &\qquad 1 = B\cdot 4 &&\Longrightarrow\quad B = \tfrac{1}{4}. \end{aligned}$$

Now integrate:

$$\int \frac{1}{(x-1)(x+3)}\,dx = \int \left(\frac{-1/4}{x+3} + \frac{1/4}{x-1}\right)\,dx = -\frac{1}{4}\ln|x+3| + \frac{1}{4}\ln|x-1| + C$$

2c(11 pts).(Source: 7.1.11) Integrate by parts:

$$\begin{matrix} u = \ln x & dv = x\,dx \\ du = x^{-1}\,dx & v = \frac{1}{2}x^2, \end{matrix} \quad \Longrightarrow \quad \begin{aligned} \int x\ln x\,dx &= \tfrac{1}{2}x^2\ln x - \int \tfrac{1}{2}x\,dx \\ &= \tfrac{1}{2}x^2\ln x - \tfrac{1}{4}x^2 + C \end{aligned}$$

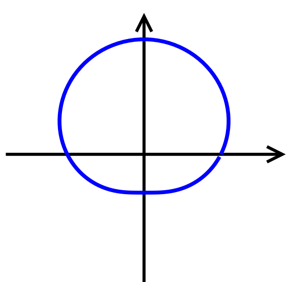
3(10 pts).(Source: Euler.9c)

$$\begin{aligned} \sin 4x \sin 6x &= \left(\frac{e^{i4x} - e^{-i4x}}{2i}\right)\left(\frac{e^{i6x} - e^{-i6x}}{2i}\right) = \frac{e^{i10x} + e^{-i10x} - e^{i2x} - e^{-i2x}}{-4} \\ &= \frac{1}{2}\left(\frac{e^{i2x} + e^{-i2x}}{2} - \frac{e^{i10x} + e^{-i10x}}{2}\right) = \frac{1}{2}(\cos 2x - \cos 10x). \end{aligned}$$

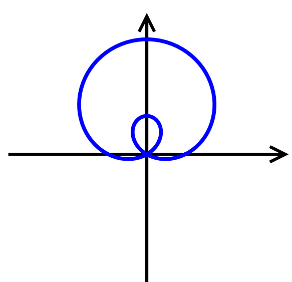
4(7 pts).(Source: 9.3.32) Along the orthogonal trajectories, $\frac{dy}{dx} = \frac{1}{y^2}$. Separate variables and integrate:

$$\int y^2\,dy = \int dx \quad \Longrightarrow \quad \frac{1}{3}y^3 = x + C$$

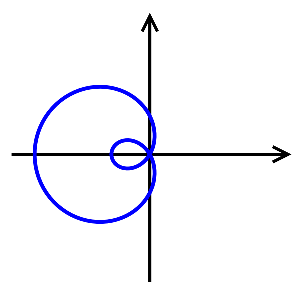
5(12 pts).(Source: 10.3.15, 31-40)



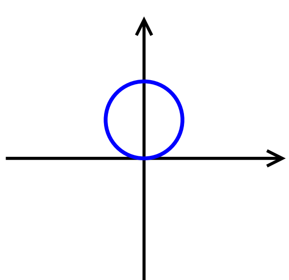
$r = 2 + \sin\theta$



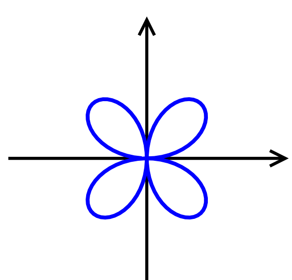
$r = 1 + 2\sin\theta$



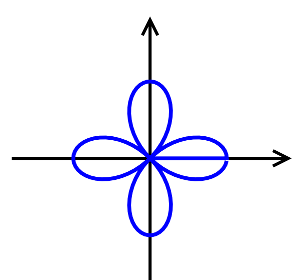
$r = 1 - 2\cos\theta$



$r = 2\sin\theta$



$r = 2\sin(2\theta)$



$r = 2\cos(2\theta)$

6a(13 pts)(Source: 10.3.59) The curve in question is parametrized by

$$x = r\cos\theta = \sin(3\theta)\cos\theta \qquad y = r\sin\theta = \sin(3\theta)\sin\theta$$

and so

$$\frac{dy}{dx} = \frac{dy/d\theta}{dx/d\theta} = \frac{3\cos(3\theta)\sin\theta + \sin(3\theta)\cos\theta}{3\cos(3\theta)\cos\theta - \sin(3\theta)\sin\theta}$$

At $\theta = \frac{\pi}{6}$, $3\theta = \frac{\pi}{2}$, and so

$$\sin(\theta) = \frac{1}{2} \qquad \cos(\theta) = \frac{\sqrt{3}}{2} \qquad \sin(3\theta) = 1 \qquad \cos(3\theta) = 0$$

Therefore

$$\frac{dy}{dx} = \frac{\sqrt{3}/2}{-1/2} = -\sqrt{3}$$

6b(16 pts)(Source: 10.4.17) One petal of this three-leafed rose occurs between any two consecutive zeros of $\sin(3\theta)$. Using $0 \le \theta \le \pi/3$* and $dA = \frac{1}{2}r^2\,d\theta$, the area is

$$\frac{1}{2}\int_0^{\pi/3} \sin^2(3\theta)\,d\theta = \frac{1}{4}\int_0^{\pi/3}(1-\cos(6\theta))\,d\theta = \frac{1}{4}(\theta - \frac{1}{6}\sin(6\theta))\Big|_0^{\pi/3} = \frac{\pi}{12}.$$

(done)

Notes:
i. You can use Euler's formula to arrive at the half-angle identity used above.
ii. To integrate by the reduction formula, it's necessary to first substitute $x = 3\theta$.
iii. Six petals are drawn between 0 to 2π, and so the area of one petal is $\frac{1}{12}\int_0^{2\pi}\sin^2(3\theta)\,d\theta$.

