1. (14 pts) Find the radius of convergence of the power series \[ \sum_{n=2}^{\infty} \frac{(x+1)^n}{3^n \ln n}. \]

2. (15 pts) Evaluate the limit. Show your work, but your answer to each should be a number, \( \infty \), \( -\infty \), or “does not exist”.
   a. \[ \lim_{n \to \infty} \frac{(n+1)!}{(n-1)!} \]
   b. \[ \lim_{n \to \infty} 3^n 5^{1-n} \]
   c. \[ \lim_{n \to \infty} \frac{(-1)^n}{1+n} \]

3. (20 pts) Find the sum, if it exists.
   a. \[ \sum_{n=0}^{\infty} ((0.1)^{n-1} - (0.9)^n) \]
   b. \[ \sum_{n=1}^{\infty} \frac{\ln(n)}{\ln(4n)} \]

4. (23 pts) Determine whether the series converges or diverges.
   a. \[ \sum_{n=1}^{\infty} \frac{\sin^2 n}{3^n + 1} \]
   b. \[ \sum_{n=0}^{\infty} \frac{n^2 + 2}{n^3 + 3} \]

5. (10 pts) Find an upper bound for the error that occurs when we approximate the sum of the series \[ \sum_{n=1}^{\infty} \frac{1}{n^4} \] with the partial sum \[ \sum_{n=1}^{100} \frac{1}{n^4}. \] (That is, find a number \# which satisfies error < \#.)

6. (18 pts) Determine whether the series \[ \sum_{n=2}^{\infty} \frac{(-1)^n}{n \ln n} \] converges absolutely, converges conditionally, or diverges.
1.(14 pts)(Source: 11.8.17) The solution to a radius-of-convergence problem typically begins with either the Root or Ratio test. Here’s a solution with Ratio:

\[
\left| \frac{(x+1)^{n+1}/3^{n+1} \ln(n+1)}{(x+1)^n/3^n \ln n} \right| = \left| \frac{x+1}{x+1}^{n+1} \cdot \frac{3^n}{3^{n+1}} \cdot \frac{\ln n}{\ln(n+1)} \right| = \frac{|x+1|}{3} \cdot \frac{\ln n}{\ln(n+1)}.
\]

To take the limit of \( \frac{\ln n}{\ln(n+1)} \), use l’Hospital’s Rule:

\[
\frac{\ln n}{\ln(n+1)} \xrightarrow{\text{HR}} \frac{\frac{1}{n}}{\frac{1}{n+1}} = \frac{n+1}{n} = 1 + \frac{1}{n} \to 1.
\]

Therefore,

\[
\lim_{n \to \infty} \frac{|x+1|}{3} \cdot \frac{\ln n}{\ln(n+1)} = \frac{|x+1|}{3}.
\]

The Ratio test tells us that the series converges (absolutely) when

\[
\frac{|x+1|}{3} < 1, \quad \text{or} \quad |x+1| < 3,
\]

and so the Radius of Convergence is 3.

If you used the Root test, you would need to take the limit of

\[
\left| \left( \frac{(x+1)^{n}/3^n \ln n} \right)^{1/n} \right| = \left| \frac{x+1}{3} \right| (\ln n)^{1/n}
\]

As \( n \to \infty \), the term \( y = (\ln n)^{1/n} \) has the indeterminate form \( \infty^0 \). Since the variable is in the exponent, take the limit of \( \ln y \), which is

\[
\ln \left( (\ln n)^{1/n} \right) = \frac{1}{n} \ln(\ln n) = \frac{\ln(\ln n)}{n} \xrightarrow{\text{HR}} \frac{1}{n \ln n} \to 0.
\]

Therefore, by l’Hôpital’s Rule, \( \ln y \to 0 \) as \( n \to \infty \), and therefore \( y = e^{\ln y} \to e^0 = 1 \).

This means that the limit in the root test is \( \frac{|x+1|}{3} \), and the rest of the solution is the same as with the Ratio test.

2a(5 pts).(Source: 11.1.37) Simplify \( \frac{(n+1)!}{(n-1)!} \) by canceling common factors:

\[
\frac{(n+1)!}{(n-1)!} = \frac{1 \cdot 2 \cdot 3 \cdots (n-1)n(n+1)}{1 \cdot 2 \cdot 3 \cdots (n-1)} = n(n+1),
\]

which goes to \( \infty \) as \( n \to \infty \).

2b(5 pts).(Source: 11.1.27) By limit 1 of Flesk*, \( \lim_{n \to \infty} 3^n 5^{1-n} = \lim_{n \to \infty} (\frac{3}{5})^n \cdot 5 = 0. \)

* https://kunklet.people.cofc.edu/MATH220/flesk.pdf
2c (5 pts). (Source: 11.1.35) \[ \frac{(-1)^n}{1+n} \] equals \( \frac{1}{n+1} \) if \( n \) is even and \(-\frac{1}{n+1}\) is \( n \) is odd, so it’s fair to say that

\[ \frac{-1}{n+1} \leq \frac{(-1)^n}{1+n} \leq \frac{1}{n+1}. \]

Since \( \lim_{n \to \infty} \frac{-1}{n+1} = \lim_{n \to \infty} \frac{1}{n+1} = 0 \), the Squeeze theorem implies that \( \lim_{n \to \infty} \frac{(-1)^n}{1+n} \) also equals 0.

