1.1: Systems of linear equations

Solutions

A **solution** to a **system of linear equations** in the variables $x_1, x_2, \ldots, x_n$

$$\begin{aligned} a_{1,1}x_1 + a_{1,2}x_2 + \cdots + a_{1,n}x_n &= b_1 \\ a_{2,1}x_1 + a_{2,2}x_2 + \cdots + a_{2,n}x_n &= b_2 \\ &\vdots \\ a_{m,1}x_1 + a_{m,2}x_2 + \cdots + a_{m,n}x_n &= b_m \end{aligned} \tag{1.1.1}$$

is a set of values of $x_1, x_2, \ldots, x_n$ that together satisfy every equation in the system. A linear system of m equations in n unknowns is sometimes called an $m \times n$ system.

1.1.re1. $(x_1, x_2) = (2, 1)$ is a solution to the 2×2 linear system

$$\begin{aligned} x_1 + 2x_2 &= 4 \\ 5x_1 - 2x_2 &= 8 \end{aligned} \tag{1.1.2}$$

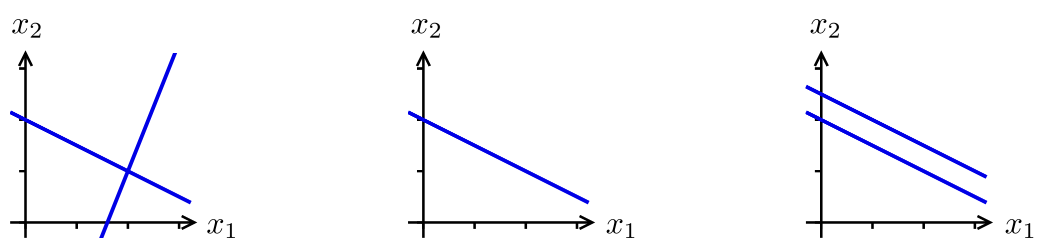
because both equations of the system are true at these x-values:

$$\begin{aligned} 2 + 2(1) &= 4 \\ 5(2) - 2(1) &= 8 \end{aligned}$$

In 1.1.re2, we see that $(x_1, x_2) = (2, 1)$ is the *only* solution to (1.1.2).

A linear system can have either no solutions, one solution, or infinitely many solutions. A system is said to be **consistent** if it has at least one solution and **inconsistent** otherwise.

1.1.re2.



$x_1 + 2x_2 = 4$	$x_1 + 2x_2 = 4$	$x_1 + 2x_2 = 4$
$5x_1 - 2x_2 = 8$	$2x_1 + 4x_2 = 8$	$2x_1 + 4x_2 = 5$
one solution	infinitely many solutions	no solutions
consistent	consistent	inconsistent

Elementary row operations

The three types of elementary row operations are

interchange:	$\mathbf{r}_i \rightleftarrows \mathbf{r}_j$
replacement:	$\mathbf{r}_i \leftarrow \mathbf{r}_i + c\,\mathbf{r}_j$
scaling:	$\mathbf{r}_i \leftarrow c\,\mathbf{r}_i \quad (c \neq 0)$

Here, $\mathbf{r}_i$ denotes the ith row in the matrix, and $\leftarrow$ denotes the assignment of a new value. An elementary row operation

(1) does not change the number of rows in the matrix, and
(2) can be undone by another elementary row operation.

1.1.re3. Perform the given elementary row operation on the matrix

$$W = \begin{bmatrix} 1 & 0 & 2 & 3 \\ 4 & 5 & 0 & 6 \\ 7 & 8 & 9 & 0 \end{bmatrix}$$

What row operation will return the matrix to its original state?

a. $\mathbf{r}_2 \leftarrow -4\,\mathbf{r}_2$ b. $\mathbf{r}_3 \leftarrow \mathbf{r}_3 + 5\,\mathbf{r}_1$ c. $\mathbf{r}_1 \rightleftarrows \mathbf{r}_3$

Augmented matrix form; row reduction

To solve the linear system (1.1.1), rewrite it in **augmented matrix form**

$$\begin{bmatrix} a_{1,1} & a_{1,2} & \cdots & a_{1,n} & b_1 \\ a_{2,1} & a_{2,2} & \cdots & a_{2,n} & b_2 \\ \vdots & \vdots & & \vdots & \vdots \\ a_{m,1} & a_{m,2} & \cdots & a_{m,n} & b_m \end{bmatrix} \tag{1.1.3}$$

and then row-reduce by a sequence of elementary row operations. (We'll discuss row reduction in greater detail in section 1.2.)

1.1.re4. Solve the linear system by row reduction.

$$\begin{aligned} x_1 + 2x_2 &= 4 \\ 5x_1 - 2x_2 &= 8 \end{aligned}$$

row operation	aug. matrix	linear system	inverse row op.
	$\begin{matrix} 1 & 2 & 4 \\ 5 & -2 & 8 \end{matrix}$	$\begin{aligned} x_1 + 2x_2 &= 4 \\ 5x_1 - 2x_2 &= 8 \end{aligned}$	$\mathbf{r}_2 \leftarrow \mathbf{r}_2 + 5\,\mathbf{r}_1$
$\mathbf{r}_2 \leftarrow \mathbf{r}_2 - 5\,\mathbf{r}_1$	$\begin{matrix} 1 & 2 & 4 \\ 0 & -12 & -12 \end{matrix}$	$\begin{aligned} x_1 + 2x_2 &= 4 \\ -12x_2 &= -12 \end{aligned}$	$\mathbf{r}_2 \leftarrow -12\,\mathbf{r}_2$
$\mathbf{r}_2 \leftarrow -\frac{1}{12}\,\mathbf{r}_2$	$\begin{matrix} 1 & 2 & 4 \\ 0 & 1 & 1 \end{matrix}$	$\begin{aligned} x_1 + 2x_2 &= 4 \\ x_2 &= 1 \end{aligned}$	$\mathbf{r}_1 \leftarrow \mathbf{r}_1 + 2\,\mathbf{r}_2$
$\mathbf{r}_1 \leftarrow \mathbf{r}_1 - 2\,\mathbf{r}_2$	$\begin{matrix} 1 & 0 & 2 \\ 0 & 1 & 1 \end{matrix}$	$\begin{aligned} x_1 &= 2 \\ x_2 &= 1 \end{aligned}$	

end example 1.1.re4

Two matrices are **row equivalent** if one can be obtained from the other by a sequence of row operations, and two *linear systems* are **row equivalent** if their augmented matrices are row equivalent. If the matrices A and B are row equivalent, we write

$$A \sim B$$

1.1.re4, continued. Because

$$\begin{bmatrix} 1 & 2 & 4 \\ 5 & -2 & 8 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 1 \end{bmatrix},$$

we say that the linear systems

$$\begin{aligned} x_1 + 2x_2 &= 4 \\ 5x_1 - 2x_2 &= 8 \end{aligned} \quad \text{and} \quad \begin{aligned} x_1 &= 4 \\ x_2 &= 1 \end{aligned}$$

are also row equivalent.

Fact 1.1.4. *If two linear systems are row equivalent, then they have the same solutions.*

In row reduction, our goal is to reduce the original system to one whose solutions are obvious.

1.1.re5. Solve the linear system.

a.

$$\begin{aligned} x_1 + 5x_2 + x_3 &= 20 \\ 2x_1 + 9x_2 + 5x_3 &= 45 \\ x_1 + 5x_2 + 2x_3 &= 23 \end{aligned}$$

b.

$$\begin{aligned} x_1 - 2x_2 + x_3 &= -5 \\ -3x_1 + 8x_2 - 2x_3 &= 17 \\ x_1 + 2x_2 + 3x_3 &= 2 \end{aligned}$$

row op.	**aug. matrix**
	$\begin{matrix} 1 & 5 & 1 & 20 \\ 2 & 9 & 5 & 45 \\ 1 & 5 & 2 & 23 \end{matrix}$
$\mathbf{r}_2 \leftarrow \mathbf{r}_2 - 2\,\mathbf{r}_1$ $\mathbf{r}_3 \leftarrow \mathbf{r}_3 - \mathbf{r}_1$	$\begin{matrix} 1 & 5 & 1 & 20 \\ 0 & -1 & 3 & 5 \\ 0 & 0 & 1 & 3 \end{matrix}$
$\mathbf{r}_2 \leftarrow \mathbf{r}_2 - 3\,\mathbf{r}_3$ $\mathbf{r}_1 \leftarrow \mathbf{r}_1 - \mathbf{r}_3$	$\begin{matrix} 1 & 5 & 0 & 17 \\ 0 & -1 & 0 & -4 \\ 0 & 0 & 1 & 3 \end{matrix}$
$\mathbf{r}_1 \leftarrow \mathbf{r}_1 + 5\,\mathbf{r}_2$ $\mathbf{r}_2 \leftarrow \, - \mathbf{r}_2$	$\begin{matrix} 1 & 0 & 0 & -3 \\ 0 & 1 & 0 & 4 \\ 0 & 0 & 1 & 3 \end{matrix}$

row op.	**aug. matrix**
	$\begin{matrix} 1 & -2 & 1 & -5 \\ -3 & 8 & -2 & 17 \\ 1 & 2 & 3 & 2 \end{matrix}$
$\mathbf{r}_2 \leftarrow \mathbf{r}_2 + 3\,\mathbf{r}_1$ $\mathbf{r}_3 \leftarrow \mathbf{r}_3 - \mathbf{r}_1$	$\begin{matrix} 1 & -2 & 1 & -5 \\ 0 & 2 & 1 & 2 \\ 0 & 4 & 2 & 7 \end{matrix}$
$\mathbf{r}_3 \leftarrow \mathbf{r}_3 - 2\,\mathbf{r}_2$	$\begin{matrix} 1 & -2 & 1 & -5 \\ 0 & 2 & 1 & 2 \\ 0 & 0 & 0 & 3 \end{matrix}$

In a., the final augmented matrix translates to the single solution:

$$x_1 = -3,\; x_2 - = 4,\; x_3 = 3.$$

But in b., the bottom row of the last matrix translates to $0x_1 + 0x_2 + 0x_3 = 3$, and so the system has no solutions.

1.1.re6. Solve the system (expressed here in augmented matrix form).

a. $\begin{bmatrix} 1 & 3 & -1 \\ -2 & -5 & 1 \end{bmatrix}$ b. $\begin{bmatrix} 3 & 2 & 22 \\ -12 & -7 & -83 \end{bmatrix}$ c. $\begin{bmatrix} 1 & -2 & -3 \\ 5 & -9 & -15 \end{bmatrix}$

d. $\begin{bmatrix} 2 & 4 & -19 \\ 2 & 5 & -24 \end{bmatrix}$ e. $\begin{bmatrix} 1 & 2 & -6 \\ -1 & -1 & -2 \\ 2 & 3 & -1 \end{bmatrix}$ f. $\begin{bmatrix} 1 & 1 & 2 \\ -1 & 1 & 8 \\ 2 & 5 & 19 \end{bmatrix}$

g. $\begin{bmatrix} 1 & 4 & 0 & 7 \\ 3 & 13 & 0 & 23 \\ 1 & 6 & 1 & 16 \end{bmatrix}$ h. $\begin{bmatrix} 1 & 1 & 3 & -1 \\ 6 & 8 & 19 & -4 \\ 1 & -13 & -4 & -12 \end{bmatrix}$ i. $\begin{bmatrix} 1 & 2 & 2 & -4 \\ -4 & -6 & -4 & 15 \\ 0 & 4 & 18 & 3 \end{bmatrix}$

1.1.re7. For what value(s) of ℓ is the linear system consistent?

a. $\begin{bmatrix} 2 & \ell & 3 \\ -4 & 5 & 1 \end{bmatrix}$ b. $\begin{bmatrix} 2 & 3 & -1 \\ -4 & 6 & \ell \end{bmatrix}$ c. $\begin{bmatrix} 2 & 3 & -1 \\ -4 & -6 & \ell \end{bmatrix}$

Answers

1.1.re.e3a. $\begin{bmatrix} 1 & 0 & 2 & 3 \\ -16 & -20 & 0 & -24 \\ 7 & 8 & 9 & 0 \end{bmatrix}$; $\mathbf{r}_2 \leftarrow -\frac{1}{4}\,\mathbf{r}_2$. 1.1.re.e3b. $\begin{bmatrix} 1 & 0 & 2 & 3 \\ 4 & 5 & 0 & 6 \\ 12 & 8 & 19 & 15 \end{bmatrix}$; $\mathbf{r}_3 \leftarrow \mathbf{r}_3 - 5\,\mathbf{r}_1$

1.1.re.e3c. $\begin{bmatrix} 7 & 8 & 9 & 0 \\ 4 & 5 & 0 & 6 \\ 1 & 0 & 2 & 3 \end{bmatrix}$; $\mathbf{r}_1 \rightleftarrows \mathbf{r}_3$. 1.1.re.e6a. $(2, -1)$ 1.1.re.e6b. $(4, 5)$ 1.1.re.e6c. $(-3, 0)$

1.1.re.e6d. $(1/2, -5)$ 1.1.re.e6e. inconsistent 1.1.re.e6f. $(-3, 5)$ 1.1.re.e6g. $(-1, 2, 5)$

1.1.re.e6h. inconsistent 1.1.re.e6i. $(-2, -3/2, 1/2)$ 1.1.re.e7a. consistent for all $\ell \neq -5/2$.

1.1.re.e7b. consistent for all ℓ. 1.1.re.e7c. consistent only if $\ell = 2$.

1.2: Row reduction, echelon forms

A row (or column) consisting entirely of zeros is called a **zero** row (or column); any other row (or column) is called **nonzero**. The **lead entry** of a nonzero row is its leftmost nonzero entry. A zero row has no lead entry.

1.2.re1. The lead entries of the matrix

$$A = \begin{bmatrix} 4 & 0 & 2 & 1 & -1 & 0 \\ 0 & 0 & 0 & -2 & 0 & 3 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 3 & 0 & -3 & 0 \end{bmatrix}$$

are 4 (first row), -2 (second row), and 1 (fourth row). The third row of A has no lead entry.

A matrix is in **row echelon form**, or **ref**, if

1. all nonzero rows are above all zero rows, and
2. the lead entry of each row lies to the right of the lead entries of the rows above it.

If a matrix is in ref, then all column entries beneath any lead entry are zeros. A matrix is in **reduced row echelon form**, or **rref**, if, in addition to 1. and 2.,

3. The lead entry of each row equals 1, and
4. Each leading entry is the only nonzero element of its column.

When a matrix is in row echelon form, the lead entries of its nonzero rows are called **pivots**, and the rows and columns containing pivots are called pivot rows and pivot columns.

1.2.re1, continued. The matrix B is in row echelon form. The matrix C is in reduced row echelon form. The pivots are marked with boxes.

$$B = \begin{bmatrix} \boxed{4} & 0 & 2 & 1 & -1 & 0 \\ 0 & \boxed{1} & 3 & 0 & -3 & 0 \\ 0 & 0 & 0 & \boxed{-2} & 0 & 3 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \qquad C = \begin{bmatrix} \boxed{1} & 0 & 1/2 & 0 & -1/4 & 3/8 \\ 0 & \boxed{1} & 3 & 0 & -3 & 0 \\ 0 & 0 & 0 & \boxed{1} & 0 & -3/2 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

1.2.re2. What row operations transform A to B and B to C? Be careful: there can be more than one correct answer, but order matters.

Row reduction

Every matrix can be row reduced to one of row echelon form, and from there to reduced row echelon form, by a process called **row reduction** or **Gaussian elimination**.

In the **forward phase**, we produce a matrix in row echelon form by using row interchanges and replacements to produce zeros below each pivot, starting at the leftmost pivot column and working to the right. Specifically,

a: interchange rows, if necessary, so that the leftmost nonzero column contains a nonzero entry in the top row, and

b: subtract multiples of the pivot row from rows beneath it to obtain zeros below the pivot.

In the **backward phase**, we use row replacements to produce zeros *above* each pivot, starting at the rightmost pivot column and working to the left.

Note:

1: In the forward phase, we always add multiples of the pivot row to the rows *beneath* it.

2: In the backward phase, we always add multiples of the pivot row to the rows *above* it.

3: We never make row interchanges in during the backward phase.

4: During either the forward or backward phase, we can rescale rows whenever it's convenient. To produce a matrix in reduced row echelon form, we *must* rescale rows to make all lead entries equal 1.

