No notes, books, electronic devices, or outside materials of any kind.

Read each problem carefully and simplify your answers.

Unless otherwise indicated, supporting work will be required on every problem worth more than 2 points.

You are expected to know the values of all trig functions at multiples of $\pi/4$ and of $\pi/6$.

1(11 pts). Find the particular solution to $(y-1)\dfrac{dy}{dx} - xy(x+1) = 0$ that passes through the point $x = 0$, $y = 1$. You are not required to express y as a function of x in your solution.

2(26 pts). Find a power series representation of the given function.

a. $\ln(2+x)$ b. $\dfrac{x^2}{2+x}$ c. $(3-x)^{-3}$ d. e^{-x^2}

3(15 pts). Use the definition of Taylor series to find the Taylor series for $\frac{1}{2+x}$ centered at $a = 1$.

4a(7 pts). Find $T_4(x)$, the Taylor polynomial of degree 4 centered at $a = 0$ for $\cos x$.

4b(10 pts). Find a positive number b so that $|\cos x - T_4(x)| \leq 10^{-3}$ for all x in $[-b, b]$. Hint: $b = 10^{-99}$ is correct but won't earn you any credit. On this problem, the larger the correct b, the better the answer.

5(25 pts). This problem is about the curve parametrized by $x = t^2 - 1$, $y = \frac{1}{3}t^3 - t$.

a. Find $\frac{dy}{dx}$ along this curve. Express your answer as a function of t.

b. At which t-values, if any, is the line tangent to the curve horizontal?

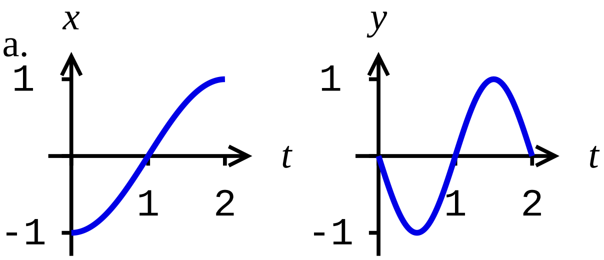
c. At which t-values, if any, is the line tangent to the curve vertical?

d. Find $\frac{d^2y}{dx^2}$ along this curve. Express your answer as a function of t.

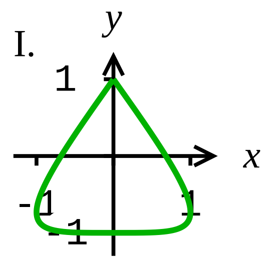
e. Find the length of this curve from $t = 0$ to $t = 1$.

6(6 pts). Match the graphs of the functions $x = f(t)$ and $y = g(t)$ in a–d with the curves given parametrically by these equations in I–IV.

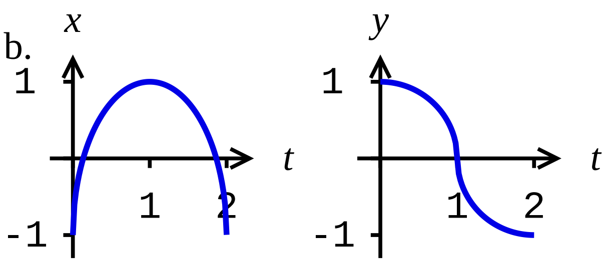
a.



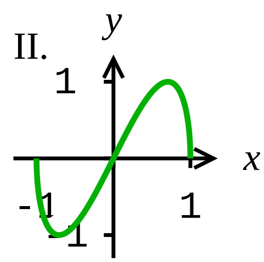
I.



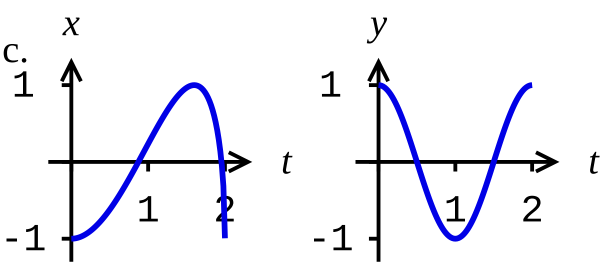
b.



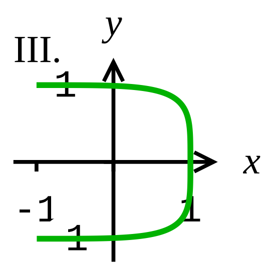
II.



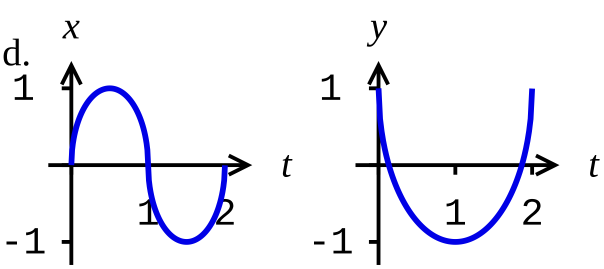
c.



