No notes, books, electronic devices, or outside materials of any kind.

Read each problem carefully and simplify your answers.

Unless otherwise indicated, supporting work will be required on every problem worth more than 2 points.

You are expected to know the values of all trig functions at multiples of $\pi/4$ and of $\pi/6$.

1(14 pts). Find the radius of convergence of the power series $\displaystyle\sum_{n=2}^{\infty} \frac{(x+1)^n}{3^n \ln n}$.

2(15 pts). Evaluate the limit. Show your work, but your answer to each should be a number, ∞, $-\infty$, or "does not exist".

a. $\displaystyle\lim_{n\to\infty} \frac{(n+1)!}{(n-1)!}$ b. $\displaystyle\lim_{n\to\infty} 3^n 5^{1-n}$ c. $\displaystyle\lim_{n\to\infty} \frac{(-1)^n}{1+n}$

3(20 pts). Find the sum, if it exists.

a. $\displaystyle\sum_{n=0}^{\infty} \left((0.1)^{n-1} - (0.9)^n\right)$ b. $\displaystyle\sum_{n=1}^{\infty} \frac{\ln(n)}{\ln(4n)}$

4(23 pts). Determine whether the series converges or diverges.

a. $\displaystyle\sum_{n=1}^{\infty} \frac{\sin^2 n}{3^n+1}$ b. $\displaystyle\sum_{n=0}^{\infty} \frac{n^2+2}{n^3+3}$

5(10 pts). Find an upper bound for the error that occurs when we approximate the sum of the series $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^4}$ with the partial sum $\displaystyle\sum_{n=1}^{100} \frac{1}{n^4}$.
(That is, find a number # which satisfies error < #.)

6(18 pts). Determine whether the series $\displaystyle\sum_{n=2}^{\infty} \frac{(-1)^n}{n \ln n}$ converges absolutely, converges conditionally, or diverges.

1(14 pts).(Source: 11.8.17) The solution to a radius-of-convergence problem typically begins with either the Root or Ratio test. Here's a solution with Ratio:

$$\frac{\left|\frac{(x+1)^{n+1}}{3^{n+1}\ln(n+1)}\right|}{\left|\frac{(x+1)^n}{3^n\ln n}\right|} = \frac{|x+1|^{n+1}}{|x+1|^n}\cdot\frac{3^n}{3^{n+1}}\cdot\frac{\ln n}{\ln(n+1)} = \frac{|x+1|}{3}\cdot\frac{\ln n}{\ln(n+1)}.$$

To take the limit of $\frac{\ln n}{\ln(n+1)}$, use l'Hospital's Rule:

$$\frac{\ln n}{\ln(n+1)} \to \text{“}\frac{\infty}{\infty}\text{”} \overset{HR}{\hookrightarrow} \frac{\frac{1}{n}}{\frac{1}{n+1}} = \frac{n+1}{n} = 1+\frac{1}{n} \to 1.$$

Therefore,

$$\lim_{n\to\infty}\frac{|x+1|}{3}\cdot\frac{\ln n}{\ln(n+1)} = \frac{|x+1|}{3}.$$

The Ratio test tells us that the series converges (absolutely) when

$$\frac{|x+1|}{3} < 1, \quad \text{or} \quad |x+1| < 3,$$

and so the Radius of Convergence is 3.

If you used the Root test, you would need to take the limit of

$$\left|\frac{(x+1)^n}{3^n\ln n}\right|^{1/n} = \frac{|x+1|}{3}(\ln n)^{1/n}$$

As $n\to\infty$, the term $y=(\ln n)^{1/n}$ has the indeterminate form ∞^0. Since the variable is in the exponent, take the limit of $\ln y$, which is

$$\ln\big((\ln n)^{1/n}\big) = \frac{1}{n}\ln(\ln n) = \frac{\ln(\ln n)}{n} \to \text{“}\frac{\infty}{\infty}\text{”} \overset{HR}{\hookrightarrow} \frac{\frac{1}{n\ln n}}{1} \to 0.$$

Therefore, by l'Hôpital's Rule, $\ln y\to 0$ as $n\to\infty$, and therefore $y=e^{\ln y}\to e^0=1$. This means that the limit in the root test is $\frac{|x+1|}{3}$, and the rest of the solution is the same as with the Ratio test.

2a(5 pts).(Source: 11.1.37) Simplify $\frac{(n+1)!}{(n-1)!}$ by canceling common factors:

$$\frac{(n+1)!}{(n-1)!} = \frac{1\cdot 2\cdot 3\cdots(n-1)n(n+1)}{1\cdot 2\cdot 3\cdots(n-1)} = n(n+1),$$

which goes to ∞ as $n\to\infty$.

2b(5 pts).(Source: 11.1.27) By limit 1 of Flesk*, $\lim_{n\to\infty}3^n5^{1-n} = \lim_{n\to\infty}\left(\frac{3}{5}\right)^n\cdot 5 = 0$.

* https://kunklet.people.cofc.edu/MATH220/flesk.pdf

2c(5 pts).(Source: 11.1.35) $\frac{(-1)^n}{1+n}$ equals $\frac{1}{n+1}$ if n is even and $\frac{-1}{n+1}$ is n is odd, so it's fair to say that

$$\frac{-1}{n+1} \leq \frac{(-1)^n}{1+n} \leq \frac{1}{n+1}.$$

Since $\lim_{n\to\infty} \frac{-1}{n+1} = \lim_{n\to\infty} \frac{1}{n+1} = 0$, the Squeeze theorem implies that $\lim_{n\to\infty} \frac{(-1)^n}{1+n}$ also equals 0.