1.2.re3. When $\begin{bmatrix} 0 & 8 & 4 & -1 & -14 \\ -6 & 2 & 3 & 1 & -5 \\ -12 & -20 & -6 & 6 & 34 \end{bmatrix}$ is row-reduced, the forward phase is

row operation	**result**
$\mathbf{r}_1 \rightleftarrows \mathbf{r}_2$	$\begin{matrix} \boxed{-6} & 2 & 3 & 1 & -5 \\ 0 & 8 & 4 & -1 & -14 \\ -12 & -20 & -6 & 6 & 34 \end{matrix}$
$\mathbf{r}_3 \leftarrow \mathbf{r}_3 - 2\,\mathbf{r}_1$	$\begin{matrix} \boxed{-6} & 2 & 3 & 1 & -5 \\ 0 & \boxed{8} & 4 & -1 & -14 \\ 0 & -24 & -12 & 4 & 44 \end{matrix}$
$\mathbf{r}_3 \leftarrow \mathbf{r}_3 + 3\,\mathbf{r}_2$	$\begin{matrix} \boxed{-6} & 2 & 3 & 1 & -5 \\ 0 & \boxed{8} & 4 & -1 & -14 \\ 0 & 0 & 0 & \boxed{1} & 2 \end{matrix}$

and the backward phase is

row operation	result				
$\mathbf{r}_2 \leftarrow \mathbf{r}_2 + \mathbf{r}_3$	$\boxed{-6}$	2	3	0	−7
$\mathbf{r}_1 \leftarrow \mathbf{r}_1 - \mathbf{r}_3$	0	$\boxed{2}$	1	0	−3
$\mathbf{r}_2 \leftarrow \frac{1}{4}\,\mathbf{r}_2$	0	0	0	$\boxed{1}$	2
	$\boxed{-6}$	0	2	0	−4
$\mathbf{r}_1 \leftarrow \mathbf{r}_1 - 2\,\mathbf{r}_2$	0	$\boxed{2}$	1	0	−3
	0	0	0	$\boxed{1}$	2

To produce the matrix in reduced row echelon form, we rescale rows 1 and 2 :

row operation	result				
$\mathbf{r}_1 \leftarrow -\,\frac{1}{6}\,\mathbf{r}_1$	$\boxed{1}$	0	−1/3	0	2/3
$\mathbf{r}_2 \leftarrow \frac{1}{2}\,\mathbf{r}_2$	0	$\boxed{1}$	1/2	0	−3/2
	0	0	0	$\boxed{1}$	2

end example 1.2.re3

1.2.re4. Here's another, larger example:

Forward phase: working left to right, identify the pivots and eliminate the nonzero terms below them:

row operation	**result**
(beginning matrix)	$\begin{matrix} 0 & -2 & 4 & 18 & -10 & 32 \\ \boxed{3} & 18 & -24 & 21 & 6 & 276 \\ -5 & -27 & 34 & -62 & 7 & -508 \\ 4 & 26 & -36 & 11 & 15 & 339 \end{matrix}$
$\mathbf{r}_1 \rightleftarrows \mathbf{r}_2$	$\begin{matrix} \boxed{3} & 18 & -24 & 21 & 6 & 276 \\ 0 & -2 & 4 & 18 & -10 & 32 \\ -5 & -27 & 34 & -62 & 7 & -508 \\ 4 & 26 & -36 & 11 & 15 & 339 \end{matrix}$
$\mathbf{r}_1 \leftarrow \frac{1}{3}\mathbf{r}_1$	$\begin{matrix} \boxed{1} & 6 & -8 & 7 & 2 & 92 \\ 0 & -2 & 4 & 18 & -10 & 32 \\ -5 & -27 & 34 & -62 & 7 & -508 \\ 4 & 26 & -36 & 11 & 15 & 339 \end{matrix}$
$\mathbf{r}_3 \leftarrow \mathbf{r}_3 + 5\,\mathbf{r}_1$ $\mathbf{r}_4 \leftarrow \mathbf{r}_4 - 4\,\mathbf{r}_1$	$\begin{matrix} \boxed{1} & 6 & -8 & 7 & 2 & 92 \\ 0 & \boxed{-2} & 4 & 18 & -10 & 32 \\ 0 & 3 & -6 & -27 & 17 & -48 \\ 0 & 2 & -4 & -17 & 7 & -29 \end{matrix}$
$\mathbf{r}_2 \leftarrow -\frac{1}{2}\,\mathbf{r}_2$	$\begin{matrix} \boxed{1} & 6 & -8 & 7 & 2 & 92 \\ 0 & \boxed{1} & -2 & -9 & 5 & -16 \\ 0 & 3 & -6 & -27 & 17 & -48 \\ 0 & 2 & -4 & -17 & 7 & -29 \end{matrix}$
$\mathbf{r}_3 \leftarrow \mathbf{r}_3 - 3\,\mathbf{r}_2$ $\mathbf{r}_4 \leftarrow \mathbf{r}_4 - 2\,\mathbf{r}_2$	$\begin{matrix} \boxed{1} & 6 & -8 & 7 & 2 & 92 \\ 0 & \boxed{1} & -2 & -9 & 5 & -16 \\ 0 & 0 & 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 1 & -3 & 3 \end{matrix}$
$\mathbf{r}_3 \rightleftarrows \mathbf{r}_4$	$\begin{matrix} \boxed{1} & 6 & -8 & 7 & 2 & 92 \\ 0 & \boxed{1} & -2 & -9 & 5 & -16 \\ 0 & 0 & 0 & \boxed{1} & -3 & 3 \\ 0 & 0 & 0 & 0 & \boxed{2} & 0 \end{matrix}$
$\mathbf{r}_4 \leftarrow \frac{1}{2}\,\mathbf{r}_4$	$\begin{matrix} \boxed{1} & 6 & -8 & 7 & 2 & 92 \\ 0 & \boxed{1} & -2 & -9 & 5 & -16 \\ 0 & 0 & 0 & \boxed{1} & -3 & 3 \\ 0 & 0 & 0 & 0 & \boxed{1} & 0 \end{matrix}$

The forward phase is done and the matrix is in row echelon form.

Now the backward phase: working from right to left, eliminate nonzero terms above each pivot:

$\mathbf{r}_3 \leftarrow \mathbf{r}_3 + 3\,\mathbf{r}_4$ $\mathbf{r}_2 \leftarrow \mathbf{r}_2 - 5\,\mathbf{r}_4$ $\mathbf{r}_1 \leftarrow \mathbf{r}_1 - 2\,\mathbf{r}_4$	$\boxed{1}$ 0 0 0	6 $\boxed{1}$ 0 0	−8 −2 0 0	7 −9 $\boxed{1}$ 0	0 0 0 $\boxed{1}$	92 −16 3 0
$\mathbf{r}_2 \leftarrow \mathbf{r}_2 + 9\,\mathbf{r}_3$ $\mathbf{r}_1 \leftarrow \mathbf{r}_1 - 7\,\mathbf{r}_3$	$\boxed{1}$ 0 0 0	6 $\boxed{1}$ 0 0	−8 −2 0 0	0 0 $\boxed{1}$ 0	0 0 0 $\boxed{1}$	71 11 3 0
$\mathbf{r}_1 \leftarrow \mathbf{r}_1 - 6\,\mathbf{r}_2$	$\boxed{1}$ 0 0 0	0 $\boxed{1}$ 0 0	4 −2 0 0	0 0 $\boxed{1}$ 0	0 0 0 $\boxed{1}$	5 11 3 0

The backward phase is done and the matrix is now in reduced row echelon form.

end example 1.2.re4

When a linear system is represented in augmented matrix form, variables whose coefficients form a pivot column are called **basic**; all other variables are called **free**.

1.2.re5. Which variables in the system

$$\begin{aligned} 8x_2 + 4x_3 - 1x_4 &= -14 \\ -6x_1 + 2x_2 + 3x_3 + x_4 &= -5 \\ -12x_1 - 20x_2 - 6x_3 + 6x_4 &= 34 \end{aligned}$$

are free and which are basic? Find the general solution of the system.

The augmented matrix form of this system was row reduced in 1.2.re3. The pivots are in columns, 1, 2, and 4, so x_1, x_2 and x_4 are basic variables and x_3 is the free variable. The reduced row echelon form of augmented matrix tells us

$$\begin{aligned} x_1 - 1/3x_3 &= 2/3 \\ x_2 + 1/2x_3 &= -3/2 \\ x_4 &= 2 \end{aligned}$$

The basic variables can be expressed in terms of the free variable:

$$\begin{aligned} &x_1 = 2/3 + 1/3x_3 \\ &x_2 = -3/2 - 1/2x_3 \\ &x_3 \text{ is free} \\ &x_4 = 2 \end{aligned}$$

This is the **parametric** form of the general solution. Since the parameter x_3 is "free" to take any value, the general solution of this system contains infinitely many individual solutions (x_1, x_2, x_3, x_4).

Fact 1.2.1. *Every matrix is row equivalent to infinitely many row echelon forms, but only one reduced row echelon form.*

1.2.re6. Give an example of two different row echelon forms of $I = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$.

1.2.re7. Find a row echelon form and the reduced row echelon form of the given matrix.

a. $\begin{bmatrix} 1 & 6 & 9 & -3 & 7 \\ -2 & 12 & 30 & -6 & -71 \\ 1 & -2 & -7 & 1 & 33 \end{bmatrix}$ b. $\begin{bmatrix} 6 & 2 & 4 & 0 & 14 \\ 24 & 8 & 18 & -5 & 58 \\ 0 & 0 & 4 & 5 & 7 \end{bmatrix}$ c. $\begin{bmatrix} 0 & 2 & -1 & -5 \\ 15 & 10 & 1 & -3 \\ 0 & -6 & 5 & 19 \end{bmatrix}$

d. $\begin{bmatrix} -4 & -8 & 3 & 6 & -12 \\ 4 & 8 & -2 & -4 & 12 \end{bmatrix}$ e. $\begin{bmatrix} 1 & -2 & 2 \\ 3 & -6 & 11 \\ 2 & 0 & 16 \end{bmatrix}$ f. $\begin{bmatrix} 4 & 2 & 4 & 1 \\ 4 & -2 & 7 & -10 \\ 4 & -2 & 8 & -11 \end{bmatrix}$

1.2.re7, continued. Find the general solution of the system represented in augmented matrix form by the given matrix.

a. (above) b. (above) c. (above)
d. (above) e. (above) f. (above)

Fact 1.2.2. *Every linear system has either 0, 1, or infinitely many solutions, depending on the locations of pivots in the row echelon form of the augmented matrix:*

Does the rightmost column contain a pivot?
Yes: The system has no solutions.
No: Are there any free variables?
Yes: the system has infinitely many solutions.
No: the system has exactly one solution.

Answers

1.2.re2. B is obtained from A by $\mathbf{r}_2 \rightleftarrows \mathbf{r}_4$, $\mathbf{r}_3 \rightleftarrows \mathbf{r}_4$. C is obtained from B by $\mathbf{r}_3 \leftarrow -\frac{1}{2}\mathbf{r}_3$, $\mathbf{r}_1 \leftarrow \mathbf{r}_1 - \mathbf{r}_3$, $\mathbf{r}_1 \leftarrow \frac{1}{4}\mathbf{r}_1$. 1.2.re6. $I \sim \begin{bmatrix} 1 & 4 & 5 \\ 0 & 2 & 6 \\ 0 & 0 & 3 \end{bmatrix} \sim \begin{bmatrix} 4 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 2 \end{bmatrix}$.

1.2.re.e7a. one possible ref $= \begin{bmatrix} 1 & 6 & 9 & -3 & 7 \\ 0 & 8 & 16 & -4 & -19 \\ 0 & 0 & 0 & 0 & 7 \end{bmatrix}$; the rref $= \begin{bmatrix} 1 & 0 & -3 & 0 & 0 \\ 0 & 1 & 2 & -1/2 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$.

1.2.re.e7b. one possible ref $= \begin{bmatrix} 6 & 2 & 4 & 0 & 14 \\ 0 & 0 & 2 & -5 & 2 \\ 0 & 0 & 0 & 5 & 1 \end{bmatrix}$; the rref $= \begin{bmatrix} 1 & 1/3 & 0 & 0 & 4/3 \\ 0 & 0 & 1 & 0 & 3/2 \\ 0 & 0 & 0 & 1 & 1/5 \end{bmatrix}$.

1.2.re.e7c. one possible ref $= \begin{bmatrix} 15 & 10 & 1 & -3 \\ 0 & 2 & -1 & -5 \\ 0 & 0 & 1 & 2 \end{bmatrix}$; the rref $= \begin{bmatrix} 1 & 0 & 0 & 2/3 \\ 0 & 1 & 0 & -3/2 \\ 0 & 0 & 1 & 2 \end{bmatrix}$.

1.2.re.e7d. one possible ref $= \begin{bmatrix} -4 & -8 & 3 & 6 & -12 \\ 0 & 0 & 1 & 2 & 0 \end{bmatrix}$; the rref $= \begin{bmatrix} 1 & 2 & 0 & 0 & 3 \\ 0 & 0 & 1 & 2 & 0 \end{bmatrix}$.

1.2.re.e7e. one possible ref $= \begin{bmatrix} 1 & -2 & 2 \\ 0 & -2 & -6 \\ 0 & 0 & 5 \end{bmatrix}$; the rref $= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$.

1.2.re.e7f. one possible ref $= \begin{bmatrix} 4 & 2 & 4 & 1 \\ 0 & 4 & -3 & 11 \\ 0 & 0 & 1 & -1 \end{bmatrix}$; the rref $= \begin{bmatrix} 1 & 0 & 0 & 1/4 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & -1 \end{bmatrix}$.

1.2.re.e7a. no solutions. 1.2.re.e7b. $(x_1, x_2, x_3, x_4) = (4/3 - 1/3x_2, \text{free}, 3/2, 1/5)$; 1.2.re.e7c. $(x_1, x_2, x_3) = (2/3, -3/2, 2)$ 1.2.re.e7d. $(x_1, x_2, x_3, x_4) = (3 - 2x_2, \text{free}, -2x_4, \text{free})$ 1.2.re.e7e. no solutions.
1.2.re.e7f. $(x_1, x_2, x_3) = (1/4, 2, -1)$

1.3: Vector equations

A matrix with m rows and n columns is said to or **size** or **dimension** $m \times n$. A matrix with only one column is called a **vector**, and its entries are called its **components**. A real number is called a **scalar**. Often, vector variables are written in **bold** and scalar variables are written in *math-italic*.

The set of all $n \times 1$ vectors is denoted $\mathbb{R}^n$. The zero vector in $\mathbb{R}^n$ is denoted $\mathbf{0}$ (with n to be made clear by context).

1.3.re1. $\begin{bmatrix} 1 \\ -2 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} 1 \\ 0 \\ -3 \end{bmatrix}$ are vectors in $\mathbb{R}^3$. $\begin{bmatrix} 0 \\ 1 \\ 1 \\ 1 \end{bmatrix}$ is in $\mathbb{R}^4$. In $\mathbb{R}^2$, the symbol $\mathbf{0}$ refers to the vector $\begin{bmatrix} 0 \\ 0 \end{bmatrix}$, but in $\mathbb{R}^3$, the same symbol $\mathbf{0}$ refers to the vector $\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$.

Vector arithmetic, linear combination, span

If

$$\mathbf{u} = \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_m \end{bmatrix} \text{ and } \mathbf{v} = \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_m \end{bmatrix}$$

are vectors of the same size, and c is a scalar, then

$$\mathbf{u} + \mathbf{v} = \begin{bmatrix} u_1 + v_1 \\ u_2 + v_2 \\ \vdots \\ u_m + v_m \end{bmatrix} \text{ and } c\mathbf{u} = \begin{bmatrix} cu_1 \\ cu_2 \\ \vdots \\ cu_m \end{bmatrix}$$

If $\mathbf{u}_1, \mathbf{u}_2, \ldots \mathbf{u}_n$ are vectors of the same size and $x_1, \ldots, x_n$ are scalars, then

$$x_1\mathbf{u}_1 + x_2\mathbf{u}_2 + \cdots + x_n\mathbf{u}_n$$

is called a **linear combination** of $\mathbf{u}_1, \mathbf{u}_2, \ldots \mathbf{u}_n$.

1.3.re2. $\begin{bmatrix} -8 \\ -5 \\ 17 \end{bmatrix}$ is a linear combination of $\begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix}$, $\begin{bmatrix} 2 \\ 1 \\ -3 \end{bmatrix}$, and $\begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$, since

$$2\begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix} - 3\begin{bmatrix} 2 \\ 1 \\ -3 \end{bmatrix} + 4\begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ -2 \\ 4 \end{bmatrix} + \begin{bmatrix} -6 \\ -3 \\ 9 \end{bmatrix} + \begin{bmatrix} -4 \\ 0 \\ 4 \end{bmatrix} = \begin{bmatrix} -8 \\ -5 \\ 17 \end{bmatrix}$$

The collection of all linear combinations of $\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_n$ is called its **span**, written $\text{span}\{\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_n\}$. If these vectors lie in $\mathbb{R}^m$, then so does their span. If *every* vector in $\mathbb{R}^m$ is a linear combination of $\mathbf{u}_1, \ldots, \mathbf{u}_n$, these vectors are said to span $\mathbb{R}^m$.

Linear systems expressed in vector form

Fact 1.3.1. *The linear system (1.1.1) is equivalent to the* **vector equation**

$$x_1\begin{bmatrix}a_{1,1}\\a_{2,1}\\\vdots\\a_{m,1}\end{bmatrix}+x_2\begin{bmatrix}a_{1,2}\\a_{2,2}\\\vdots\\a_{m,2}\end{bmatrix}+\cdots+x_n\begin{bmatrix}a_{1,n}\\a_{2,n}\\\vdots\\a_{m,n}\end{bmatrix}=\begin{bmatrix}b_1\\b_2\\\vdots\\b_m\end{bmatrix}.$$

This system is consistent iff

$$\begin{bmatrix}b_1\\b_2\\\vdots\\b_m\end{bmatrix}\in\text{span}\left\{\begin{bmatrix}a_{1,1}\\a_{2,1}\\\vdots\\a_{m,1}\end{bmatrix},\begin{bmatrix}a_{1,2}\\a_{2,2}\\\vdots\\a_{m,2}\end{bmatrix},\ldots,\begin{bmatrix}a_{1,n}\\a_{2,n}\\\vdots\\a_{m,n}\end{bmatrix}\right\}$$

1.3.re3. Express the first vector as a linear combination of the others or explain why this is not possible.

a. $\begin{bmatrix}-3\\-15\\18\end{bmatrix},\left\{\begin{bmatrix}1\\4\\-2\end{bmatrix},\begin{bmatrix}6\\18\\6\end{bmatrix},\begin{bmatrix}5\\16\\2\end{bmatrix}\right\}$ b. $\begin{bmatrix}11\\20\\-4\end{bmatrix},\left\{\begin{bmatrix}6\\12\\0\end{bmatrix},\begin{bmatrix}1\\1\\-2\end{bmatrix},\begin{bmatrix}6\\11\\0\end{bmatrix}\right\}$

c. $\begin{bmatrix}20\\43\\14\end{bmatrix},\left\{\begin{bmatrix}6\\12\\0\end{bmatrix},\begin{bmatrix}1\\1\\-2\end{bmatrix},\begin{bmatrix}6\\11\\0\end{bmatrix},\begin{bmatrix}11\\20\\-4\end{bmatrix}\right\}$ d. $\begin{bmatrix}-6\\-7\\44\end{bmatrix},\left\{\begin{bmatrix}-1\\-2\\5\end{bmatrix},\begin{bmatrix}-1\\-1\\9\end{bmatrix},\begin{bmatrix}-2\\-1\\22\end{bmatrix}\right\}$

1.3.re4. Find an equation in x, y, z that is equivalent to the consistency of the the system represented below in augmented matrix form.

$$\begin{bmatrix}1&7&4&x\\0&5&3&y\\2&9&5&z\end{bmatrix}$$

Answers

1.3.re.e3a. $\begin{bmatrix}1&6&5&-3\\4&18&16&-15\\-2&6&2&18\end{bmatrix}\sim\begin{bmatrix}1&6&5&-3\\0&6&4&3\\0&0&0&-1\end{bmatrix}$. Not in the span. 1.3.re.e3b. The given vector is in the span, since it $=\frac{3}{2}\begin{bmatrix}6\\12\\0\end{bmatrix}+2\begin{bmatrix}1\\1\\-2\end{bmatrix}$. 1.3.re.e3c. The given vector is in the span, since it $=\frac{1}{2}\begin{bmatrix}6\\12\\0\end{bmatrix}-7\begin{bmatrix}1\\1\\-2\end{bmatrix}+4\begin{bmatrix}6\\11\\0\end{bmatrix}$. 1.3.re.e3d. $\begin{bmatrix}-1&-1&-2&-6\\-2&-1&-1&-7\\5&9&22&44\end{bmatrix}\sim\begin{bmatrix}-1&-1&-2&-6\\0&1&3&5\\0&0&0&-3\end{bmatrix}$. Not in the span.