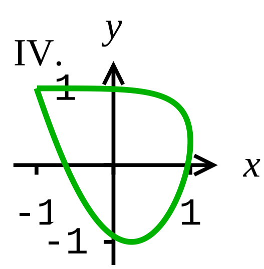
III.



d.



IV.



1(11 pts).(Source: 9.3.3,6,14) To find the general solution, separate variables and integrate:

$$\frac{y-1}{y}\,dy = x(x+1)\,dx \qquad \int (1-\frac{1}{y})\,dy = \int (x^2+x)\,dx \qquad y - \ln|y| = \frac{1}{3}x^3 + \frac{1}{2}x^2 + C$$

To find the particular solution in question, set $x = 0$ and $y = 1$ and solve for C:

$$1 - 0 = 0 + C \quad\Longrightarrow\quad y - \ln|y| = \frac{1}{3}x^3 + \frac{1}{2}x^2 + 1.$$

2ab(15 pts). $\dfrac{1}{2+x} = \dfrac{1}{2}\cdot\dfrac{1}{1-(\frac{-x}{2})} = \dfrac{1}{2}\displaystyle\sum_{n=0}^{\infty}\left(\frac{-x}{2}\right)^n = \sum_{n=0}^{\infty}(-1)^n\frac{x^n}{2^{n+1}}.$

2a.(Source: 11.9.15,11.10.12) Integrate:

$$\ln(2+x) = \int \frac{1}{2+x}\,dx = \sum_{n=0}^{\infty}(-1)^n \int \frac{x^n}{2^{n+1}}\,dx = C + \sum_{n=0}^{\infty}(-1)^n\frac{x^{n+1}}{(n+1)2^{n+1}}.$$

Evaluate at $x = 0$ to find C:

$$\ln 2 = C + \sum_{n=0}^{\infty} 0 \quad\Longrightarrow\quad \ln(2+x) = \ln 2 + \sum_{n=0}^{\infty}(-1)^n\frac{x^{n+1}}{(n+1)2^{n+1}}.$$

2b.(Source: 11.9.9) Multiply the series for $\frac{1}{2+x}$ by x^2:

$$\frac{x^2}{2+x} = x^2\sum_{n=0}^{\infty}(-1)^n\frac{x^n}{2^{n+1}} = \sum_{n=0}^{\infty}(-1)^n\frac{x^{n+2}}{2^{n+1}}.$$

2c(6 pts). Solution one (Source: 11.9.13,18) Differentiate $(1-x)^{-1}$ twice and divide by 2:

$$(1-x)^{-1} = \sum_{n=0}^{\infty} x^n \qquad\qquad 2(1-x)^{-3} = \sum_{n=2}^{\infty} n(n-1)x^{n-2}$$

$$(1-x)^{-2} = \sum_{n=1}^{\infty} nx^{n-1} \qquad\qquad (1-x)^{-3} = \sum_{n=2}^{\infty}\frac{n(n-1)}{2}x^{n-2}$$

Now $(3-x)^{-3} = \dfrac{1}{3^3}(1-\dfrac{x}{3})^{-3} = \dfrac{1}{3^3}\displaystyle\sum_{n=2}^{\infty}\frac{n(n-1)}{2}\left(\frac{x}{3}\right)^{n-2} = \sum_{n=2}^{\infty}\frac{n(n-1)}{2}\frac{x^{n-2}}{3^{n+1}}.$ (1)

Solution two (Source: 11.10.31,33) Use the binomial series.

$$\frac{1}{3^3}\left(1+\left(-\frac{x}{3}\right)\right)^{-3} = \frac{1}{3^3}\sum_{n=0}^{\infty}\binom{-3}{n}\left(-\frac{x}{3}\right)^n = \sum_{n=0}^{\infty}(-1)^n\binom{-3}{n}\frac{x^n}{3^{n+3}} \qquad (2)$$

To see that (1) and (2) are the same series, reindex (1) $\sum_{n=0}^{\infty} \frac{(n+2)(n+1)}{2} \frac{x^n}{3^{n+2}}$. Now observe that, if $n = 0$, then $(-1)^0\binom{-3}{0} = 1 = \frac{1\cdot 2}{2}$, and if n is not zero, then

$$\begin{aligned}(-1)^n\binom{-3}{n} &= (-1)^n\frac{(-3)(-4)\cdots(-3-(n-1))}{1\cdot 2\cdots(n-1)n} \\ &= \frac{3\cdot 4\cdot\cdots n(n+1)(n+2)}{1\cdot 2\cdot 3\cdot 4\cdots n} = \frac{(n+1)(n+2)}{2}\end{aligned}$$

2d(5 pts).(Source: 11.10.35) $e^x = \sum_{n=0}^{\infty}\frac{x^n}{n!}$, so $e^{-x^2} = \sum_{n=0}^{\infty}\frac{(-x^2)^n}{n!} = \sum_{n=0}^{\infty}(-1)^n\frac{x^{2n}}{n!}$.