3a(10 pts). (Source: 11.2.32) $\sum_{n=0}^{\infty}(0.1)^{n-1}$ and $\sum_{n=0}^{\infty}(0.9)^n$ are both convergent geometric series (with $r = 0.1$ in the first case and 0.9 in the second, both between -1 and 1). Therefore

$$\sum_{n=0}^{\infty}\left((0.1)^{n-1} - (0.9)^n\right) = \sum_{n=0}^{\infty}(0.1)^{n-1} - \sum_{n=0}^{\infty}(0.9)^n = \frac{a}{1-0.1} - \frac{\tilde{a}}{1-0.9},$$

where a and $\tilde{a}$ are the first terms of each series. Therefore the sum is $\frac{10}{0.9} - \frac{1}{0.1}$ (which could be rewritten $\frac{100}{9} - 10 = \frac{10}{9}$).

3b(10 pts).(Source: 11.2.37,11.1.38) $\lim_{n\to\infty} \frac{\ln n}{\ln(4n)} = \lim_{n\to\infty} \frac{\ln n}{\ln n + \ln 4} = \lim_{n\to\infty} \frac{1}{1+\frac{\ln 4}{\ln n}} = \frac{1}{1+0} = 1$. You could instead take this limit using l'Hospital's Rule:

$$\frac{\ln n}{\ln(4n)} \to \text{“}\frac{\infty}{\infty}\text{”} \overset{HR}{\hookrightarrow} \frac{\frac{1}{n}}{4\cdot\frac{1}{4n}} = 1 \to 1.$$

Since this limit is nonzero, $\sum_{n=1}^{\infty} \frac{\ln(n)}{\ln(4n)}$ diverges by the nth term test (also known as the Test for Divergence) [7], p. 713 of our text.

4a(10 pts). (Source: 11.4.10) Since $0 \leq \sin^2 n \leq 1$,

$$0 \leq \frac{\sin^2 n}{3^n+1} \leq \frac{1}{3^n+1} \leq \frac{1}{3^n},$$

and since $\sum_{n=0}^{\infty}(\frac{1}{3})^n$ is a convergent geometric series (convergent because $r = \frac{1}{3}$ is between -1 and 1), $\sum_{n=1}^{\infty} \frac{\sin^2 n}{3^n+1}$ is convergent by the Comparison test.

4b(13 pts).(Source: 11.4.16) Limit-compare this positive series with the Harmonic series $\sum_{n=1}^{\infty} \frac{1}{n}$:

$$\lim_{n\to\infty} \frac{n^2+2}{n^3+3} \div \frac{1}{n} = \lim_{n\to\infty} \frac{n^3+2n}{n^3+3}.$$

Either apply l'Hôpital's Rule to this or, as shown below, divide top and bottom by n^3:

$$\lim_{n\to\infty} \frac{1+\frac{2}{n^2}}{1+\frac{3}{n^3}} = 1$$

Since this limit is positive and finite, and since the Harmonic series is known to diverge, $\sum_{n=0}^{\infty} \frac{n^2+2}{n^3+3}$ also diverges by the Limit Comparison test.

5(10 pts).(Source: 11.3.more) Because the function $\frac{1}{x^4}$ is positive and decreasing on $[1, \infty)$, the Integral test implies that the error

$$\sum_{n=1}^{\infty} \frac{1}{n^4} - \sum_{n=1}^{100} \frac{1}{n^4} = \sum_{n=101}^{\infty} \frac{1}{n^4} \leq \int_{100}^{\infty} \frac{1}{x^4}\, dx.$$

(See page 44 of https://kunklet.people.cofc.edu/MATH220/220review.pdf.) We calculate the improper integral to be

$$\lim_{b\to\infty} \int_{100}^{b} x^{-4}\, dx = \lim_{b\to\infty} -\frac{1}{3}x^{-3}\Big|_{100}^{b} = \lim_{b\to\infty} -\frac{1}{3}b^{-3} + \frac{1}{3}\cdot 100^{-3} = \frac{1}{3}\cdot 100^{-3},$$

and therefore the error is $\leq \frac{1}{3}\cdot 10^{-6}$.

6(18 pts). (Source: 11.6.38,11.3.21,11.5.6) First test for absolute convergence. Since $(x\ln x)^{-1}$ is positive and decreasing on $[2, \infty)$, the integral test says that the series $\sum_{n=2}^{\infty}(n\ln n)^{-1}$ and the improper integral $\int_2^{\infty}(x\ln x)^{-1}\, dx$ either both converge or both diverge. The indefinite integral $\int (x\ln x)^{-1}\, dx = \ln(\ln x) + C$, and so the improper integral

$$\int_2^{\infty} \frac{1}{x\ln x}\, dx = \lim_{c\to\infty} \ln(\ln x)\Big|_2^c = \lim_{c\to\infty} \big(\ln(\ln c) - \ln(\ln 2)\big) = \infty.$$

Therefore $\sum_{n=2}^{\infty}(n\ln n)^{-1}$ also diverges, and the $\sum_{n=2}^{\infty}(-1)^n(n\ln n)^{-1}$ fails to converge absolutely.
Now test the original series for convergence using the Alternating Series test. $(n\ln n)^{-1}$ decreases and goes to zero as $n \to \infty$, so the AST tells us that $\sum_{n=2}^{\infty}(-1)^n(n\ln n)^{-1}$ converges. Since it does not converge absolutely, the series converges conditionally.