3a (10 pts). (Source: 11.2.32) \[
\sum_{n=0}^{\infty} \left(0.1\right)^{n-1} \quad \text{and} \quad \sum_{n=0}^{\infty} \left(0.9\right)^{n} \]
are both convergent geometric series (with \( r = 0.1 \) in the first case and 0.9 in the second, both between \(-1 \) and 1). Therefore

\[
\sum_{n=0}^{\infty} \left[ \left(0.1\right)^{n-1} - \left(0.9\right)^{n} \right] = \sum_{n=0}^{\infty} \left(0.1\right)^{n-1} - \sum_{n=0}^{\infty} \left(0.9\right)^{n} = \frac{a}{1-0.1} - \frac{\tilde{a}}{1-0.9},
\]

where \( a \) and \( \tilde{a} \) are the first terms of each series. Therefore the sum is \( \frac{10}{0.9} - \frac{1}{0.1} \) (which could be rewritten \( \frac{100}{9} - 10 = 10 \)).

3b (10 pts). (Source: 11.2.37, 11.1.38) \[
\lim_{n \to \infty} \frac{\ln n}{\ln(4n)} = \lim_{n \to \infty} \frac{\ln n}{\ln n + \ln 4} = \lim_{n \to \infty} \frac{1}{1 + \frac{\ln 4}{\ln n}} = \frac{1}{1+0} = 1.
\]
You could instead take this limit using l’Hospital’s Rule:

\[
\lim_{n \to \infty} \frac{\ln n}{\ln(4n)} \rightarrow \frac{\infty}{\infty} \quad \text{by} \quad \text{HR} \quad \rightarrow \quad \frac{1}{4} \cdot \frac{n}{\ln n} \rightarrow 1.
\]

Since this limit is nonzero, \( \sum_{n=1}^{\infty} \frac{\ln(n)}{\ln(4n)} \) diverges by the nth term test (also known as the Test for Divergence) p. 713 of our text.

4a (10 pts). (Source: 11.4.10) Since \( 0 \leq \sin^2 n \leq 1 \),

\[
0 \leq \frac{\sin^2 n}{3^n + 1} \leq \frac{1}{3^n + 1} \leq \frac{1}{3^n},
\]

and since \( \sum_{n=0}^{\infty} \left(\frac{1}{3}\right)^n \) is a convergent geometric series (convergent because \( r = \frac{1}{3} \) is between \(-1 \) and 1), \( \sum_{n=1}^{\infty} \frac{\sin^2 n}{3^n + 1} \) is convergent by the Comparison test.

4b (13 pts). (Source: 11.4.16) Limit-compare this positive series with the Harmonic series \( \sum_{n=1}^{\infty} \frac{1}{n} \):

\[
\lim_{n \to \infty} \frac{n^2 + 2}{n^3 + 3} = \lim_{n \to \infty} \frac{n^3 + 2n}{n^3 + 3}.
\]

Either apply l’Hôpital’s Rule to this or, as shown below, divide top and bottom by \( n^3 \):

\[
\lim_{n \to \infty} \frac{1 + \frac{2}{n^2}}{1 + \frac{3}{n^3}} = 1
\]

Since this limit is positive and finite, and since the Harmonic series is known to diverge, \( \sum_{n=0}^{\infty} \frac{n^2 + 2}{n^3 + 3} \) also diverges by the Limit Comparison test.
5. (10 pts) (Source: 11.3.3) Because the function \( \frac{1}{x^4} \) is positive and decreasing on \([1, \infty)\), the Integral test implies that the error

\[
\sum_{n=1}^{\infty} \frac{1}{n^4} - \sum_{n=1}^{100} \frac{1}{n^4} = \sum_{n=101}^{\infty} \frac{1}{n^4} \leq \int_{100}^{\infty} \frac{1}{x^4} \, dx.
\]

(See page 44 of https://kunklet.people.cofc.edu/MATH220/220review.pdf.) We calculate the improper integral to be

\[
\lim_{b \to \infty} \int_{100}^{b} x^{-4} \, dx = \lim_{b \to \infty} -\frac{1}{3} x^{-3} \bigg|_{100}^{b} = \lim_{b \to \infty} \left( -\frac{1}{3} b^{-3} + \frac{1}{3} \cdot 100^{-3} \right) = \frac{1}{3} \cdot 100^{-3},
\]

and therefore the error is \( \leq \frac{1}{3} \cdot 10^{-6} \).

6. (18 pts) (Source: 11.6.38,11.3.21,11.5.6) First test for absolute convergence. Since \((x \ln x)^{-1}\) is positive and decreasing on \([2, \infty)\), the integral test says that the series \(\sum_{n=2}^{\infty} (n \ln n)^{-1}\) and the improper integral \(\int_{2}^{\infty} (x \ln x)^{-1} \, dx\) either both converge or both diverge. The indefinite integral \(\int (x \ln x)^{-1} \, dx = \ln(\ln x) + C\), and so the improper integral

\[
\int_{2}^{\infty} \frac{1}{x \ln x} \, dx = \lim_{c \to \infty} \ln(\ln x) \bigg|_{2}^{c} = \lim_{c \to \infty} \left( \ln(\ln c) - \ln(\ln 2) \right) = \infty.
\]

Therefore \(\sum_{n=2}^{\infty} (n \ln n)^{-1}\) also diverges, and the \(\sum_{n=2}^{\infty} (-1)^n (n \ln n)^{-1}\) fails to converge absolutely.

Now test the original series for convergence using the Alternating Series test. \((n \ln n)^{-1}\) decreases and goes to zero as \(n \to \infty\), so the AST tells us that \(\sum_{n=2}^{\infty} (-1)^n (n \ln n)^{-1}\) converges. Since it does not converge absolutely, the series converges conditionally.