No notes, books, electronic devices, or outside materials of any kind.

Read each problem carefully and simplify your answers.

Unless otherwise indicated, supporting work will be required on every problem worth more than 2 points.

You are expected to know the values of all trig functions at multiples of $\pi/4$ and of $\pi/6$.

You may use, without proof, any of these **reduction formulas** that are relevant.

$$\int \sin^n x\,dx = -\frac{1}{n}\sin^{n-1} x\cos x + \frac{n-1}{n}\int \sin^{n-2} x\,dx$$
$$\int \cos^n x\,dx = \frac{1}{n}\cos^{n-1} x\sin x + \frac{n-1}{n}\int \cos^{n-2} x\,dx$$
$$\int \tan^n x\,dx = \frac{1}{n-1}\tan^{n-1} x - \int \tan^{n-2} x\,dx$$
$$\int \sec^n x\,dx = \frac{1}{n-1}\sec^{n-2} x\tan x + \frac{n-2}{n-1}\int \sec^{n-2} x\,dx \qquad (n \neq 1)$$

1(11 pts). Evaluate the limit: $\lim_{x\to 1} x^{\frac{1}{1-x}}$

2(16 pts). Find the length of the curve $y = \ln(\sec x)$ for $0 \le x \le \frac{\pi}{4}$.

3(28 pts). Evaluate the indefinite integral:

a. $\displaystyle\int \frac{dx}{(x^2-4)^{3/2}}$ b. $\displaystyle\int \frac{x^2+2x}{x-2}\,dx$

4(13 pts). Find the partial fraction decomposition of $\dfrac{5x^2-15x+7}{(x+1)(x-2)^2}$. You are not required to integrate this function.

5(13 pts). Evaluate the improper integral $\displaystyle\int_0^3 \frac{1}{(x-2)^3}\,dx$ or show that it diverges.

6(10 pts). Approximate the integral $\displaystyle\int_{1/2}^2 \sqrt{1+\ln x}\,dx$ using Simpson's Rule with $n = 6$ subintervals. You are not required to express your answer in decimal form.

7(9 pts). Suppose

$$|f^{(2)}(x)| \le 7 \qquad |f^{(3)}(x)| \le 9 \qquad |f^{(4)}(x)| \le 16 \qquad |f^{(5)}(x)| \le 31$$

on $[10, 20]$.

How large an n must we use to approximate $\int_{10}^{20} f(x)\,dx$ using the trapezoid rule with an absolute error at most 10^{-5}? Write your answer in an inequality of the form $n \ge \#$ for some number #. You are not required to express your answer in decimal form.

Hint:

$$|E_T| \leq \frac{K(b-a)^3}{12n^2} \qquad\qquad |E_S| \leq \frac{L(b-a)^5}{180n^4}$$

1(11 pts).(Source: 4.4.61) This limit is of the indeterminate form 1^∞. Let $y = x^{\frac{1}{1-x}}$ and take the limit as $x \to 1$ of

$$\ln y = \ln x^{\frac{1}{1-x}} = \frac{\ln x}{1-x} \to \text{“}\frac{0}{0}\text{”} \overset{HR}{\hookrightarrow} \frac{x^{-1}}{-1},$$

which goes to -1 as $x \to 1$. Therefore, by l'Hospital's Rule, $\lim_{x\to 1} \ln y$ also equals -1, and so $\lim_{x\to 1} y = \lim_{x\to 1} e^{\ln y} = e^{-1}$.

2(16 pts).(Source: 8.1.14,15) Recall that $\frac{d}{dx} \ln|\sec x| = \tan x$.
The required arclength equals $\int ds = \int_{x=0}^{x=\pi/4} \sqrt{1 + (\frac{dy}{dx})^2}\, dx =$

$$\int_0^{\pi/4} \sqrt{1 + (\tan x)^2}\, dx = \int_0^{\pi/4} \sqrt{\sec^2 x}\, dx.$$

Since $\sec x > 0$ on the interval $[0, \pi/4]$, the integral equals

$$\int_0^{\pi/4} \sec x\, dx = \ln|\tan x + \sec x|\Big|_0^{\pi/4}$$
$$= \ln\left|\tan\frac{\pi}{4} + \sec\frac{\pi}{4}\right| - \ln|\tan 0 + \sec 0| = \ln|1+\sqrt{2}| - \ln|0+1| = \ln(1+\sqrt{2})$$

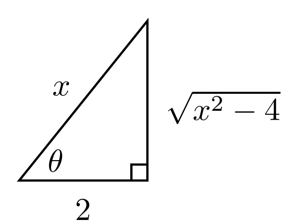
3a(18 pts). (Source: 7.3.18) We want $x^2 - 4 = 4\sec^2\theta - 4$, which equals $4\tan^2\theta$, so let $x = 2\sec\theta$. This necessitates $dx = 2\sec\theta\tan\theta\, d\theta$, and the integral becomes

$$\int \frac{2\sec\theta\tan\theta}{(4\tan^2\theta)^{3/2}}\, d\theta = \int \frac{2\sec\theta\tan\theta}{8\tan^3\theta}\, d\theta = \frac{1}{4}\int \frac{\sec\theta}{\tan^2\theta}\, d\theta.$$