1.3.re4. $z+y-2x=0$

1.4: The product of a matrix and a vector

If the matrix A has matrix with n columns and the vector $\mathbf{x}$ has n rows:

$$A = [\,\mathbf{a}_1 \quad \mathbf{a}_2 \quad \cdots \quad \mathbf{a}_n\,] \qquad \mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$$

then the product of A and $\mathbf{x}$ is defined to be

$$A\mathbf{x} = x_1\mathbf{a}_1 + x_2\mathbf{a}_2 + \cdots + x_n\mathbf{a}_n,$$

that is, the linear combination of the columns of A with the weights $x_1, x_2, \ldots, x_n$.

Dimensions in matrix-vector products:

$$(m \times n) \cdot (n \times 1) = (m \times 1)$$

1.4.re1. Find the product, if it exists.

a. $\begin{bmatrix} 1 & -2 \\ -1 & 1 \\ 0 & 1 \end{bmatrix} [\,4 \quad -3\,]$ b. $\begin{bmatrix} 1 & -2 \\ -1 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 4 \\ -3 \end{bmatrix}$ c. $\begin{bmatrix} 1 & 2 & 3 \\ 3 & 0 & -1 \end{bmatrix} \begin{bmatrix} 1 \\ -2 \\ 4 \end{bmatrix}$

d. $\begin{bmatrix} 1 & 2 \\ 2 & 3 \\ 3 & -1 \end{bmatrix} \begin{bmatrix} 2 \\ 4 \\ 3 \end{bmatrix}$ e. $\begin{bmatrix} 2 \\ 4 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 2 & 3 \\ 3 & 0 \end{bmatrix}$ f. $[\,1 \quad 2 \quad 3\,] \begin{bmatrix} 1 \\ -2 \\ 4 \end{bmatrix}$

Fact 1.4.1. *If A is $m \times n$ and $\mathbf{u}$ and $\mathbf{v}$ are $n \times 1$, and if c is a scalar, then*

$$A(\mathbf{u} + \mathbf{v}) = A\mathbf{u} + A\mathbf{v}$$
$$A(c\mathbf{u}) = c(A\mathbf{u})$$

Equivalent formations of a linear system

The system of linear equations

$$\begin{aligned}
a_{1,1}x_1 + a_{1,2}x_2 + \cdots a_{1,n}x_n &= b_1 \\
a_{2,1}x_1 + a_{2,2}x_2 + \cdots a_{2,n}x_n &= b_2 \\
&\vdots \\
a_{m,1}x_1 + a_{m,2}x_2 + \cdots a_{m,n}x_n &= b_m
\end{aligned}$$

can be expressed in **augmented matrix form**:

$$\begin{bmatrix} a_{1,1} & a_{1,2} & \cdots & a_{1,n} & b_1 \\ a_{2,1} & a_{2,2} & \cdots & a_{2,n} & b_2 \\ \vdots & \vdots & & \vdots & \vdots \\ a_{m,1} & a_{m,2} & \cdots & a_{m,n} & b_m \end{bmatrix}$$

in **vector form**:

$$x_1 \begin{bmatrix} a_{1,1} \\ a_{2,1} \\ \vdots \\ a_{m,1} \end{bmatrix} + x_2 \begin{bmatrix} a_{1,2} \\ a_{2,2} \\ \vdots \\ a_{m,2} \end{bmatrix} + \cdots + x_n \begin{bmatrix} a_{1,n} \\ a_{2,n} \\ \vdots \\ a_{m,n} \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}$$

or in **matrix form**:

$$A\mathbf{x} = \mathbf{b}$$

where

$$A = \begin{bmatrix} a_{1,1} & a_{1,2} & \cdots & a_{1,n} \\ a_{2,1} & a_{2,2} & \cdots & a_{2,n} \\ \vdots & \vdots & & \vdots \\ a_{m,1} & a_{m,2} & \cdots & a_{m,n} \end{bmatrix} \qquad \mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} \qquad \mathbf{b} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}$$

The system is consistent iff $\mathbf{b}$ lies in the span of the columns of A.

Spanning sets for $\mathbb{R}^m$

Fact 1.4.2. *If A is a matrix with m rows, then the following are equivalent:*

1. *The columns of A span $\mathbb{R}^m$.*
2. *The system $A\mathbf{x} = \mathbf{b}$ is consistent for every $\mathbf{b} \in \mathbb{R}^m$.*
3. *The row echelon form of A has no row of zeros.*
4. *The row echelon form of A has a pivot in every row.*

1.4.re2. Determine whether the given set of vectors spans $\mathbb{R}^3$.

a. $\left\{ \begin{bmatrix} 3 \\ 3 \\ 3 \end{bmatrix}, \begin{bmatrix} -2 \\ -4 \\ -2 \end{bmatrix}, \begin{bmatrix} 3 \\ 1 \\ 4 \end{bmatrix} \right\}$

b. $\left\{ \begin{bmatrix} 1 \\ 5 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ 9 \\ -1 \end{bmatrix}, \begin{bmatrix} -1 \\ -1 \\ 8 \end{bmatrix} \right\}$

c. $\left\{ \begin{bmatrix} 2 \\ 2 \\ 4 \end{bmatrix}, \begin{bmatrix} 3 \\ 6 \\ 15 \end{bmatrix}, \begin{bmatrix} 1 \\ 3 \\ 8 \end{bmatrix} \right\}$

d. $\left\{ \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} -2 \\ -6 \\ -4 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -4 \\ -10 \\ -7 \end{bmatrix} \right\}$

e. $\left\{ \begin{bmatrix} 2 \\ -4 \\ -6 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -10 \\ 11 \\ 3 \end{bmatrix} \right\}$

1.4.re3. Explain why four vectors in $\mathbb{R}^5$ cannot form a spanning set for $\mathbb{R}^5$.

Answers

1.4.re.e1a. d.n.e. 1.4.re.e1b. $\begin{bmatrix} 10 \\ -7 \\ -3 \end{bmatrix}$ 1.4.re.e1c. $\begin{bmatrix} 9 \\ -1 \end{bmatrix}$ 1.4.re.e1d. dne. 1.4.re.e1e. dne. 1.4.re.e1f. [9]

1.4.re.e2a. $\begin{bmatrix} 3 & -2 & 3 \\ 3 & -4 & 1 \\ 3 & -2 & 4 \end{bmatrix} \sim \begin{bmatrix} 3 & -2 & 3 \\ 0 & 2 & 2 \\ 0 & 0 & 1 \end{bmatrix}$. Spans. 1.4.re.e2b. $\begin{bmatrix} 1 & 2 & -1 \\ 5 & 9 & -1 \\ 1 & -1 & 8 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ Spans.
1.4.re.e2c. $\begin{bmatrix} 2 & 3 & 1 \\ 2 & 6 & 3 \\ 4 & 15 & 8 \end{bmatrix} \sim \begin{bmatrix} 2 & 0 & -1 \\ 0 & 3 & 2 \\ 0 & 0 & 0 \end{bmatrix}$. Does not span. 1.4.re.e2d. $\begin{bmatrix} 1 & -2 & 1 & -4 \\ 2 & -6 & 0 & -10 \\ 1 & -4 & 0 & -7 \end{bmatrix}$
$\sim \begin{bmatrix} 1 & -2 & 1 & -4 \\ 0 & 2 & 2 & 2 \\ 0 & 0 & 1 & -1 \end{bmatrix}$. Spans. 1.4.re.e2e. $\begin{bmatrix} 2 & -1 & -10 \\ -4 & 1 & 11 \\ -6 & 0 & 3 \end{bmatrix} \sim \begin{bmatrix} 2 & -1 & -10 \\ 0 & -1 & -9 \\ 0 & 0 & 0 \end{bmatrix}$. Does not span.
1.4.re3. a. No row or column can contain more than one pivot, and so the number of pivots in a matrix cannot is exceed its number of rows or number of columns. A 4×5 matrix contains at most 4 pivots and consequently must have a row with no pivot.

1.5: Homogeneous linear systems

A **homogeneous** linear system is one of the form

$$A\mathbf{x} = \mathbf{0}.$$

If A is $m \times n$, then the $\mathbf{0}$ on the right side is in $\mathbb{R}^m$. A homogenous system always has the **trivial** solution $\mathbf{x} = \mathbf{0}$ (in $\mathbb{R}^n$); its has infinitely many solutions iff it has free variables. (See Fact 1.2.2.)

1.5.re1. Solve the system

$$\begin{bmatrix} 3 & -1 & 5 & 1 & -3 \\ 6 & 0 & 12 & 3 & 0 \\ 9 & -5 & 13 & 3 & -11 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} -2 \\ -3 \\ -12 \end{bmatrix}$$

Express the general solution in **parametric vector form**.

Solution:

row operation	**result**
	$\begin{matrix} 3 & -1 & 5 & 1 & -3 & -2 \\ 6 & 0 & 12 & 3 & 0 & -3 \\ 9 & -5 & 13 & 3 & -11 & -12 \end{matrix}$
$\mathbf{r}_2 \leftarrow \mathbf{r}_2 - 2\,\mathbf{r}_1$ $\mathbf{r}_3 \leftarrow \mathbf{r}_3 - 3\,\mathbf{r}_1$	$\begin{matrix} 3 & -1 & 5 & 1 & -3 & -2 \\ 0 & 2 & 2 & 1 & 6 & 1 \\ 0 & -2 & -2 & 0 & -2 & -6 \end{matrix}$
$\mathbf{r}_3 \leftarrow \mathbf{r}_3 + \mathbf{r}_2$	$\begin{matrix} 3 & -1 & 5 & 1 & -3 & -2 \\ 0 & 2 & 2 & 1 & 6 & 1 \\ 0 & 0 & 0 & 1 & 4 & -5 \end{matrix}$
$\mathbf{r}_2 \leftarrow \mathbf{r}_2 - \mathbf{r}_3$ $\mathbf{r}_1 \leftarrow \mathbf{r}_1 - \mathbf{r}_3$	$\begin{matrix} 3 & -1 & 5 & 0 & -7 & 3 \\ 0 & 2 & 2 & 0 & 2 & 6 \\ 0 & 0 & 0 & 1 & 4 & -5 \end{matrix}$
$\mathbf{r}_2 \leftarrow \frac{1}{2}\mathbf{r}_2$ $\mathbf{r}_1 \leftarrow \mathbf{r}_1 + \mathbf{r}_2$ $\mathbf{r}_1 \leftarrow \frac{1}{3}\mathbf{r}_1$	$\begin{matrix} 1 & 0 & 2 & 0 & -2 & 2 \\ 0 & 1 & 1 & 0 & 1 & 3 \\ 0 & 0 & 0 & 1 & 4 & -5 \end{matrix}$

As in example 1.2.re5, the solution in parametric form is:

$$\begin{aligned} x_1 &= 2 - 2x_3 + 2x_5 \\ x_2 &= 3 - x_3 - x_5 \\ x_3 &\text{ is free.} \\ x_4 &= -5 - 4x_5 \\ x_5 &\text{ is free.} \end{aligned}$$

To write it in **parametric vector form** means to express the solution $\mathbf{x}$ as a vector and to use parameters to stand for the values of the free variables. If we let $s = x_3$ and $t = x_5$, then

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} 2 \\ 3 \\ 0 \\ -5 \\ 0 \end{bmatrix} + s \begin{bmatrix} -2 \\ -1 \\ 1 \\ 0 \\ 0 \end{bmatrix} + t \begin{bmatrix} 2 \\ -1 \\ 0 \\ -4 \\ 1 \end{bmatrix}$$

end example 1.5.re1

1.5.re2. Solve the homogeneous system associated to the nonhomogeneous system in 1.5.re1:

$$\begin{bmatrix} 3 & -1 & 5 & 1 & -3 \\ 6 & 0 & 12 & 3 & 0 \\ 9 & -5 & 13 & 3 & -11 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Augment the coefficient matrix with a column of three zeros, and perform the same row operations as in 1.5.re1, resulting in

$$\begin{bmatrix} 1 & 0 & 2 & 0 & -2 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 4 & 0 \end{bmatrix}$$

and so the parametric vector form of the solution to the homogenous system is

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = s \begin{bmatrix} -2 \\ -1 \\ 1 \\ 0 \\ 0 \end{bmatrix} + t \begin{bmatrix} 2 \\ -1 \\ 0 \\ -4 \\ 1 \end{bmatrix}$$

Note that the only difference between this and the solution to 1.5.re1 is the term $\begin{bmatrix} 2 \\ 3 \\ 0 \\ -5 \\ 0 \end{bmatrix}$, which is itself a solution to 1.5.re1.

Fact 1.5.1. *Suppose $A\mathbf{u} = \mathbf{b}$. Then the solution set to $A\mathbf{x} = \mathbf{b}$ is*

$$\{\mathbf{u} + \mathbf{v} \mid A\mathbf{v} = \mathbf{0}\}.$$

That is, if $\mathbf{u}$ is any solution to $A\mathbf{x} = \mathbf{b}$, then the solutions are exactly those vectors which can be expressed as $\mathbf{u}$ plus a solution $\mathbf{v}$ to the associated homogeneous system $A\mathbf{x} = \mathbf{0}$.

1.5.re3. Write the solution set to the system (expressed here in augmented matrix form) in parametric vector form. Then give the solution to the associated homogeneous system.

a. $\begin{bmatrix} 1 & 1 & 1 & 1 & 3 \\ 2 & 1 & 0 & 1 & 4 \\ 1 & 5 & 9 & 5 & 11 \end{bmatrix}$

b. $\begin{bmatrix} 1 & -1 & 9 & 6 \\ 1 & -4 & 18 & 12 \\ -2 & -23 & 72 & 43 \end{bmatrix}$

c. $\begin{bmatrix} -2 & 7 & 12 & -42 & -3 & 18 \\ 1 & -1 & -1 & 6 & 2 & 2 \\ 3 & -18 & -33 & 108 & 6 & -54 \end{bmatrix}$

d. $\begin{bmatrix} -1 & 1 & 4 & -3 & -1 \\ -2 & 4 & 14 & -8 & 26 \\ -1 & 0 & 1 & -2 & 0 \end{bmatrix}$

e. $\begin{bmatrix} 3 & 2 & -1 & 1 \end{bmatrix}$

Answers

1.5.re.e3a. nonhomo: $\mathbf{x} = \begin{bmatrix} 1 \\ 2 \\ 0 \\ 0 \end{bmatrix} + s \begin{bmatrix} 1 \\ -2 \\ 1 \\ 0 \end{bmatrix} + t \begin{bmatrix} 0 \\ -1 \\ 0 \\ 1 \end{bmatrix}$. homo: $\mathbf{x} = s \begin{bmatrix} 1 \\ -2 \\ 1 \\ 0 \end{bmatrix} + t \begin{bmatrix} 0 \\ -1 \\ 0 \\ 1 \end{bmatrix}$. 1.5.re.e3b. nonhomo: no free variables. only solution is $\mathbf{x} = \begin{bmatrix} 2 \\ -1 \\ 1/3 \end{bmatrix}$. homo: no free variables. only solution is $\mathbf{x} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$.

1.5.re.e3c. nonhomo: $\mathbf{x} = \begin{bmatrix} 2 \\ 4 \\ 0 \\ 0 \\ 2 \end{bmatrix} + s \begin{bmatrix} -1 \\ -2 \\ 1 \\ 0 \\ 0 \end{bmatrix} + t \begin{bmatrix} 0 \\ 6 \\ 0 \\ 1 \\ 0 \end{bmatrix}$. homo: $\mathbf{x} = s \begin{bmatrix} -1 \\ -2 \\ 1 \\ 0 \\ 0 \end{bmatrix} + t \begin{bmatrix} 0 \\ 6 \\ 0 \\ 1 \\ 0 \end{bmatrix}$. 1.5.re.e3d. nonhomo: no solutions. homo: $\mathbf{x} = s \begin{bmatrix} 1 \\ -3 \\ 1 \\ 0 \end{bmatrix} + t \begin{bmatrix} -2 \\ 1 \\ 0 \\ 1 \end{bmatrix}$. 1.5.re.e3e. nonhomo: $\mathbf{x} = \begin{bmatrix} 1/3 \\ 0 \\ 0 \end{bmatrix} + s \begin{bmatrix} -2/3 \\ 1 \\ 0 \end{bmatrix} + t \begin{bmatrix} 1/3 \\ 0 \\ 1 \end{bmatrix}$. homo: $\mathbf{x} = s \begin{bmatrix} -2/3 \\ 1 \\ 0 \end{bmatrix} + t \begin{bmatrix} 1/3 \\ 0 \\ 1 \end{bmatrix}$.