6c(7 pts)(Source: 10.4.45-48) The arclength equals

$$\int_0^{\pi/3}\sqrt{r^2 + \frac{dr}{d\theta}^2}\,d\theta = \int_0^{\pi/3}\sqrt{\sin^2(3\theta) + 9\cos^2(3\theta)}\,d\theta$$

7(13 pts).(Source: 11.8.13)

$$\lim_{n\to\infty}\left|\frac{(x+5)^{n+1}}{3^{n+1}(n+1)(n+2)}\right| \div \left|\frac{(x+5)^n}{3^n n(n+1)}\right| = \lim_{n\to\infty}\frac{|x+5|^{n+1}}{|x+5|^n}\frac{3^n}{3^{n+1}}\frac{n(n+1)}{(n+1)(n+2)}$$
$$= \lim_{n\to\infty}\frac{|x+5|}{3}\frac{n}{n+1} = \frac{|x+5|}{3}$$

(using limit 5 from FLESK https://kunklet.people.cofc.edu/MATH220/flesk.pdf). By the Ratio test, the power series diverges if $\frac{|x+5|}{3} > 1$ and converges absolutely if $\frac{|x+5|}{3} < 1$, or $|x+5| < 3$. Therefore, the radius of convergence is 3.

* https://www.desmos.com/calculator/ccceq4zkbp

8(20 pts)(Source: 11.6.38,11.5.6,11.3.21) First test for absolute convergence. The function $f(x) = \frac{1}{x \ln x}$ is positive on $[2, \infty)$. It's also decreasing on $[2, \infty)$, since $x \ln x$ is increasing on that interval. The Integral Test then tells us that the infinite series and the improper integral

$$\sum_{n=2}^{\infty} \frac{1}{n \ln n} \qquad \text{and} \qquad \int_2^{\infty} \frac{1}{x \ln x}\, dx$$

must either both converge or both diverge. Calculate the improper integral directly.
First find the antiderivative: if we let $v = \ln x$, then $dv = \frac{1}{x}\, dx$, and the indefinite integral becomes $\int v^{-1}\, dv = \ln |v| + C = \ln |\ln x| + C$. Now evaluate the improper integral:

$$\begin{aligned} \int_2^{\infty} \frac{1}{x \ln x}\, dx &= \lim_{B \to \infty} \int_2^{B} \frac{1}{x \ln x}\, dx = \lim_{B \to \infty} \ln |\ln x| \Big|_2^B \\ &= \lim_{B \to \infty} (\ln |\ln B| - \ln |\ln 2|) = \infty. \end{aligned}$$

Since the improper integral diverges, so does $\sum_{n=2}^{\infty} \frac{1}{n \ln n}$. Therefore, $\sum_{n=2}^{\infty} (-1)^n \frac{1}{n \ln n}$ fails to converge absolutely.
Next we test for conditional convergence. It has been noted above that $\frac{1}{n \ln n}$ is decreasing. Furthermore, $\lim_{n \to \infty} \frac{1}{n \ln n} = 0$, since $\lim_{n \to \infty} n \ln n = \infty$. Theefore, $\sum_{n=2}^{\infty} (-1)^n \frac{1}{n \ln n}$ converges by the alternating series test. Conclusion: The original series converges conditionally, that is, converges, but not absolutely.

9(14 pts).(Source: 11.10.21,22) **Solution one.** Calculate the first few terms of the Taylor series:

n	$f^{(n)}(x)$	$f^{(n)}(2)/n!$
0	x^{-1}	2^{-1}
1	$-x^{-2}$	-2^{-2}
2	$2x^{-3}$	2^{-3}
3	$-3!x^{-4}$	-2^{-4}
4	$4!x^{-5}$	2^{-5}
5	$-5!x^{-6}$	-2^{-6}

The Maclaurin series equals

$$\begin{aligned} &\frac{1}{2} - \frac{1}{2^2}(x-2) \\ &\quad + \frac{1}{2^3}(x-2)^2 - \frac{1}{2^4}(x-2)^3 \\ &\qquad + \frac{1}{2^5}(x-2)^4 - \frac{1}{2^6}(x-2)^5 + \cdots \\ &= \sum_{n=0}^{\infty} (-1)^n \frac{(x-2)^n}{2^{n+1}} \end{aligned}$$

Solution two.

$$\frac{1}{x} = \frac{1}{2-(2-x)} = \frac{1}{2} \cdot \frac{1}{1-\frac{1}{2}(2-x)} = \frac{1}{2} \sum_{n=0}^{\infty} \frac{(2-x)^n}{2^n} = \sum_{n=0}^{\infty} \frac{(-1)^n (x-2)^n}{2^{n+1}}$$

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