Rewrite in terms of sine and cosine and substitute $u = \sin\theta$:

$$= \frac{1}{4}\int \frac{\cos\theta}{\sin^2\theta}\, d\theta = \frac{1}{4}\int u^{-2}\, du = -\frac{1}{4}u^{-1} + C = -\frac{1}{4}(\sin\theta)^{-1} + C.$$

To rewrite this answer in terms of the original variable x, draw a right triangle with interior angle θ. Label two sides using $\sec\theta = x/2$, and then find the third side by the Pythagorean theorem, as shown in the figure. Consequently, the integral equals



$$-\frac{1}{4}(\sin\theta)^{-1} = -\frac{1}{4}\frac{x}{\sqrt{x^2-4}} + C.$$

3b(10 pts).(Source: 7.4.7,8) When the degree of the numerator fails to be less than that of the demominator, perform long division:

$$\begin{array}{r|l} & x+4 \\ x-2 & x^2+2x \\ & -(x^2-2x) \\ & \quad 4x \\ & \quad -(4x-8) \\ & \qquad 8 \end{array}$$

Long division continues until the degree of the remainder is less than that of the divisor. Now integrate:

$$\int \frac{x^2+2x}{x-2}\,dx = \int\left(x+4+\frac{8}{x-2}\right)dx$$
$$= \frac{1}{2}x^2 + 4x + 8\ln|x-2| + C$$

(corrected) ($\frac{8}{x-2}$ already has the form $\frac{A}{x-2}$, so there's no need to search for its PFD.)

4(13 pts).(Source: 7.4.19) The partial fraction decomposition has the form

$$\frac{5x^2-15x+7}{(x+1)(x-2)^2} = \frac{A}{x+1} + \frac{B}{x-2} + \frac{C}{(x-2)^2}.$$

Multiply both sides by $(x+1)(x-2)^2$:

$$5x^2-15x+7 = A(x-2)^2 + B(x+1)(x-2) + C(x+1).$$

Now find a system of three equations in A, B, and C. Solving will be easy if we generate two of these by evaluating at $x=-1$ and $x=2$. For the third, I evaluated at $x=0$, but there are many different correct third equations.

$$\begin{array}{lrcl} \text{at } x=-1, & 27 = 9A & & A=3 \\ \text{at } x=2, & -3 = C & \Longrightarrow & B=2 \\ \text{at } x=0, & 7 = 4A-2B+C & & C=-1 \\ & = 12-2B-1 & & \end{array}$$

and so the PFD is $\frac{3}{x+1} + \frac{2}{x-2} - \frac{1}{(x-2)^2}$.

5(13 pts).(Source: 7.8.29,21,33) The integrand is unbounded at x near 2, so we must break up the integral into two:

$$\int_0^2 \frac{1}{(x-2)^3}\,dx + \int_2^3 \frac{1}{(x-2)^3}\,dx.$$

For the original integral to converge, we insist that both of these converge.

The first equals the limit

$$\int_0^2 \frac{1}{(x-2)^3}\,dx = \lim_{\zeta\to 2^-}\int_0^\zeta (x-2)^{-3}\,dx = \lim_{\zeta\to 2^-} -\frac{1}{2}(x-2)^{-2}\Big|_0^\zeta$$
$$= \lim_{\zeta\to 2^-} -\frac{1}{2}(\zeta-2)^{-2} + \frac{1}{2}(-2)^{-2}.$$

Since $(\zeta-2)^{-2} = \frac{1}{(\zeta-2)^2} \to \infty$ as $\zeta \to 2$, the improper integral $\int_0^2 \frac{1}{(x-2)^3}\,dx$ diverges (to $-\infty$). Therefore, $\int_0^3 \frac{1}{(x-2)^3}\,dx$ also diverges.

6(10 pts).(Source: 7.7.17) Divide the interval $[1/2, 2]$ into 6 subintervals length $\Delta x = (2 - \frac{1}{2})/6 = 1/4$, and their endpoints will be 1/2, 3/4, 1, 5/4, 3/2, 7/4, and 2. According to Simpson's rule,

$$\int_{1/2}^2 \sqrt{1+\ln x}\,dx \approx \frac{1/4}{3}\Big(\sqrt{1+\ln(1/2)} + 4\sqrt{1+\ln(3/4)} + 2\sqrt{1+\ln(1)} + 4\sqrt{1+\ln(5/4)}$$
$$+2\sqrt{1+\ln(3/2)} + 4\sqrt{1+\ln(7/4)} + \sqrt{1+\ln(2)}\Big)$$

7(9 pts).(Source: 7.7.more)

The absolute error in the trapezoid rule $|E_T|$ is bounded above by $\frac{K(b-a)^3}{12n^2}$, where K is an upper bound on $|f^{(2)}(x)|$ on $[10, 20]$. So, using $K = 7$, to ensure that $|E_T| \le 10^{-5}$, we'll choose n so as to ensure

$$\frac{7(20-10)^3}{12n^2} \le 10^{-5}.$$

To solve this inequality for n, multiply by $10^5 n^2$ and take square roots. Because multiplication by a positive number and the square root are both increasing functions, applying them to both sides preserves the direction of the inequality:

$$\frac{7(10)^3}{12n^2}10^5 \le n^2 \quad\Longrightarrow\quad \sqrt{\frac{7(10)^8}{12}} \le n,$$

or

$$n \ge \sqrt{\frac{7}{12}}10^4$$