1.7: Linear independence

Definition 1.7.1. *A set of vectors* $\{\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_n\}$ *in* $\mathbb{R}^m$ *is* **linearly independent** *if the only solution to*

$$x_1\mathbf{a}_1 + x_2\mathbf{a}_2 + \cdots + x_n\mathbf{a}_n = \mathbf{0} \quad (\textit{in } \mathbb{R}^m) \tag{1.7.2}$$

is $\mathbf{x} = \mathbf{0}$ *(in* $\mathbb{R}^n$*). A set of vectors is* **linearly dependent** *if it is not linearly independent.*

Fact 1.7.3. *If* A *is a matrix, then the following are equivalent:*

1. *The columns of* A *are linearly independent .*
2. *The homogeneous system* $A\mathbf{x} = \mathbf{0}$ *has only the trivial solution* $\mathbf{x} = \mathbf{0}$.
3. $A\mathbf{x}$ *has no free variables.*
4. *The row echelon form of* A *has a pivot in every column.*

1.7.re1. Prove each of the following.

a. Any set of more than m vectors in $\mathbb{R}^m$ must be linearly dependent.

b. A set of two or more vectors is linearly independent iff some vector in that set is a linear combination of the others.

c. The linear dependence or independence of a set of vectors $\{\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_n\}$ is independent of the order in which we write the vectors.

d. A set of two vectors is linearly dependent iff one vector is a scalar multiple of the other.

e. Any set containing $\mathbf{0}$ in linearly dependent.

f. If A is a set of vectors and $B \subset A$, then

i. B is linearly dependent $\Longrightarrow$ A is linearly dependent.

ii. A is linearly independent $\Longrightarrow$ B is linearly independent.

1.7.re2. Determine whether the given set of vectors is linearly independent.

a. $\left\{ \begin{bmatrix} 3 \\ 3 \\ 3 \end{bmatrix}, \begin{bmatrix} -2 \\ -4 \\ -2 \end{bmatrix}, \begin{bmatrix} 3 \\ 1 \\ 4 \end{bmatrix} \right\}$

b. $\left\{ \begin{bmatrix} 1 \\ 5 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ 9 \\ -1 \end{bmatrix}, \begin{bmatrix} -1 \\ -1 \\ 8 \end{bmatrix} \right\}$

c. $\left\{ \begin{bmatrix} 2 \\ 2 \\ 4 \end{bmatrix}, \begin{bmatrix} 3 \\ 6 \\ 15 \end{bmatrix}, \begin{bmatrix} 1 \\ 3 \\ 8 \end{bmatrix} \right\}$

d. $\left\{ \begin{bmatrix} 0 \\ 3 \\ 7 \end{bmatrix}, \begin{bmatrix} -2 \\ \pi \\ 0 \end{bmatrix}, \begin{bmatrix} 3 \\ 0 \\ 0 \end{bmatrix} \right\}$

e. $\left\{ \begin{bmatrix} -1 \\ -2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 4 \\ -2 \\ 0 \end{bmatrix} \right\}$

f. $\left\{ \begin{bmatrix} -1 \\ -2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 3 \\ 0 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ -2 \\ 2 \\ 1 \end{bmatrix} \right\}$

g. $\left\{ \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 0 \\ -1 \\ e \end{bmatrix}, \begin{bmatrix} 2 \\ -\sqrt{2} \\ 1 \\ 1 \end{bmatrix} \right\}$

h. $\left\{ \begin{bmatrix} -1 \\ -2 \\ -1 \end{bmatrix}, \begin{bmatrix} 1 \\ 4 \\ 0 \end{bmatrix}, \begin{bmatrix} 4 \\ 14 \\ 1 \end{bmatrix}, \begin{bmatrix} -3 \\ -8 \\ -2 \end{bmatrix} \right\}$

1.7.re3. For what values of h is the given set linearly independent?

a. $\left\{\begin{bmatrix}1\\1\\-2\end{bmatrix}, \begin{bmatrix}-1\\0\\4\end{bmatrix}, \begin{bmatrix}2\\5\\h\end{bmatrix}\right\}$ b. $\left\{\begin{bmatrix}0\\1\\1\\0\end{bmatrix}, \begin{bmatrix}0\\2\\h\\-2\end{bmatrix}\right\}$ c. $\left\{\begin{bmatrix}0\\2\\4\end{bmatrix}, \begin{bmatrix}0\\-1\\-2\end{bmatrix}, \begin{bmatrix}h\\h\\0\end{bmatrix}\right\}$

Answers

1.7.re.e1a. Consider the vectors as columns of a matrix. The number of pivots in an $m \times n$ matrix cannot exceed m or n, since each column or row has at most one pivot. A matrix with more than m columns must therefore contain a column with no pivot. 1.7.re.e1b. First suppose that $\{\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_n\}$ is linearly independent. Then there exists scalars $x_1, x_2, \ldots, x_n$, not all of which are zero, for which $x_1\mathbf{a}_1 + x_2\mathbf{a}_2 + \cdots + x_n\mathbf{a}_n = \mathbf{0}$. Renumber these terms, if necessary, so that $x_1 \neq 0$. Then $\mathbf{a}_1$ is a linear combination of $\mathbf{a}_2, \ldots, \mathbf{a}_n$, since $\mathbf{a}_1 = -\frac{x_2}{x_1}\mathbf{a}_2 - \cdots - \frac{x_n}{x_1}\mathbf{a}_n$.
Now suppose that one of $\{\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_n\}$ is a linear combination of the others. Renumber the vectors, if necessary, so that $\mathbf{a}_1 = c_2\mathbf{a}_2 + \cdots + c_n\mathbf{a}_n$. Then $\mathbf{a}_1 - c_2\mathbf{a}_2 - \cdots - c_n\mathbf{a}_n = \mathbf{0}$. Since at least one of the scalars $1, -c_2, \ldots, -c_n$ is nonzero, $\{\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_n\}$ is linearly dependent. 1.7.re.e1c. The sum on the right side of $x_1\mathbf{a}_1 + x_2\mathbf{a}_2 + \cdots + x_n\mathbf{a}_n = \mathbf{0}$. is unaffected by a reordering of its terms. 1.7.re.e1d. The vectors are linearly independent iff one vector is a linear combination of the other, but a linear combination of one vector is just a scalar multiple of that vector. 1.7.re.e1e. $1 \cdot \mathbf{0} + 0\mathbf{a}_1 + 0\mathbf{a}_2 + \cdots + 0\mathbf{a}_n = \mathbf{0}$ and the scalar $1 \neq 0$. 1.7.re.e1f. i. If B is linearly dependent, then some element of B is a linear combination of the other elements of B. But elements of B are also elements of A, so some element of A is a linear combination of the other elements of A, and therefore A is linearly dependent. ii. By i., if A fails to be linearly dependent, then B could not have been linearly dependent. 1.7.re.e2a. $\begin{bmatrix}3&-2&3\\3&-4&1\\3&-2&4\end{bmatrix} \sim \begin{bmatrix}3&-2&3\\0&2&2\\0&0&1\end{bmatrix}$. pivot in each column; linearly independent. 1.7.re.e2b. $\begin{bmatrix}1&2&-1\\5&9&-1\\1&-1&8\end{bmatrix} \sim \begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix}$. pivot in each column; linearly independent. 1.7.re.e2c. $\begin{bmatrix}2&3&1\\2&6&3\\4&15&8\end{bmatrix} \sim \begin{bmatrix}2&0&-1\\0&3&2\\0&0&0\end{bmatrix}$. Third column contains no pivot; linearly dependent. 1.7.re.e2d. Reorder columns to $\begin{bmatrix}3&-2&0\\0&\pi&3\\0&0&7\end{bmatrix}$ and observe a pivot in every column; linearly independent. 1.7.re.e2e. Observe 2rd vector is -2 times the first. Linearly dependent. 1.7.re.e2f. Observe 3rd vector is the sum of the first two. Linearly dependent. 1.7.re.e2g. Set contains $\mathbf{0}$; linearly dependent. 1.7.re.e2h. 4 vectors in $\mathbb{R}^3$ must be linearly dependent.

1.7.re.e3a. $\begin{bmatrix}1&-1&2\\1&0&5\\-2&4&h\end{bmatrix} \sim \begin{bmatrix}1&-1&2\\0&1&3\\0&0&h-2\end{bmatrix}$. Linearly independent for all $h \neq 2$.

1.7.re.e3b. $\begin{bmatrix}0&0\\1&2\\1&h\\0&-2\end{bmatrix} \sim \begin{bmatrix}1&2\\0&-2\\1&h\\0&0\end{bmatrix} \sim \begin{bmatrix}1&2\\0&1\\1&h\\0&0\end{bmatrix} \sim \begin{bmatrix}1&0\\0&1\\0&0\\0&0\end{bmatrix}$. Linearly independent for all h.

1.7.re.e3c. First vector is -2 times second regardless of value of h. Linearly dependent for all h.

1.8: Linear Transformations

Domain, codomain and range of a transformation

Transformation is another word for function, a map from one set to another. We use the symbols

$$T : U \to V$$

to indicate that T is a transformation from a set U, called its **domain**, to a set V, called its **codomain**. The symbols

$$T : \mathbf{x} \mapsto T(\mathbf{x})$$

can be used to give an explicit formula for $T(\mathbf{x})$, sometimes called the "image" of $\mathbf{x}$ under T. We sometimes put both together, as in

$$T : U \to V : \mathbf{x} \mapsto T(\mathbf{x}).$$

The collection of all images under T is a subset of the codomain called the **range** of T. The domain, codomain, and range of T can be written

$$\operatorname{dom} T \qquad \operatorname{codom} T \qquad \operatorname{ran} T$$

1.8.re1. The transformation $W : \mathbb{R}^2 \to \mathbb{R}^3 : \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \mapsto \begin{bmatrix} x_1 \\ x_1 - x_2 \\ x_2 \end{bmatrix}$ maps its domain $\mathbb{R}^2$ to its codomain $\mathbb{R}^3$. The vector $\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ isn't in $\operatorname{ran} W$, since the linear system

$$\begin{aligned} x_1 &= 1 \\ x_1 - x_2 &= 0 \\ x_2 &= 0 \end{aligned}$$

has no solution (as you should check for yourself). Therefore, $\operatorname{ran} W$ is a proper subset of $\mathbb{R}^3$.

1.8.re2. (An example from Calculus III) Points (x, y, z) in $\mathbb{R}^3$ can be represented by their spherical coordinates (ρ, ϕ, θ), where

$$x = \rho \sin\phi \cos\theta \qquad y = \rho \sin\phi \sin\theta \qquad z = \rho \cos\phi$$

The transformation

$$S : \mathbb{R}^2 \to \mathbb{R}^3 : \begin{bmatrix} \phi \\ \theta \end{bmatrix} \mapsto \begin{bmatrix} \sin\phi\cos\theta \\ \sin\phi\sin\theta \\ \cos\phi \end{bmatrix}$$

maps $\mathbb{R}^2$ into $\mathbb{R}^3$, and its range is the sets of all points in $\mathbb{R}^3$ for which $\rho = 1$, which is the unit sphere:

$$\left\{ \begin{bmatrix} x \\ y \\ z \end{bmatrix} : x^2 + y^2 + z^2 = 1 \right\}$$

Linear transformations

A transformation $T : \mathbb{R}^n \to \mathbb{R}^m$ is **linear** if

$$T(\mathbf{u}+\mathbf{v}) = T(\mathbf{u}) + T(\mathbf{v}) \quad \text{for all } \mathbf{u} \text{ and } \mathbf{v} \in \mathbb{R}^n \qquad \text{(additivity)}$$

and

$$T(c\mathbf{u}) = cT(\mathbf{u}) \quad \text{for all } \mathbf{u} \in \mathbb{R}^n \text{ and } c \in \mathbb{R} \qquad \text{(homogeneity)}$$

Fact 1.8.1. *If A is an $m \times n$ matrix, then $T : \mathbf{x} \mapsto A\mathbf{x}$ is a linear transformation from $\mathbb{R}^n$ to $\mathbb{R}^m$. (See 1.4.1.) The range of T is exactly the span of the columns of A.*

1.8.re1, continued. The transformation V is a linear transformation, since

$$V\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) = \begin{bmatrix} x_1 \\ x_1 - x_2 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 1 & -1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}.$$

The range of V is the span of the two columns of this matrix. By 1.4.2, two vectors cannot span all of $\mathbb{R}^3$; this explains why were were able to find a vector in $\mathbb{R}^3$ not in $\operatorname{ran} V$.

In the next section, we'll see that the converse of Fact 1.8.1 is also true.

1.8.re3. The linear transformation $T(\mathbf{x}) = \begin{bmatrix} 1 & -1 \\ 2 & 0 \\ 3 & -2 \end{bmatrix} \mathbf{x}$ maps $\mathbb{R}^2$ into $\mathbb{R}^3$. The point $\begin{bmatrix} 5 \\ 6 \\ 13 \end{bmatrix}$ is in the range of T, since $\begin{bmatrix} 1 & -1 \\ 2 & 0 \\ 3 & -2 \end{bmatrix} \mathbf{x} = \begin{bmatrix} 5 \\ 6 \\ 13 \end{bmatrix}$ has the solution $\mathbf{x} = \begin{bmatrix} 3 \\ -2 \end{bmatrix}$ (as can be found by row elimination). However, $\begin{bmatrix} 1 & -1 \\ 2 & 0 \\ 3 & -2 \end{bmatrix} \sim \begin{bmatrix} 1 & -1 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$, and so there are vectors in $\mathbb{R}^3$ that are not equal $T(\mathbf{x})$ for *any* $\mathbf{x} \in \mathbb{R}^2$.

1.8.re4. For the given matrix A and vector $\mathbf{u}$, determine whether $\mathbf{u}$ lies in the domain, the codomain, or range of the transformation $T(\mathbf{x}) = A\mathbf{x}$.

a. $A = \begin{bmatrix} 1 & -1 & 9 \\ 1 & -4 & 27 \\ -3 & 3 & -12 \end{bmatrix}; \mathbf{u} = \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix}$

b. $A = \begin{bmatrix} -1 & 1 \\ -5 & 7 \\ 2 & -3 \end{bmatrix}; \mathbf{u} = \begin{bmatrix} 4 \\ 30 \\ -14 \end{bmatrix}$

c. $A = \begin{bmatrix} 2 & 3 & -1 \\ -4 & -3 & 5 \end{bmatrix}; \mathbf{u} = \begin{bmatrix} 7 \\ -17 \end{bmatrix}$

d. $A = \begin{bmatrix} 1 & -1 & 0 \\ 1 & -4 & -18 \\ 0 & -3 & -15 \\ 4 & -4 & 3 \end{bmatrix}; \mathbf{u} = \begin{bmatrix} 3 \\ 0 \\ -2 \\ 13 \end{bmatrix}$

Answers

1.8.re.e4a. dom $= \mathbb{R}^3 =$ codom, so $\mathbf{u}$ is in both the domain and codomain. $A \sim \begin{bmatrix} 1 & -1 & 9 \\ 0 & -1 & 6 \\ 0 & 0 & 3 \end{bmatrix}$, and so the columns of A span $\mathbb{R}^3$. Therefore $\mathbf{u}$ is in the range of T. 1.8.re.e4b. $\mathbf{u} \notin$ dom $= \mathbb{R}^2$; $\mathbf{u} \in$ codom $= \mathbb{R}^3$. $\begin{bmatrix} -1 & 1 & 4 \\ -5 & 7 & 30 \\ 2 & -3 & -14 \end{bmatrix} \sim \begin{bmatrix} -1 & 1 & 4 \\ 0 & 1 & 5 \\ 0 & 0 & 1 \end{bmatrix}$, so $\mathbf{u}$ is not in the range of T. 1.8.re.e4c. $\mathbf{u} \notin$ dom $= \mathbb{R}^3$; $\mathbf{u} \in$ codom $= \mathbb{R}^2$. Since $A \sim \begin{bmatrix} 1 & 0 & -2 \\ 0 & 1 & 1 \end{bmatrix}$ has a pivot in every row, the range of T is also $= \mathbb{R}^2$, and therefore $\mathbf{u}$ is in the range of T. 1.8.re.e4d. $\mathbf{u} \notin$ dom $= \mathbb{R}^3$. $\mathbf{u} \in$ codom $= \mathbb{R}^4$. $\mathbf{u}$ is in the range of T, since there's no pivot in the last column of the augmented matrix:

$$\begin{bmatrix} 1 & -1 & 0 & 3 \\ 1 & -4 & -18 & 0 \\ 0 & -3 & -15 & -2 \\ 4 & -4 & 3 & 13 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 3 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

1.9: The matrix representation of a linear transformation

The identity matrix

The $n \times n$ **identity matrix** I_n is the square matrix with ones on its main diagonal and zeros elsewhere:

$$I_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \quad I_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad I_4 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

If the value of n is clear from context, the identity matrix is simply referred to as I.
The columns of I_n are called $\mathbf{e}_1$, $\mathbf{e}_2$, etc.. For instance, if $n = 4$, then

$$\mathbf{e}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} \quad \mathbf{e}_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} \quad \mathbf{e}_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} \quad \mathbf{e}_4 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

The identity matrix is so named because of the property that

$$\begin{aligned} I_n\mathbf{x} &= x_1\mathbf{e}_1 + x_2\mathbf{e}_2 + \cdots + x_n\mathbf{e}_n \\ &= \mathbf{x} \text{ for all } \mathbf{x} \in \mathbb{R}^n. \end{aligned}$$

The standard matrix

Fact 1.9.1. *If T is a linear transformation from $\mathbb{R}^n$ into $\mathbb{R}^m$, then there exists an $m \times n$ matrix A, called the* **standard matrix** *for T, so that*

(1.9.2) $$T(\mathbf{x}) = A\mathbf{x} \text{ for all } \mathbf{x} \in \mathbb{R}^n.$$

The columns of A are the image under T of the columns of I_n:

$$A = [\, T(\mathbf{e}_1) \quad T(\mathbf{e}_2) \quad \cdots \quad T(\mathbf{e}_n) \,]$$

1.9.re1. To find the standard matrix for the linear transformation

$$U : \mathbb{R}^3 \to \mathbb{R}^2 : \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \mapsto \begin{bmatrix} x_1 + 2x_2 - 3x_3 \\ -2x_1 - 4x_2 \end{bmatrix},$$

calculate

$$U\left(\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} 1 \\ -2 \end{bmatrix} \qquad U\left(\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} 2 \\ -4 \end{bmatrix} \qquad U\left(\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} -3 \\ 0 \end{bmatrix}$$

Therefore, the standard matrix is

$$\begin{bmatrix} 1 & 2 & -3 \\ -2 & -4 & 0 \end{bmatrix}.$$

Indeed,

$$\begin{bmatrix} 1 & 2 & -3 \\ -2 & -4 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} x_1 + 2x_2 - 3x_3 \\ -2x_1 - 4x_2 \end{bmatrix} = U(\mathbf{x}).$$

One-to-one and onto transformations

A transformation

$$T : \mathbb{R}^n \to \mathbb{R}^m$$

is **one-to-one** (sometimes written "1-1") if $T(\mathbf{u}) = T(\mathbf{v})$ implies $\mathbf{u} = \mathbf{v}$.
T maps $\mathbb{R}^n$ **onto** $\mathbb{R}^m$ (or simply "T is onto") if $\operatorname{ran} T$ equals all of $\mathbb{R}^m$, that is, if every $\mathbf{b} \in \mathbb{R}^m$ equals $T(\mathbf{x})$ for some $\mathbf{x} \in \mathbb{R}^n$

Fact 1.9.3. *If the transformation $T : \mathbb{R}^n \to \mathbb{R}^m$ is linear and A is its standard matrix, then*

T is onto
iff the columns of A span $\mathbb{R}^m$.
iff the system $A\mathbf{x} = \mathbf{b}$ is consistent for every $\mathbf{b} \in \mathbb{R}^m$.
iff the row echelon form of A has no row of zeros.
iff ref *A has a pivot in every row.*

and

T is one-to-one
iff the columns of A are linearly independent.
iff the system $A\mathbf{x} = \mathbf{0}$ has only the trivial solution $\mathbf{x} = \mathbf{0}$.
iff Ax has no free variables.
iff ref *A has a pivot in every column.*

(Compare with 1.4.2 and 1.7.3.)