3(15 pts).(Source: 11.10.22) Let $g(x) = \frac{1}{2+x}$ and calculate Taylor coefficients at $a = 1$ until the pattern in clear.

n	$g^{(n)}(x)$	$g^{(n)}(1)$	$g^{(n)}(1)/n!$
0	$(2+x)^{-1}$	$\frac{1}{3}$	$\frac{1}{3}$
1	$-(2+x)^{-2}$	$-\frac{1}{3^2}$	$-\frac{1}{3^2}$
2	$2(2+x)^{-3}$	$2\frac{1}{3^3}$	$\frac{1}{3^3}$
3	$-3\cdot 2(2+x)^{-4}$	$-3!\frac{1}{3^4}$	$-\frac{1}{3^4}$

The Taylor series is

$$\frac{1}{3} - \frac{1}{3^2}(x-1) + \frac{1}{3^3}(x-1)^2 - \frac{1}{3^4}(x-1)^3 + \cdots = \sum_{n=0}^{\infty}(-1)^n\frac{(x-1)^n}{3^{n+1}}.$$

4a(7 pts).(Source: 11.11.4) Either calculate the first four Taylor coefficients as in 3, or remember that

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots$$

and so

$$T_4(x) = 1 - \frac{x^2}{2!} + \frac{x^4}{4!}.$$

4b(10 pts).(Source: 11.11.28) Observe that, for this function, $T_4(x) = T_5(x)$. Taylor's Theorem tells us that, for some c between x and 0,

$$|\cos x - T_5(x)| = \left|\frac{\cos^{(6)} c}{6!}x^6\right| = \frac{|\cos^{(6)} c|}{6!}|x|^6 \le \frac{1}{6!}b^6.$$

To ensure that $|\cos x - T_5(x)| \le 10^{-3}$, set $\frac{1}{6!}b^6 \le 10^{-3}$ and solve to find $b \le \sqrt[6]{6!\cdot 10^{-3}}$. For the best answer, take $b = \sqrt[6]{6!\cdot 10^{-3}}$.

5.(Source: 10.2.12,17)

5a(5 pts). $\dfrac{dy}{dx} = \dfrac{\frac{dy}{dt}}{\frac{dx}{dt}} = \dfrac{t^2-1}{2t} = \dfrac{1}{2}t - \dfrac{1}{2}t^{-1}$.

5b(2 pts). The tangent line is horizontal when $\frac{dy}{dt} = 0 \neq \frac{dx}{dt}$, at $t = \pm 1$.

5c(2 pts). The tangent line is vertical when $\frac{dy}{dt} \neq 0 = \frac{dx}{dt}$, at $t = 0$.

5d(5 pts). $\dfrac{d^2y}{dx^2} = \dfrac{d}{dx}\left(\dfrac{dy}{dx}\right) = \dfrac{\frac{d}{dt}\left(\frac{dy}{dx}\right)}{\frac{dx}{dt}} = \dfrac{\frac{d}{dt}(\frac{1}{2}t - \frac{1}{2}t^{-1})}{2t} = \dfrac{\frac{1}{2} + \frac{1}{2}t^{-2}}{2t} = \dfrac{1}{4}t^{-1} + \dfrac{1}{4}t^{-3}$.

5e(11 pts). $\displaystyle\int_0^1 \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\,dt = \int_0^1 \sqrt{(2t)^2 + (t^2-1)^2}\,dt$

$$= \int_0^1 \sqrt{4t^2 + t^4 - 2t^2 + 1}\,dt = \int_0^1 \sqrt{t^4 + 2t^2 + 1}\,dt$$
$$= \int_0^1 \sqrt{(t^2+1)^2}\,dt = \int_0^1 (t^2+1)\,dt = \left(\frac{1}{3}t^3 + t\right)\Big|_0^1 = \frac{4}{3}.$$

6(6 pts).(Source: 10.1.24) a:II b:III c:IV d:I
a is the only curve that passes though $(0, 0)$.
b is the only curve that passes through $(-1, 1)$ and $(-1, -1)$.
Of c and d, only c passes through $(-1, 1)$.
a begins and ends with a vertical tangent and has two horizontal tangents in between.
b begins and ends with a horizontal tangent and has one vertical tangent in between.
Excluding the point $(-1, 1)$, c has one horizontal and one vertical tangent.
Excluding the point $(0, 1)$, d has one horizontal and two vertical tangents.

Complete your course-instructor evaluations at `https://coursereview.cofc.edu/`