1.9.re1, continued. Applying the row operation $\mathbf{r}_2 \leftarrow \mathbf{r}_2 + 2\,\mathbf{r}_1$ to standard matrix of U shows that

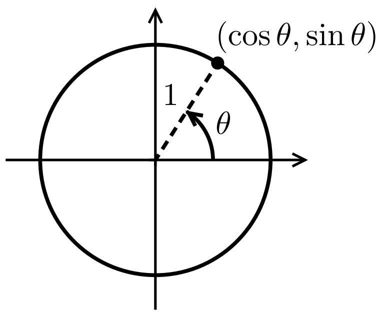
$$\begin{bmatrix} 1 & 2 & -3 \\ -2 & -4 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & -3 \\ 0 & 0 & -6 \end{bmatrix}.$$

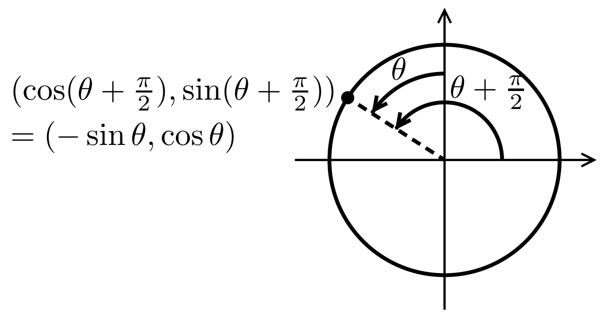
This row echelon form contains a pivot in every row, so U maps $\mathbb{R}^3$ onto $\mathbb{R}^2$. However, there's not a pivot in every column, so U fails to be one-to-one. For instance, U sends both $\begin{bmatrix} 2 \\ 0 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$ to $\begin{bmatrix} 2 \\ -4 \end{bmatrix}$

Linear transformations in $\mathbb{R}^2$

See Examples 2-3 and Tables 1-4 in section 1.9 of our text (pages 76-80) for some common geometric transformations from $\mathbb{R}^2$ into $\mathbb{R}^2$ that are linear. Any such transformation has a 2×2 standard matrix.

By their definition, $\cos\theta$ and $\sin\theta$ are the coordinates of the point θ radians from the positive x-axis on the unit circle, in the counterclockwise direction if $\theta > 0$ and clockwise if $\theta < 0$.





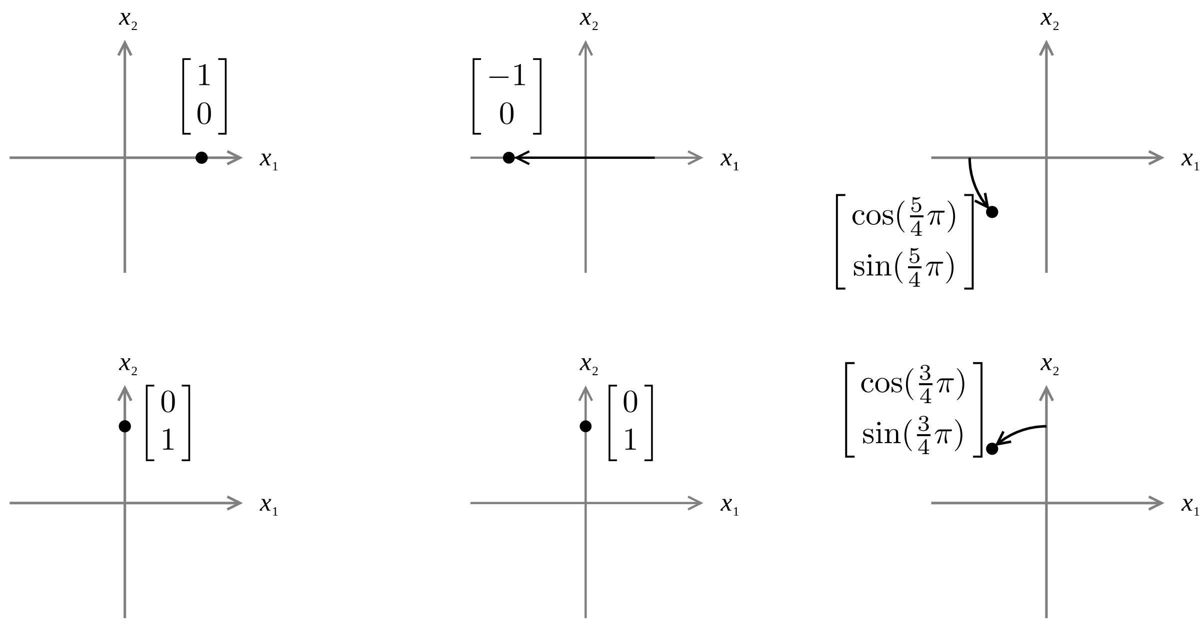
Consequently, the standard matrix for rotation by θ radians is

$$\begin{bmatrix} \cos\theta & \cos(\theta + \frac{\pi}{2}) \\ \sin\theta & \sin(\theta + \frac{\pi}{2}) \end{bmatrix}, \quad \text{or} \quad \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}.$$

1.9.re2. Find the standard matrix for the linear transformation $T : \mathbb{R}^2 \to \mathbb{R}^2$ that reflects across the x_2-axis and the rotates the result by $\frac{\pi}{4}$ radians in the positive (counterclockwise) direction. Is T one-to-one? Is it onto?

To find the columns of its standard matrix, find the image under T of $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$:

Starting point — reflected about $x_1 = 0$ — then rotated $\frac{\pi}{4}$ radians:



Therefore the standard matrix is $A = \begin{bmatrix} \cos(\frac{5}{4}\pi) & \cos(\frac{3}{4}\pi) \\ \sin(\frac{5}{4}\pi) & \sin(\frac{3}{4}\pi) \end{bmatrix} = \frac{1}{\sqrt{2}} \begin{bmatrix} -1 & -1 \\ -1 & 1 \end{bmatrix}$.

T is one-to-one since the reduced row echelon form of its standard matrix, $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, has a pivot in every column. Because there's also a pivot in every row, T is onto.

1.9.re3. Find the standard matrix of the given linear transformation. Is the transformation one-to-one? Is it onto?

a. $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \mapsto \begin{bmatrix} x_1 + 2x_3 \\ x_1 + 2x_2 + x_3 \end{bmatrix}$

b. $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \mapsto \begin{bmatrix} x_2 \\ -x_1 \\ x_1 + x_2 \end{bmatrix}$

c. $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \mapsto \begin{bmatrix} x_1 + 2x_3 \\ x_1 + 2x_2 + x_3 \\ 2x_2 - x_3 \end{bmatrix}$

d. $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \mapsto \begin{bmatrix} x_2 + x_3 \\ -x_1 - x_3 \\ x_1 + x_2 + x_3 \end{bmatrix}$

e. The transformation that rotates points in $\mathbb{R}^2$ $-\pi/3$ radians about the origin.

f. The transformation that sends each point in $\mathbb{R}^2$ to its reflection across the line $x_1 = -x_2$.

g. The transformation that rotates points in $\mathbb{R}^2$ $\pi/4$ radians about the origin, and then projects the result onto the line $x_1 = 0$.

h. The horizontal sheer in $\mathbb{R}^2$ that leaves points on the x_1-axis unchanged, but sends points $(0, x_2)$ to $(\frac{1}{2}x_2, x_2)$.

i. The vertical sheer in $\mathbb{R}^2$ that leaves points on the x_2-axis unchanged, but sends points $(x_1, 0)$ to $(x_1, 2x_1)$.

Answers

1.9.re.e3a. $\begin{bmatrix} 1 & 0 & 2 \\ 1 & 2 & 1 \end{bmatrix}$. onto, not 1-1. 1.9.re.e3b. $\begin{bmatrix} 0 & 1 \\ -1 & 0 \\ 1 & 1 \end{bmatrix}$. 1-1, not onto. 1.9.re.e3c. $\begin{bmatrix} 1 & 0 & 2 \\ 1 & 2 & 1 \\ 0 & 2 & -1 \end{bmatrix}$. neither 1-1 nor onto. 1.9.re.e3d. $\begin{bmatrix} 0 & 1 & 1 \\ -1 & 0 & -1 \\ 1 & 1 & 1 \end{bmatrix}$. both 1-1 and onto. 1.9.re.e3e. $\frac{1}{2}\begin{bmatrix} 1 & \sqrt{3} \\ -\sqrt{3} & 1 \end{bmatrix}$. both 1-1 and onto. 1.9.re.e3f. $\begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}$. both 1-1 and onto. 1.9.re.e3g. $\frac{1}{\sqrt{2}}\begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix}$. neither 1-1 nor onto. 1.9.re.e3h. $\begin{bmatrix} 1 & \frac{1}{2} \\ 0 & 1 \end{bmatrix}$. 1-1 and onto. 1.9.re.e3i. $\begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix}$. 1-1 and onto.

2.1: Matrix arithmetic

Matrix addition, scalar multiplication

To multiply a matrix by a scalar is to multiply each element by the scalar.
Addition of two matrices is defined only when the matrices are of the same size. In that case, their sum is calculated term-by-term.

2.1.re1. Find the following, if it exists.

a. $\begin{bmatrix} 1 & -1 & 2 \\ 6 & -2 & -3 \end{bmatrix} + \begin{bmatrix} 0 & 8 & 1 \\ -2 & -3 & 10 \end{bmatrix}$ b. $-2\begin{bmatrix} 1 & 0 & 2 \\ -3 & 2 & 1 \\ 0 & 2 & -1 \end{bmatrix}$ c. $\begin{bmatrix} 1 & 2 \\ 2 & 1 \\ -1 & 1 \end{bmatrix} + \begin{bmatrix} 5 \\ 0 \\ 1 \end{bmatrix}$

Matrix multiplication

If the number of columns of A equals the number of rows of B, then the jth column of their product AB is defined as the A times the jth column of B:

$$A\,[\,\mathbf{b}_1 \quad \mathbf{b}_2 \quad \cdots \quad \mathbf{b}_p\,] = [\,A\mathbf{b}_1 \quad A\mathbf{b}_2 \quad \cdots \quad A\mathbf{b}_p\,]$$

1. The columns of AB are linear combinations of the columns of A.
2. The rows of AB are linear combinations of the rows of B.
3. The i, j element of AB is the ith row of A times the j column of B.
4. If A is an $m \times p$ matrix and B is $p \times n$ then their product AB is $m \times n$.

2.1.re2. Find the following, if it exists.

a. $\begin{bmatrix} 1 & -1 & 3 \\ 2 & 4 & 0 \end{bmatrix} \begin{bmatrix} 4 & 4 \\ 2 & 1 \\ 2 & -3 \end{bmatrix}$

b. $\begin{bmatrix} 1 & -1 & 2 \\ 1 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & -1 \\ -1 & 1 \\ -1 & 1 \end{bmatrix}$

c. $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & -1 & 3 & 4 \\ 2 & 0 & 1 & 1 \\ 9 & 8 & 7 & 6 \end{bmatrix}$

d. $\begin{bmatrix} 2 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 2 & 1 \\ 0 & 1 \end{bmatrix}$

e. $\begin{bmatrix} 2 & 1 & 0 \\ 1 & -1 & 3 \end{bmatrix} \begin{bmatrix} 1 & 3 \\ 3 & 4 \\ 4 & 0 \end{bmatrix}$

f. $\begin{bmatrix} 5 & 10 \\ 10 & -1 \end{bmatrix} \begin{bmatrix} 2 & 0 \\ 1 & 1 \end{bmatrix}$

g. $\begin{bmatrix} 1 & 3 \\ 3 & 4 \\ 4 & 0 \end{bmatrix} \begin{bmatrix} 2 & 0 \\ 1 & 1 \end{bmatrix}$

h. $\begin{bmatrix} 2 & 1 & 0 \\ 1 & -1 & 3 \end{bmatrix} \begin{bmatrix} 5 & 3 \\ 10 & 4 \\ 8 & 0 \end{bmatrix}$

Matrix multiplication is defined so that, if A is the standard matrix of the linear operator T, and B is standard matrix of the operator S, then AB is the standard matrix of the composition $T \circ S$. That is, $T\big(S(\mathbf{x})\big) = (AB)\mathbf{x}$ for all $\mathbf{x}$ (in the domain of S).

2.1.re3. Find the standard matrices A and B of the given linear operators on $\mathbb{R}^2$. Then compute the product BA and see that equals the answer to 1.9.re2.

a. Reflection across the x_2-axis. b. Rotation about the origin by $\pi/4$.

We use the symbol 0 to denote the all-zero matrix (of size made clear by context).
Matrix arithmetic obeys laws similar to the laws of real number arithmetic. See Theorems 1 and 2 in the text. Some important exceptions:

1. AB is not necessarily equal BA.
2. If $AB = 0$, neither A nor B is necessarily equal 0.

For examples of these, see 2.1.re2.b and d.

Transpose

If A is a matrix, its transpose is the matrix defined by $(A^T)_{i,j} = A_{j,i}$.

2.1.re4. $\begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 2 \end{bmatrix}^T = \begin{bmatrix} 1 & 0 \\ 2 & 1 \\ 3 & 2 \end{bmatrix}$.

See Theorem 3 in the text for important algebraic laws involving the transpose.

2.1.re5. Compute the following.

a. $\begin{bmatrix} 2 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} -1 & 1 \\ 2 & 1 \\ -1 & 1 \end{bmatrix}$ b. $\begin{bmatrix} -1 & 2 & -1 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 2 & 1 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}$

Answers

2.1.re.e1a. $\begin{bmatrix} 1 & 7 & 3 \\ 4 & -5 & 7 \end{bmatrix}$ 2.1.re.e1b. $\begin{bmatrix} -2 & 0 & -4 \\ 6 & -4 & -2 \\ 0 & -4 & 2 \end{bmatrix}$. 2.1.re.e1c. d.n.e. 2.1.re.e2a. $\begin{bmatrix} 8 & -6 \\ 16 & 12 \end{bmatrix}$

2.1.re.e2b. $\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$. 2.1.re.e2c. $\begin{bmatrix} 1 & -1 & 3 & 4 \\ 2 & 0 & 1 & 1 \\ 9 & 8 & 7 & 6 \end{bmatrix}$ 2.1.re.e2d. $\begin{bmatrix} 0 & 2 \\ 0 & 0 \end{bmatrix}$ (not zero!) 2.1.re.e2e. $\begin{bmatrix} 5 & 10 \\ 10 & -1 \end{bmatrix}$

2.1.re.e2f. $\begin{bmatrix} 20 & 10 \\ 19 & -1 \end{bmatrix}$ 2.1.re.e2g. $\begin{bmatrix} 5 & 3 \\ 10 & 4 \\ 8 & 0 \end{bmatrix}$ 2.1.re.e2h. $\begin{bmatrix} 20 & 10 \\ 19 & -1 \end{bmatrix}$ (which equals f.)

2.1.re.e3a. $\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$ 2.1.re.e3b. $\frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}$ 2.1.re.e5a. $\begin{bmatrix} 0 & 3 \\ -2 & 2 \end{bmatrix}$ 2.1.re.e5b. $\begin{bmatrix} 0 & -2 \\ 3 & 2 \end{bmatrix}$

2.2: The inverse matrix

An $n \times n$ matrix A is said to be **invertible** if there's an $n \times n$ matrix A^{-1} so that

$$A^{-1}A = AA^{-1} = I \tag{2.2.1}$$

Note that only a square matrix can be invertible.
A square matrix that is not invertible is said to be **singular**.

2.2.re1. $\begin{bmatrix} 2 & 3 \\ 3 & 5 \end{bmatrix}^{-1} = \begin{bmatrix} 5 & -3 \\ -3 & 2 \end{bmatrix}$ and $\begin{bmatrix} 5 & -3 \\ -3 & 2 \end{bmatrix}^{-1} = \begin{bmatrix} 2 & 3 \\ 3 & 5 \end{bmatrix}$ because

$$\begin{bmatrix} 2 & 3 \\ 3 & 5 \end{bmatrix} \begin{bmatrix} 5 & -3 \\ -3 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 5 & -3 \\ -3 & 2 \end{bmatrix} \begin{bmatrix} 2 & 3 \\ 3 & 5 \end{bmatrix}$$

Fact 2.2.2.

1. *If A is invertible, then its inverse matrix is unique.*
2. *If A is invertible, then so is A^{-1} and $(A^{-1})^{-1} = A$.*
3. *If A and B are invertible (and the same size), then so is AB, and*

$$(AB)^{-1} = B^{-1}A^{-1}$$

In the homework in 2.3, we'll see that the converse to 3 is also true:

4. *If A and B are square and AB is invertible, then A and B are both invertible.*

Finding the inverse matrix

Fact 2.2.3. *The matrix A is invertible iff it is row equivalent to I, and in that case A^{-1} can be found by row reduction:*

$$[\,A \mid I\,] \sim [\,I \mid A^{-1}\,]$$

That is, the same row operations that reduce A to I will transform I to A^{-1}.

2.2.re2. To find the inverse of $\begin{bmatrix} -1 & 1 & 0 \\ 2 & -4 & 8 \\ 0 & 3 & -11 \end{bmatrix}$, augment with the 3×3 identity and row reduce.

$$(1)\ \begin{bmatrix} -1 & 1 & 0 & 1 & 0 & 0 \\ 2 & -4 & 8 & 0 & 1 & 0 \\ 0 & 3 & -11 & 0 & 0 & 1 \end{bmatrix}$$

$$(2)\ \begin{bmatrix} 1 & -1 & 0 & -1 & 0 & 0 \\ 2 & -4 & 8 & 0 & 1 & 0 \\ 0 & 3 & -11 & 0 & 0 & 1 \end{bmatrix}$$

$$(3)\ \begin{bmatrix} 1 & -1 & 0 & -1 & 0 & 0 \\ 0 & -2 & 8 & 2 & 1 & 0 \\ 0 & 3 & -11 & 0 & 0 & 1 \end{bmatrix}$$

$$(4)\ \begin{bmatrix} 1 & -1 & 0 & -1 & 0 & 0 \\ 0 & 1 & -4 & -1 & -\frac{1}{2} & 0 \\ 0 & 3 & -11 & 0 & 0 & 1 \end{bmatrix}$$

$$(5)\ \begin{bmatrix} 1 & -1 & 0 & -1 & 0 & 0 \\ 0 & 1 & -4 & -1 & -\frac{1}{2} & 0 \\ 0 & 0 & 1 & 3 & \frac{3}{2} & 1 \end{bmatrix}$$

$$(6)\ \begin{bmatrix} 1 & -1 & 0 & -1 & 0 & 0 \\ 0 & 1 & 0 & 11 & \frac{11}{2} & 4 \\ 0 & 0 & 1 & 3 & \frac{3}{2} & 1 \end{bmatrix}$$

$$(7)\ \begin{bmatrix} 1 & 0 & 0 & 10 & \frac{11}{2} & 4 \\ 0 & 1 & 0 & 11 & \frac{11}{2} & 4 \\ 0 & 0 & 1 & 3 & \frac{3}{2} & 1 \end{bmatrix}$$

Therefore $\begin{bmatrix} -1 & 1 & 0 \\ 2 & -4 & 8 \\ 0 & 3 & -11 \end{bmatrix}^{-1} = \begin{bmatrix} 10 & \frac{11}{2} & 4 \\ 11 & \frac{11}{2} & 4 \\ 3 & \frac{3}{2} & 1 \end{bmatrix}$. To check this answer, just confirm that the matrix product $\begin{bmatrix} -1 & 1 & 0 \\ 2 & -4 & 8 \\ 0 & 3 & -11 \end{bmatrix} \begin{bmatrix} 10 & \frac{11}{2} & 4 \\ 11 & \frac{11}{2} & 4 \\ 3 & \frac{3}{2} & 1 \end{bmatrix}$ equals $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$.

2.2.re3. Find the inverse of the given matrix or explain why it does not exist.

a. $\begin{bmatrix} -1 & 1 & 0 \\ 2 & -4 & -12 \\ 0 & 3 & 21 \end{bmatrix}$ b. $\begin{bmatrix} -1 & 0 & -3 \\ 1 & 0 & 6 \\ -1 & -1 & -6 \end{bmatrix}$ c. $\begin{bmatrix} 1 & -1 & 0 & 0 \\ 1 & -2 & -1 & 0 \\ 1 & -1 & 0 & -1 \\ 0 & 1 & 3 & 2 \end{bmatrix}$

d. $\begin{bmatrix} 2 & -2 & 4 & 8 \\ 1 & -2 & 1 & 5 \\ 0 & 2 & 3 & -2 \\ 1 & -1 & 5 & 4 \end{bmatrix}$ e. $\begin{bmatrix} 3 & 6 \\ -1 & -1 \end{bmatrix}$ f. $\begin{bmatrix} 0 & -1 & -4 \\ -1 & 1 & 1 \\ -1 & 3 & 9 \end{bmatrix}$

The inverse matrix and solutions to linear systems

Fact 2.2.4. *If the $n \times n$ matrix A is invertible, then for every $\mathbf{b} \in \mathbb{R}^n$, the vector $A^{-1}\mathbf{b}$ is the unique solution to $A\mathbf{x} = \mathbf{b}$.*

2.2.re4. Use the inverses you found in 2.2.re3 to solve the system $A\mathbf{x} = \mathbf{b}$ for the given matrix A and vector $\mathbf{b}$.

a. $\begin{bmatrix} 3 & 6 \\ -1 & -1 \end{bmatrix}, \begin{bmatrix} 3 \\ 2 \end{bmatrix}$

b. $\begin{bmatrix} -1 & 0 & -3 \\ 1 & 0 & 6 \\ -1 & -1 & -6 \end{bmatrix}, \begin{bmatrix} -1 \\ -1 \\ 2 \end{bmatrix}$

c. $\begin{bmatrix} -1 & 1 & 0 \\ 2 & -4 & -12 \\ 0 & 3 & 21 \end{bmatrix}, \begin{bmatrix} 1 \\ -4 \\ 0 \end{bmatrix}$

d. $\begin{bmatrix} 1 & -1 & 0 & 0 \\ 1 & -2 & -1 & 0 \\ 1 & -1 & 0 & -1 \\ 0 & 1 & 3 & 2 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ -1 \\ 0 \end{bmatrix}$

e. $\begin{bmatrix} -1 & 1 & 0 \\ 2 & -4 & -12 \\ 0 & 3 & 21 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 2 \end{bmatrix}$

f. $\begin{bmatrix} 1 & -1 & 0 & 0 \\ 1 & -2 & -1 & 0 \\ 1 & -1 & 0 & -1 \\ 0 & 1 & 3 & 2 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 2 \\ 8 \end{bmatrix}$

Answers

2.2.re.e3a. $\begin{bmatrix} -8 & -\frac{7}{2} & -2 \\ -7 & -\frac{7}{2} & -2 \\ 1 & \frac{1}{2} & \frac{1}{3} \end{bmatrix}$ 2.2.re.e3b. $\begin{bmatrix} -2 & -1 & 0 \\ 0 & -1 & -1 \\ \frac{1}{3} & \frac{1}{3} & 0 \end{bmatrix}$. 2.2.re.e3c. $\begin{bmatrix} \frac{7}{2} & -\frac{3}{2} & -1 & -\frac{1}{2} \\ \frac{5}{2} & -\frac{3}{2} & -1 & -\frac{1}{2} \\ -\frac{3}{2} & \frac{1}{2} & 1 & \frac{1}{2} \\ 1 & 0 & -1 & 0 \end{bmatrix}$

2.2.re.e3d. dne. rref of given matrix is $\begin{bmatrix} 1 & 0 & 0 & 3 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$ 2.2.re.e3e. $\begin{bmatrix} -\frac{1}{3} & -2 \\ \frac{1}{3} & 1 \end{bmatrix}$. 2.2.re.e3f. dne. rref of given matrix is $\begin{bmatrix} 1 & 0 & 3 \\ 0 & 1 & 4 \\ 0 & 0 & 0 \end{bmatrix}$ 2.2.re.e4a. $\begin{bmatrix} -5 \\ 3 \end{bmatrix}$. 2.2.re.e4b. $\begin{bmatrix} 3 \\ -1 \\ -\frac{2}{3} \end{bmatrix}$. 2.2.re.e4c. $\begin{bmatrix} 6 \\ 7 \\ -1 \end{bmatrix}$

2.2.re.e4d. $\begin{bmatrix} -\frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} & 1 \end{bmatrix}^T$ 2.2.re.e4e. $\begin{bmatrix} -4 & -4 & -\frac{2}{3} \end{bmatrix}^T$ 2.2.re.e4f. $\begin{bmatrix} -\frac{5}{2} & -\frac{7}{2} & \frac{9}{2} & -1 \end{bmatrix}^T$

2.3: The invertible matrix theorem

The invertible matrix theorem 2.3.1. *Let A be an $n \times n$ matrix. The following statements are equivalent. That is, for any given A, if any one of these is true, then all of them are true.*

1. *A is invertible.*
2. *$A \sim I$.*
3. *A has n pivots*
4. *A has a pivot in every row.*
    5. *The columns of A span $\mathbb{R}^n$.*
    6. *$A\mathbf{x} = \mathbf{b}$ is consistent for every $\mathbf{b} \in \mathbb{R}^n$.*
    7. *The transformation $T : \mathbb{R}^n \to \mathbb{R}^n : \mathbf{x} \mapsto A\mathbf{x}$ is **onto***.*
8. *A has a pivot in every column.*
    9. *The columns of A are linearly independent.*
    10. *The only solution to $A\mathbf{x} = \mathbf{0}$ is the trivial solution $\mathbf{x} = \mathbf{0}$.*
    11. *The transformation $T : \mathbb{R}^n \to \mathbb{R}^n : \mathbf{x} \mapsto A\mathbf{x}$ is **one-to-one***.*
12. *There's a matrix B for which $AB = I$.*
13. *There's a matrix C for which $CA = I$.*
14. *A^T is invertible.*

*A transformation that is both one-to-one and onto is said to be **invertible**. The above implies that the transformation

$$T : \mathbb{R}^n \to \mathbb{R}^n : \mathbf{x} \mapsto A\mathbf{x}$$

is invertible iff its standard matrix A is invertible. In that case, the transformation

$$T^{-1} : \mathbb{R}^n \to \mathbb{R}^n : \mathbf{x} \mapsto A^{-1}\mathbf{x}$$

is called the **inverse** of T and satisfies

$$T(T^{-1}(\mathbf{x})) = T^{-1}(T(\mathbf{x})) = \mathbf{x} \quad \text{for all } \mathbf{x} \in \mathbb{R}^n.$$

Diagonal and triangular matrices

The **main diagonal** of a matrix A is the set of its elements $\{A_{1,1}, A_{2,2}, A_{3,3}, \ldots\}$. A **diagonal matrix** is a square matrix whose only nonzero elements lie on the main diagonal.

An **upper triangular matrix** is one whose every element below the main diagonal is zero. A **lower triangular matrix** is one whose every element above the main diagonal is zero. A **triangular matrix** is one that is either upper or lower triangular.

A triangular matrix need not be square. A diagonal matrix is both upper and lower triangular.

2.3.re1. Identify the given matrix as either upper triangular, lower triangular, diagonal, or none of these.

a. $\begin{bmatrix} -1 & 1 & 0 & 3 \\ 0 & -4 & 12 & 7 \\ 0 & 0 & 9 & -1 \end{bmatrix}$ b. $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix}$ c. $\begin{bmatrix} 9 & 0 & 0 & 0 \\ 0 & -2 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 2 \end{bmatrix}$

d. $\begin{bmatrix} 1 & 0 & 0 & 0 \\ 2 & 3 & 0 & 0 \\ 1 & -1 & 5 & 0 \end{bmatrix}$ e. $\begin{bmatrix} 3 & 6 \\ -6 & 2 \end{bmatrix} + \begin{bmatrix} 1 & 4 \\ -6 & 3 \end{bmatrix}^T$ f. $\begin{bmatrix} -1 & 0 & -3 \\ 0 & 0 & 6 \\ 0 & 0 & -6 \end{bmatrix}$

g. $\begin{bmatrix} 0 & 0 & 1 \\ -1 & 0 & 1 \\ 1 & 1 & 1 \end{bmatrix}$

Fact 2.3.2. *A square triangular matrix is invertible if and only if its diagonal elements are all nonzero.*

2.3.re2. Which of the matrices in 2.3.re1 are invertible?

2.3.re3. Suppose the linear transformation T maps $\mathbb{R}^3$ onto itself. How many solutions are there to $T(\mathbf{x}) = [0\ 1\ 2]^T$?

2.3.re4. Suppose A is an $n \times n$ matrix whose first and second columns have the same sum as its third and fourth columns. Must there be a $\mathbf{b} \in \mathbb{R}^n$ for which ...

i. $A\mathbf{x} = \mathbf{b}$ has more than one solution?
ii. $A\mathbf{x} = \mathbf{b}$ has no solution?

2.3.re5. Suppose S maps $\mathbb{R}^3$ into itself but that $[1\ 0\ 1]^T$ fails to be in the range of S. Could S still be one-to-one?

2.3.re6. Suppose $R : \mathbb{R}^3 \to \mathbb{R}^4$ is linear and one-to-one. Could R also be onto?

2.3.re7. Suppose A is a square matrix whose row echelon form includes a row of zeros. Must the row echelon form of A^T also include a row of zeros?

2.3.re8. Suppose $V : \mathbb{R}^3 \to \mathbb{R}^2$ is linear and but not one-to-one. Could V be onto?

2.3.re9. Suppose A is a square matrix and that $AB = AC$ but $B \neq C$. What can you say about the invertibility of A?

2.3.re10. Suppose A is a not-necessarily-square matrix and that $AB = 0$ but $B \neq 0$ (the zero matrix). What do you know about the linear independence of the columns of A?

2.3.re11. If the columns of the square matrix A are linearly independent, what do we know about the linear dependence/independence of the rows of A?

2.3.re12. Suppose the transformation $\mathbf{x} \mapsto A\mathbf{x}$ is an invertible transformation from $\mathbb{R}^n$ to $\mathbb{R}^n$. What do we know about the number of pivots of the matrix A^3?

Answers

2.3.re.e1a. upper triangular. 2.3.re.e1b. lower triangular. 2.3.re.e1c. diagonal. 2.3.re.e1d. lower triangular 2.3.re.e1e. $= \begin{bmatrix} 4 & 0 \\ -2 & 5 \end{bmatrix}$, lower triangular. 2.3.re.e1f. upper triangular. 2.3.re.e1g. neither triangular nor diagonal. 2.3.re2. Not a or d, since an invertible matrix must be square; not f, since it is triangular and has a 0 on its main diagonal. b, c, d, and e are invertible because these triangular matrices have no zeros along their main diagonal. Even though some of its diagonal elements are zero, g is invertible, because it row reduces to I. 2.3.2 doesn't apply to g since that matrix isn't triangular. 2.3.re3. One. There's one solution since T is onto. Since T is onto, it's also $1-1$, so there can't be two solutions. 2.3.re4. a. Yes. $A\mathbf{x} = \mathbf{0}$ has at least two solutions, because $A\mathbf{0}$ and $A(\mathbf{e}_1 + \mathbf{e}_2 - \mathbf{e}_3 - \mathbf{e}_4)$ both equal $\mathbf{0}$. b. Yes. As shown in a., the collumns of A are linearly dependent. By the invertible matrix theorem, since A is square, its columns must also fail to span $\mathbb{R}^n$, meaning that there are $\mathbf{b} \in \mathbb{R}^n$ which can't be expressed as linear combinations of the columns of A. 2.3.re5. No. The invertible matrix theorem promises that a linear map form $\mathbb{R}^n$ into itself is one-to-one iff its is onto. Since S fails to be onto, it cannot be one-to-one 2.3.re6. No. The standard matrix of R has four rows and three columns. For R to be onto would require that there be a pivot in every row of its standard matrix. Since there can be at most pivot per column, there must be at least one row without a pivot. 2.3.re7. Yes, by several applications of the invertible matrix theorem. If A does not have a pivot in every row, then A is singular (i.e., not invertible). Therefore A^T is also singular, its row echelon form must include a row with no pivot, that is, a row of zeros. 2.3.re8. Yes. All that's necessary is that the 2×3 standard matrix of V have a pivot in every row. $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \end{bmatrix}$ is one such matrix. 2.3.re9. A is not invertible If A were invertible and $AB = AC$, then we could multiply both sides by A^{-1} to conclude that $B = C$:

$$A^{-1}AB = A^{-1}AC \implies IB = IC \implies B = C.$$

2.3.re10. The columns of A must be linear dependent. Since $B \neq 0$, there must be at least one j which the jth column of B does not equal the zero vector $\mathbf{0}$; that is $B\mathbf{e}_j \neq \mathbf{0}$. But $A(B\mathbf{e}_j) = (AB)\mathbf{e}_j = 0\mathbf{e}_j = \mathbf{0}$. That is, $A\mathbf{x} = \mathbf{0}$ has a nontrivial solution. 2.3.re11. By the invertible matrix theorem, the linear independence of the columns of A implies that A is invertible, which in turn implies that A^T is invertible, which implies that its columns are linearly independent. Since the columns of A^T are the rows of A, the rows of A must also be linearly dependent. 2.3.re12. By the invertible matrix theorem, A is invertible. Since the product of invertible matrices is invertible, $A^3 = AAA$ is also invertible and hence has n pivots.

3.1: Introduction to determinants

The **determinant** is a function whose domain is the set of all square matrices and whose range is $\mathbb{R}$. The determinant of a matrix A is denoted either $\det(A)$ or $|A|$; despite the resemblance to the absolute value function, $|A|$ can be negative.
The determinant of a scalar, that is, a 1×1 matrix, is scalar itself: $\det(a) = a$
The determinant of larger matrices can be written in terms of determinants of smaller submatrices by **cofactor expansion** along any row of column. For example, expanding a 2×2 determinant along the first column looks like this:

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} = +\,a|d| - c|b| = ad - cb$$

Expanding a 3×3 along the first row looks like this:

$$\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} = +\,a\begin{vmatrix} e & f \\ h & i \end{vmatrix} - b\begin{vmatrix} d & f \\ g & i \end{vmatrix} + c\begin{vmatrix} d & e \\ g & h \end{vmatrix}$$

Expanding the same 3×3 along the middle column looks like this:

$$\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} = -\,b\begin{vmatrix} d & f \\ g & i \end{vmatrix} + e\begin{vmatrix} a & c \\ g & i \end{vmatrix} - h\begin{vmatrix} a & c \\ d & f \end{vmatrix}$$

When we expand along a row or column, we attach the signs seen above according to this pattern:

$$\begin{matrix} + & - & + & - & \cdots \\ - & + & - & + & \cdots \\ + & - & + & - & \cdots \\ - & + & - & + & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{matrix} \tag{3.1.1}$$

3.1.re1. Find the determinant by expansion along the first row and again along the first column.

a. $\begin{vmatrix} -1 & 1 \\ 2 & -4 \end{vmatrix}$ b. $\begin{vmatrix} 2 & 3 \\ 4 & -1 \end{vmatrix}$ c. $\begin{vmatrix} 1 & 2 & 3 \\ 4 & 0 & -1 \\ 5 & -2 & -3 \end{vmatrix}$ d. $\begin{vmatrix} -1 & 2 & 3 \\ 0 & 5 & -6 \\ 0 & 0 & 6 \end{vmatrix}$

In general, cofactor expansion along the ith row of an $n \times n$ matrix A tells us that

$$\det(A) = \sum_{j=1}^{n} (-1)^{i+j} a_{i,j} \det(A_{i,j})$$

where $a_{i,j}$ stands for the ith row, jth column element of A, and $A_{i,j}$ stands for the submatrix of A obtained by deleting its ith row and jth column. Expansion along the jth column says that

$$\det(A) = \sum_{i=1}^{n} (-1)^{i+j} a_{i,j} \det(A_{i,j}).$$

The (i, j)-**cofactor** of A is defined as

$$C_{i,j} = (-1)^{i+j} \det(A_{i,j}).$$

The signs of $(-1)^{i+j}$ are seen in (3.1.1). These are not necessarily the signs of $C_{i,j}$

Fact 3.1.2. *The determinant of a square matrix can be computed by cofactor expansion along any row or column (and the result will be the same).*

Fact 3.1.3. *The determinant of a square triangular matrix is the product of its main diagonal elements.*

Note that 3.1.3 applies to diagonal matrices. The proof of 3.1.2 goes beyond the scope of MATH 203, but, assuming these, you obtain 3.1.3 by expansion along the first column of an upper triangular matrix, or first row of a lower triangular matrix.

3.1.re2. Find the determinant.

a. $\begin{vmatrix} -1 & 0 \\ 2 & -4 \end{vmatrix}$ b. $\begin{vmatrix} 1 & 2 & 3 \\ 0 & 0 & -1 \\ 0 & 0 & 9 \end{vmatrix}$ c. $\begin{vmatrix} 1 & 0 & 0 & 0 \\ 0 & 9 & 0 & 0 \\ -1 & 2 & 7 & 0 \\ 8 & 0 & 0 & \frac{1}{3} \end{vmatrix}$ d. $\begin{vmatrix} 3 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & -5 & 0 \\ 0 & 0 & 0 & -7 \end{vmatrix}$

Answers

3.1.re.e1a. $(-1)(-4) - 1(2) = 2 = (-1)(-4) - 2(1)$ 3.1.re.e1b. $2(-1) - 3(4) = -14 = 2(-1) - 4(3)$

3.1.re.e1c. $1\begin{vmatrix} 0 & -1 \\ -2 & -3 \end{vmatrix} - 2\begin{vmatrix} 4 & -1 \\ 5 & -3 \end{vmatrix} + 3\begin{vmatrix} 4 & 0 \\ 5 & -2 \end{vmatrix} = 1(-2) - 2(-12+5) - 3(-8) = -12.$

$1\begin{vmatrix} 0 & -1 \\ -2 & -3 \end{vmatrix} - 4\begin{vmatrix} 2 & 3 \\ -2 & -3 \end{vmatrix} + 5\begin{vmatrix} 2 & 3 \\ 0 & -1 \end{vmatrix} = 1(-2) - 4(0) + 5(-2)$ also $= -12.$

3.1.re.e1d. $-1\begin{vmatrix} 5 & -6 \\ 0 & 6 \end{vmatrix} - 2\begin{vmatrix} 0 & -6 \\ 0 & 6 \end{vmatrix} + 3\begin{vmatrix} 0 & 5 \\ 0 & 0 \end{vmatrix} = -1(30) - 2(0) + 3(0) = -30.$

$-1\begin{vmatrix} 5 & -6 \\ 0 & 6 \end{vmatrix} - 0\begin{vmatrix} -2 & 3 \\ 0 & 6 \end{vmatrix} + 0\begin{vmatrix} 2 & 3 \\ 5 & -6 \end{vmatrix} = -1(30) + 0 + 0$ also $= -30.$

3.1.re.e2a. 4 3.1.re.e2b. 0. 3.1.re.e2c. 21. 3.1.re.e2d. 210.

3.2: Properties of determinants

Elementary matrices

An **elementary matrix** is any matrix obtained from an identity matrix by one row operation.
Any elementary row operation is linear transformation from $\mathbb{R}^n$ into itself, and the elementary matrix obtained by performing the operation on the identity is the standard matrix of that transformation. That is, if E is the elementary matrix corresponding to a particular row operation, then EA is the matrix obtained by performing that operation on the matrix A.
A square matrix is invertible iff it row reduces to the identity iff it is a product of elementary matrices.

3.2.re1. Find the 3×3 elementary matrix E corresponding to the given row operation.

a. $\mathbf{r}_1 \leftarrow \mathbf{r}_1 - 2\,\mathbf{r}_3$ b. $\mathbf{r}_2 \rightleftarrows \mathbf{r}_3$ c. $\mathbf{r}_3 \leftarrow 4\,\mathbf{r}_3$

Then verify that the product EA is the same as the matrix obtained from

$$A = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}$$

by performing the given row operation.

3.2.re1, continued. Find the determinants of the elementary matrices found above.

3.2.re2. What row operation transforms the first matrix to the second? Compute and compare their determinants.

a. $A = \begin{vmatrix} -1 & 1 \\ 2 & -4 \end{vmatrix}$, $\tilde{A} = \begin{vmatrix} -1 & 1 \\ -1 & 2 \end{vmatrix}$ b. $B = \begin{vmatrix} 2 & 3 & 3 \\ 4 & 0 & 3 \\ 1 & 0 & -1 \end{vmatrix}$, $\tilde{B} = \begin{vmatrix} 2 & 3 & 3 \\ 1 & 0 & -1 \\ 4 & 0 & 3 \end{vmatrix}$

c. $C = \begin{vmatrix} 1 & 2 & 3 \\ 4 & 0 & -1 \\ 0 & 0 & 1 \end{vmatrix}$, $\tilde{C} = \begin{vmatrix} 1 & 2 & 3 \\ 4 & 0 & -1 \\ 2 & 4 & 7 \end{vmatrix}$ d. $D = \begin{vmatrix} a & b \\ c & d \end{vmatrix}$, $\tilde{D} = \begin{vmatrix} a-c & b-d \\ c & d \end{vmatrix}$

The determinants found in 3.2.re1 and in 3.2.re2 illustrate this next fact.

The effect of row operations on the determinant 3.2.1.

1. *If B is obtained from A by a row-interchange $\mathbf{r}_i \rightleftarrows \mathbf{r}_j$, then $|B| = -|A|$.*
2. *If B is obtained from A by a row replacement $\mathbf{r}_i \leftarrow \mathbf{r}_i + c\mathbf{r}_j$, then $|B| = |A|$.*
3. *If B is obtained from A by scaling a row $\mathbf{r}_i \leftarrow c\mathbf{r}_i$, then $|B| = c|A|$.*

Calculating determinants by row reduction

The most efficient algorithm to compute the determinant of a matrix is to row-reduce it to an upper triangular form while keeping track of changes to the determinant according to 3.2.1 and then using 3.1.3.

3.2.re3. Calculate the determinant of the 3×3 matrix by row reduction.

$$\begin{vmatrix} 1 & 2 & 3 \\ 4 & 0 & -1 \\ 5 & -2 & -3 \end{vmatrix} \stackrel{1}{=} \begin{vmatrix} 1 & 2 & 3 \\ 0 & -8 & -13 \\ 0 & -12 & -18 \end{vmatrix} \stackrel{2}{=} -8 \begin{vmatrix} 1 & 2 & 3 \\ 0 & 1 & \frac{13}{8} \\ 0 & -12 & -18 \end{vmatrix} \stackrel{1}{=} -8 \begin{vmatrix} 1 & 2 & 3 \\ 0 & 1 & \frac{13}{8} \\ 0 & 0 & \frac{3}{2} \end{vmatrix}$$

$$\stackrel{3}{=} -8 \cdot 1 \cdot 1 \cdot \frac{3}{2} = -12$$

Notes:
[1] Row replacements; no change to determinant.
[2] Factor -8 out of row 2.
[3] The determinant of a triangular matrix equals the product of its main diagonal.

3.2.re4. Calculate the determinant of the 4×4 matrix by row reduction.

$$\begin{vmatrix} 0 & 1 & 2 & -15 \\ -1 & 2 & 2 & 1 \\ 3 & -1 & 4 & 6 \\ 1 & -1 & 0 & 0 \end{vmatrix} \stackrel{0}{=} - \begin{vmatrix} 1 & -1 & 0 & 0 \\ -1 & 2 & 2 & 1 \\ 3 & -1 & 4 & 6 \\ 0 & 1 & 2 & -15 \end{vmatrix} \stackrel{1}{=} - \begin{vmatrix} 1 & -1 & 0 & 0 \\ 0 & 1 & 2 & 1 \\ 0 & 2 & 4 & 6 \\ 0 & 1 & 2 & -15 \end{vmatrix}$$

$$\stackrel{1}{=} - \begin{vmatrix} 1 & -1 & 0 & 0 \\ 0 & 1 & 2 & 1 \\ 0 & 0 & 0 & 4 \\ 0 & 0 & 0 & -16 \end{vmatrix}$$

$$\stackrel{2}{=} -1 \cdot 1 \cdot 0 \cdot (-16) = 0$$

Notes:
[0] Interchange; determinant changed by a sign.
[1] Replacements; no change to determinant.
[2] The determinant of a triangular matrix equals the product of its main diagonal.

Because every square matrix can be row-reduced to an upper triangular matrix, 2.3.2 and 3.1.3 imply the following.

Fact 3.2.2. *A is invertible iff* $\det(A) \neq 0$.

Fact 3.2.3. $\det(AB) = \det(A)\det(B)$.

3.2.re5. Explain why $\det(A^{-1}) = \det(A)^{-1}$ if A is invertible.

3.2.re6. Find the given determinants, if

$$A = \begin{bmatrix} 1 & 2 & 0 \\ 0 & 3 & 9 \\ 0 & 0 & 2 \end{bmatrix} \qquad B = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 4 & 5 & 3 \end{bmatrix} \qquad C = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ -1 & 0 & -1 \end{bmatrix}$$

a. $|A^T B|$ b. $|A^3|$ c. $|CA|$ d. $|B^{-1}|$

3.2.re7. Give an example of two matrices A and B for which $|A + B|$ is **not** equal $|A| + |B|$.

Answers

3.2.re.e1a. $E = \begin{bmatrix} 1 & 0 & -2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$; $EA = \begin{bmatrix} a-2g & b-2h & c-2i \\ d & e & f \\ g & h & i \end{bmatrix}$. 3.2.re.e1b. $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$; $EA = \begin{bmatrix} a & b & c \\ g & h & i \\ d & e & f \end{bmatrix}$. 3.2.re.e1c. $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 4 \end{bmatrix}$;s $EA = \begin{bmatrix} a & b & c \\ d & e & f \\ 4g & 4h & 4i \end{bmatrix}$. 3.2.re.e1. a. $|E| = 1$. b. $|E| = -1$. c. $|E| = 4$. 3.2.re.e2a. $\tilde{A}$ is obtained by from A by $\mathbf{r}_2 \leftarrow -\frac{1}{2}\mathbf{r}_2$. $|A| = 2$; $|\tilde{A}| = -1 = -\frac{1}{2} \cdot 2$. 3.2.re.e2b. $\tilde{B}$ obtained from B by $\mathbf{r}_2 \rightleftarrows \mathbf{r}_3$. $|B| = 21$; $|\tilde{B}| = -21$. 3.2.re.e2c. $\tilde{C}$ obtained from C by $\mathbf{r}_3 \leftarrow \mathbf{r}_3 + 2\mathbf{r}_1$; $|C| = -8 = |\tilde{C}|$ 3.2.re.e2d. $\tilde{D}$ is obtained by from D by $\mathbf{r}_1 \leftarrow \mathbf{r}_1 - \mathbf{r}_2$; $|D| = ad - bc$; $|\tilde{D}| = (a-c)d - (b-d)c = ad - bc$. 3.2.re5. $AA^{-1} = I$, and so by 3.2.3, $\det(A)\det(A^{-1}) = \det(I) = 1$. Now divide both sides by $\det(A)$ to obtain $\det(A^{-1}) = 1/\det(A) = (\det(A))^{-1}$. 3.2.re.e6a. $|A^T||B| = |A||B| = 6 \cdot 3 = 18$. 3.2.re.e6b. $= |A|^3 = 6^3$, or 216. 3.2.re.e6c. By row-reduction, calculate $|C| = 0$. Then $|CA| = |C||A| = 0 \cdot 6 = 0$. 3.2.re.e6d. $|BB^{-1}| = |I| = 1$, so $|B^{-1}| = |B|^{-1} = \frac{1}{3}$. 3.2.re7. There are many such examples. For instance, let $A = \begin{bmatrix} 1 & 0 \\ 0 & 5 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 0 \\ 0 & -2 \end{bmatrix}$. Then $|A+B| = \begin{vmatrix} 2 & 0 \\ 0 & 3 \end{vmatrix} = 6$, but $|A|+|B| = 5+(-2) = 3$.

3.3: Cramer's rule

The adjugate

If A is a square matrix and

$$C_{i,j} = (-1)^{i+j} \det(A_{i,j})$$

is the (i,j)-cofactor of A, then the **adjugate** of A is the transpose of the matrix of cofactors of A:

$$\operatorname{adj}(A) = [C_{i,j}]^T.$$

The adjugate has the property that

$$A \operatorname{adj}(A) = \det(A) I.$$

3.3.re1. Find the adjugate of $X = \begin{bmatrix} 1 & 1 & 0 \\ 1 & -1 & -1 \\ -1 & -3 & 0 \end{bmatrix}$ and calculate $X \operatorname{adj}(X)$.

The matrix of cofactors of X is

$$\begin{bmatrix} \begin{vmatrix} -1 & -1 \\ -3 & 0 \end{vmatrix} & -\begin{vmatrix} 1 & -1 \\ -1 & 0 \end{vmatrix} & \begin{vmatrix} 1 & -1 \\ -1 & -3 \end{vmatrix} \\ -\begin{vmatrix} 1 & 0 \\ -3 & 0 \end{vmatrix} & \begin{vmatrix} 1 & 0 \\ -1 & 0 \end{vmatrix} & -\begin{vmatrix} 1 & 1 \\ -1 & -3 \end{vmatrix} \\ \begin{vmatrix} 1 & 0 \\ -1 & -1 \end{vmatrix} & -\begin{vmatrix} 1 & 0 \\ 1 & -1 \end{vmatrix} & \begin{vmatrix} 1 & 1 \\ 1 & -1 \end{vmatrix} \end{bmatrix} = \begin{bmatrix} -3 & 1 & -4 \\ 0 & 0 & 2 \\ -1 & 1 & -2 \end{bmatrix}.$$

The adjugate is the transpose of the above:

$$\operatorname{adj}(X) = \begin{bmatrix} -3 & 0 & -1 \\ 1 & 0 & 1 \\ -4 & 2 & -2 \end{bmatrix}.$$

As expected, $X \operatorname{adj}(X) =$

$$\begin{bmatrix} 1 & 1 & 0 \\ 1 & -1 & -1 \\ -1 & -3 & 0 \end{bmatrix} \begin{bmatrix} -3 & 0 & -1 \\ 1 & 0 & 1 \\ -4 & 2 & -2 \end{bmatrix} = \begin{bmatrix} -2 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & -2 \end{bmatrix}.$$

3.3.re2. Find the adjugate of the given matrix.

a. $\begin{bmatrix} 4 & -5 \\ -2 & 3 \end{bmatrix}$ b. $\begin{bmatrix} 1 & 1 & -2 \\ 2 & 1 & 1 \\ -1 & 0 & -1 \end{bmatrix}$ c. $\begin{bmatrix} 1 & 1 & 1 \\ 2 & 5 & -1 \\ -1 & -4 & 2 \end{bmatrix}$ d. $\begin{bmatrix} 1 & 2 & 0 & -4 \\ 0 & 2 & 1 & 1 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

Cramer's Rule 3.3.1. *If* $\det(A) \neq 0$*, then*

$$A^{-1} = \frac{1}{\det(A)}\,\mathrm{adj}(A).$$

Furthermore, if $A\mathbf{x} = \mathbf{b}$*, then for every* $1 \leq i \leq n$*,*

$$x_i = \frac{1}{\det(A)}\det(A_i(\mathbf{b})),$$

where $A_i(\mathbf{b})$ *is the matrix obtained by replacing column* i *of* A *with the vector* $\mathbf{b}$*.*

3.3.re2, continued. Find the inverses of the matrices in 3.3.re2 or explain why they do not exist.

3.3.re3. Find x_3 and x_4 if

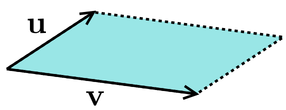
$$\begin{bmatrix} 1 & 2 & 3 & 1 \\ 0 & 1 & 0 & -4 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 3 & -1 \end{bmatrix} \mathbf{x} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 2 \end{bmatrix}.$$

Area and volume as determinants

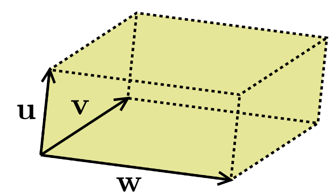
It is useful to think of a vector in $\mathbb{R}^n$ not just as the coordinates of a point but also as an arrow from the origin to that point.

Fact 3.3.2. *If* $\mathbf{u}$ *and* $\mathbf{v}$ *are in* $\mathbb{R}^2$*, then the area of the parallelogram with edges* $\mathbf{u}$ *and* $\mathbf{v}$ *is the absolute value of the determinant of the* 2×2 *matrix with columns* $\mathbf{u}$ *and* $\mathbf{v}$.

If $\mathbf{u}$, $\mathbf{v}$ *and* $\mathbf{w}$ *are in* $\mathbb{R}^3$*, then the volume of the parallelepiped with edges* $\mathbf{u}$, $\mathbf{v}$, *and* $\mathbf{w}$ *is the absolute value of the determinant of the* 3×3 *matrix with columns* $\mathbf{u}$, $\mathbf{v}$,$\mathbf{w}$.



$\det[\mathbf{u}\ \ \mathbf{v}] = \pm$ area



$\det[\mathbf{u}\ \ \mathbf{v}\ \ \mathbf{w}] = \pm$ volume

3.3.re4. Find the area of the parallelogram with the given vertices.

a. $(0,0)$, $(3,1)$, $(2,-3)$, $(5,-2)$ b. $(7,-1)$, $(12,2)$, $(3,1)$, $(8,4)$

3.3.re5. Find the volume of the parallelepiped with the given three parallel sides.

a. $(1,0,1)$, $(1,2,-1)$, $(3,2,0)$ b. $(-1,7,2)$, $(0,4,0)$, $(8,9,10)$

Answers

3.3.re.e2a. $\begin{bmatrix} 3 & 5 \\ 2 & 4 \end{bmatrix}$. 3.3.re.e2b. $\begin{bmatrix} -1 & 1 & 3 \\ 1 & -3 & -5 \\ 1 & -1 & -1 \end{bmatrix}$. 3.3.re.e2c. $\begin{bmatrix} 6 & -6 & -6 \\ -3 & 3 & 3 \\ -3 & 3 & 3 \end{bmatrix}$.

3.3.re.e2d. $\begin{bmatrix} -2 & 2 & 2 & -10 \\ 0 & -1 & -1 & 1 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & -2 \end{bmatrix}$. 3.3.re.e2. a. $\frac{1}{2}\begin{bmatrix} 3 & 5 \\ 2 & 4 \end{bmatrix}$, or $\begin{bmatrix} 3/2 & 5/2 \\ 1 & 2 \end{bmatrix}$. b. $-\frac{1}{2}\begin{bmatrix} -1 & 1 & 3 \\ 1 & -3 & -5 \\ 1 & -1 & -1 \end{bmatrix}$.

c. Inv. does not exist. Expanding along first row, det. of the original matrix $= [1 \quad 1 \quad 1]\begin{bmatrix} 6 \\ -3 \\ -3 \end{bmatrix} = 0$.

d. $-\frac{1}{2}\begin{bmatrix} -2 & 2 & 2 & -10 \\ 0 & -1 & -1 & 1 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & -2 \end{bmatrix}$. 3.3.re3. $x_3 = \begin{vmatrix} 1 & 2 & 1 & 1 \\ 0 & 1 & 0 & -4 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 2 & -1 \end{vmatrix} \div \begin{vmatrix} 1 & 2 & 3 & 1 \\ 0 & 1 & 0 & -4 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 3 & -1 \end{vmatrix} = (-2)/(-4) = 1/2$.

$x_4 = \begin{vmatrix} 1 & 2 & 3 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 3 & 2 \end{vmatrix} \div \begin{vmatrix} 1 & 2 & 3 & 1 \\ 0 & 1 & 0 & -4 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 3 & -1 \end{vmatrix} = 2/(-4) = -1/2$. 3.3.re.e4a. 11. 3.3.re.e4b. 22. 3.3.re.e5a. 2. 3.3.re.e5b. 104.

4.1: Abstract vector spaces

We use the symbol $\forall$ to stand for the words "for every" and the symbol $\exists$ to stand for the words "there exists." (These two symbols are called *quantifiers*.)

Definition 4.1.1. *A* **vector space** *is a set of objects, which are called* **vectors**, *on which are defined two operations, called* **vector addition** *and* **scalar multiplication**, *meaning*

1. $\forall \mathbf{u}$ *and* $\mathbf{v}$ *in* V, $\exists$ *a corresponding element called* $\mathbf{u}+\mathbf{v} \in V$, *and*
2. $\forall \mathbf{u} \in V$ *and* $\forall c \in \mathbb{R}$, $\exists$ *a corresponding element called* $c\mathbf{u} \in V$.

Also, $\exists$ *an element* $\mathbf{0}$ *in* V *with the property that*

3. $\forall \mathbf{u} \in V$, $\mathbf{u}+\mathbf{0}=\mathbf{u}$,

and $\forall \mathbf{u} \in V$, $\exists$ *an element* $-\mathbf{u} \in V$ *with the property that*

4. $\forall \mathbf{u} \in V$, $\mathbf{u}+(-\mathbf{u})=\mathbf{0}$.

Furthermore, the following properties are true $\forall \mathbf{u}, \mathbf{v}$, *and* $\mathbf{w}$ *in* V *and* $\forall c$ *and* d *in* $\mathbb{R}$.

5. $\mathbf{u}+\mathbf{v}=\mathbf{v}+\mathbf{u}$
6. $\mathbf{u}+(\mathbf{v}+\mathbf{w})=(\mathbf{u}+\mathbf{v})+\mathbf{w}$
7. $c(\mathbf{u}+\mathbf{v})=c\mathbf{u}+c\mathbf{v}$
8. $(c+d)\mathbf{u}=c\mathbf{u}+d\mathbf{u}$
9. $c(d\mathbf{u})=(cd)\mathbf{u}$
10. $1\mathbf{u}=\mathbf{u}$

Fact 4.1.2. *If* V *is a vector space, then*

1. *the vector* $\mathbf{0}$ *is unique.*

Furthermore, the following are true $\forall \mathbf{u} \in V$ *and* $\forall c \in \mathbb{R}$.

2. *The vector* $-\mathbf{u}$ *is unique.*
3. $c\,\mathbf{0}=\mathbf{0}$
4. $0\mathbf{u}=\mathbf{0}$
5. $(-1)\mathbf{u}=-\mathbf{u}$.

Definition 4.1.3. *If* H *is a subset of the vector space* V, *then* H *is a* **subpace** *of* V *if*

1. $\mathbf{0} \in H$
2. $\forall \mathbf{u}$ *and* $\mathbf{v}$ *in* H, $\mathbf{u}+\mathbf{v} \in H$
3. $\forall \mathbf{u} \in H$ *and* $\forall c \in \mathbb{R}$, $c\mathbf{u} \in H$

When 2 and 3 are true, we say that H is **closed** under vector addition and scalar multiplication.
If H is a subspace of V, then H is also a vector space with the same vector addition and scalar multiplication as in V.

4.1.re1. Prove that the following are subspaces of the given vector space.

a. $Z = \{\mathbf{0}\}$ (in any vector space)

b. $U = \{f \in C[0,1] \mid f(1/2) = 0\}$

c. $W = \left\{ \begin{bmatrix} a & b \\ 0 & c \end{bmatrix} \middle| \; a, b, c \in \mathbb{R} \right\}$

4.1.re2. Prove that the following are *not* subspaces of the given vector space.

a. $Q = \left\{ \begin{bmatrix} x \\ y \end{bmatrix} \middle| \; x = 0 \text{ or } y \neq 0 \right\}$

b. $X = \{p \in \mathbb{P} \mid \int_0^2 p(x)\,dx \leq 1\}$

See https://kunklet.people.cofc.edu/MATH203/ssproofs.pdf for some tips on writing subspace proofs.

Linear combination, span

Compare the following with section 1.3:

Definition 4.1.4. *If $\mathbf{u}_1, \mathbf{u}_2, \ldots \mathbf{u}_n$ are vectors in a vector space V and $x_1, \ldots, x_n$ are scalars, then*

$$x_1\mathbf{u}_1 + x_2\mathbf{u}_2 + \cdots + x_n\mathbf{u}_n$$

is called a **linear combination** *of $\mathbf{u}_1, \mathbf{u}_2, \ldots \mathbf{u}_n$. The collection of all linear combinations of $\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_n$ is called its* **span***, written*

$$\operatorname{span}\{\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_n\}.$$

If $\operatorname{span}\{\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_n\} = V$*, we say that the vectors $\mathbf{u}_1, \mathbf{u}_2, \ldots \mathbf{u}_n$* **span** *$V$.*

4.1.re3. The polynomial $5 - 4x + 3x^2$ is a linear combination of the polynomials 1, x, and x^2. The span $\{1, x, x^2\}$, equals $\mathbb{P}_2$, the vector space of all polynomials of degree at most two. Explain why $5 - 4x + 3x^2$ is also a linear combination of 1, x, x^2, and x^3

Fact 4.1.5. *If $\{\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_n\} \subset V$, a vector space, then* $\operatorname{span}\{\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_n\}$ *is a subspace of V.*

4.1.re4. Prove that the set is a subspace by showing it to be the span of a set of vectors.

a. $W = \left\{ \begin{bmatrix} a & b \\ 0 & c \end{bmatrix} \middle| \; a, b, c \in \mathbb{R} \right\}$

b. $\Omega = \left\{ \begin{bmatrix} -x + 2y + 3z \\ y - z \end{bmatrix} \middle| \; x, y, z \in \mathbb{R} \right\}$

Answers

4.1.re.e1a. $\mathbf{0} \in Z$. Since Z contains only one element, to show its closure under addition and multiplication, we need only note $\mathbf{0} + \mathbf{0} = \mathbf{0} \in Z$, and $\forall c \in \mathbb{R}$, $c\mathbf{0} = \mathbf{0} \in Z$. 4.1.re.e1b. In $C[0,1]$ (the vector space of all functions that are continuous on the interval $[0,1]$), $\mathbf{0}$ is the function given by the rule $\mathbf{0}(x) = 0$ for all x. Therefore $\mathbf{0}(1/2) = 0$. If f and g are in U and c is any real number, then $f + g$ is also in U because $(f+g)(1/2) = f(1/2)+g(1/2) = 0+0 = 0$, and cf is in U because $(cf)(1/2) = cf(1/2) = c0 = 0$. 4.1.re.e1c. ($W \subset M_{2\times 2}$ is the set of all 2×2 upper triangular matrices.) $\begin{bmatrix} a & b \\ 0 & c \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = \mathbf{0}$ when $a = b = c = 0$. For any $\begin{bmatrix} a & b \\ 0 & c \end{bmatrix}$ and $\begin{bmatrix} d & e \\ 0 & f \end{bmatrix}$ in W, their sum $\begin{bmatrix} a+d & b+e \\ 0 & c+f \end{bmatrix}$ is also in W; additionally, if g is any scalar, then $g\begin{bmatrix} a & b \\ 0 & c \end{bmatrix} = \begin{bmatrix} ga & gb \\ 0 & gc \end{bmatrix}$ is also in W. 4.1.re.e2a. $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ and $\begin{bmatrix} 1 \\ -1 \end{bmatrix}$ are in Q, since their second coordinates are not zero, but their sum $\begin{bmatrix} 2 \\ 0 \end{bmatrix}$ is not in Q because its second coordinate is zero and its first coordinate is not. 4.1.re.e2b. The polynomial x is in X, since $\int_0^2 x\,dx = \frac{1}{2}x^2|_0^2 = 1$. But the scalar multiple $4x$ in not in X because $\int_0^2 4x\,dx = 4\int_0^1 x\,dx = 4 \not\leq 1$. 4.1.re3. Because it equals $5 - 4x + 3x^2 + 0x^3$. 4.1.re.e4a. W is a vector space, since it equals span $\left\{\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}\right\}$.
4.1.re.e4b. $\begin{bmatrix} -x+2y+3z \\ y-z \end{bmatrix} = x\begin{bmatrix} -1 \\ 0 \end{bmatrix} + y\begin{bmatrix} 2 \\ 1 \end{bmatrix} + z\begin{bmatrix} 3 \\ -1 \end{bmatrix}$, so $\Omega = \text{span}\left\{\begin{bmatrix} -1 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 3 \\ -1 \end{bmatrix}\right\}$ is a subspace of $\mathbb{R}^3$.

4.2: Linear transformations, column space, row space, and null space.

If V and W are vector spaces, we sometimes denote their zero-vectors as $\mathbf{0}_V$ and $\mathbf{0}_W$.

Definition 4.2.1. *A transformation $T : V \to W$ from a vector space V to a vector space W is* **linear** *if, $\forall \mathbf{u}$ and $\mathbf{v}$ in V and $\forall c \in \mathbb{R}$,*

$$T(\mathbf{u}+\mathbf{v}) = T(\mathbf{u}) + T(\mathbf{v})$$
$$T(c\mathbf{u}) = c\,T(\mathbf{u})$$

The **kernel** *or* **null space** *of T is the set*

$$\ker T = \{\, \mathbf{v} \in V \mid T(\mathbf{v}) = \mathbf{0}_W \,\} \subset V$$

and the **range** *of T is the set*

$$\operatorname{ran} T = \{\, T(\mathbf{v}) \mid \mathbf{v} \in V \} \subset W.$$

4.2.re1. Let $T : \mathbb{P}_2 \to \mathbb{R}^2 : p(t) \mapsto \begin{bmatrix} p(0) \\ p'(0) \end{bmatrix}$. $\begin{bmatrix} 1 \\ -2 \end{bmatrix} \in \operatorname{ran} T$ because $T(1-2t) = \begin{bmatrix} 1 \\ -2 \end{bmatrix}$.
$q(t) = 5t^2$ is in $\ker T$ because both $q(t)$ an $q'(t) = 10t$ equal zero at $t = 0$.
Prove that T maps $\mathbb{P}_2$ onto $\mathbb{R}^2$ by finding a polynomial $r(t)$ (depending on a and b) for which $T(r(t)) = \begin{bmatrix} a \\ b \end{bmatrix}$.

Fact 4.2.2. *If $T : V \to W$ is linear, then*

a. $T(\mathbf{0}_V) = \mathbf{0}_W$

b. $\ker T$ *is a subspace of* V.

c. $\operatorname{ran} T$ *is a subspace of* W.

4.2.re2. Prove the three statements in 4.2.2.

Hints: a: start with $\mathbf{0}_V = \mathbf{0}_V + \mathbf{0}_V$. You'll use part a to prove b&c. b: see the proof of Thm.2 in 4.2 of the text. c: every element in $\operatorname{ran} T$ can be written $T(\mathbf{u})$ for some $\mathbf{u} \in V$.

Definition 4.2.3. *If $A \in M_{m\times n}$, then its* **null space** *is the set*

$$\operatorname{Nul} A = \{\, \mathbf{x} \in \mathbb{R}^n \mid A\mathbf{x} = \mathbf{0} \,\} \subset \mathbb{R}^n.$$

Its **column space** *and* **row space** *are the spans of its columns and rows, respectively:*

$$\operatorname{Col} A = \{\, A\mathbf{u} \mid \mathbf{u} \in \mathbb{R}^n \,\} \subset \mathbb{R}^m$$
$$\operatorname{Row} A = \{\, A^T\mathbf{v} \mid \mathbf{v} \in \mathbb{R}^m \,\} \subset \mathbb{R}^n$$

$\operatorname{Nul} A$ *and* $\operatorname{Col} A$ *are the null space and range, respectively, of the linear transformation* $T : \mathbb{R}^n \to \mathbb{R}^m : \mathbf{x} \mapsto A\mathbf{x}$, *and* $\operatorname{Row} A$ *is the column space of* A^T.

4.2.re3. Let $C = \begin{bmatrix} 1 & 0 & 2 \\ 2 & 1 & 5 \\ -1 & 1 & -1 \end{bmatrix}$. Then $\begin{bmatrix} 2 \\ 2 \\ -1 \end{bmatrix}$ is in $\operatorname{Nul} C$ because $C\begin{bmatrix} 2 \\ 2 \\ -1 \end{bmatrix} = \mathbf{0}$, and $\begin{bmatrix} 3 \\ 6 \\ -3 \end{bmatrix}$ is in $\operatorname{Col} C$ because it equals $C\begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$. Show that $\begin{bmatrix} 3 \\ 8 \\ 1 \end{bmatrix}$ is not in $\operatorname{Col} C$ by showing that the system $C\mathbf{x} = \begin{bmatrix} 3 \\ 8 \\ 1 \end{bmatrix}$ is inconsistent.

4.2.re4. We can express the null space of

$$B = \begin{bmatrix} 1 & 0 & -1 & 3 & 3 \\ 2 & 1 & 0 & 7 & 7 \\ -1 & 1 & 3 & -2 & -1 \end{bmatrix}$$

as a span of vectors in $\mathbb{R}^5$ by first reducing B to its reduced row echelon form:

$$B \sim \begin{bmatrix} 1 & 0 & -1 & 3 & 0 \\ 0 & 1 & 2 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

In parametric vector form, the typical element of $\operatorname{Nul} B$ is

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} x_3 - 3x_4 \\ -2x_3 - x_4 \\ x_3 \\ x_4 \\ 0 \end{bmatrix} = x_3\begin{bmatrix} 1 \\ -2 \\ 1 \\ 0 \\ 0 \end{bmatrix} + x_4\begin{bmatrix} -3 \\ -1 \\ 0 \\ 1 \\ 0 \end{bmatrix}.$$

Therefore

$$\operatorname{Nul} B = \operatorname{span}\left\{ \begin{bmatrix} 1 \\ -2 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -3 \\ -1 \\ 0 \\ 1 \\ 0 \end{bmatrix} \right\}$$

4.2.re5. Express the null space of the given matrix as a span of vectors.

a. $\begin{bmatrix} 0 & 1 & -3 & 2 & -2 \\ 1 & 2 & -1 & 0 & 0 \\ 1 & 3 & -4 & 4 & -2 \end{bmatrix}$ b. $\begin{bmatrix} 1 & -1 & 2 & 5 \\ 1 & -1 & 4 & 9 \\ -2 & 2 & -3 & -8 \end{bmatrix}$ c. $\begin{bmatrix} 1 & -1 & 0 & 1 & 3 \\ 2 & -2 & 1 & 4 & 6 \\ 0 & 0 & -1 & -2 & 0 \\ 1 & -1 & 0 & 1 & 3 \end{bmatrix}$

4.2.re6. Either express the given set as the null space or column space of a matrix or determine that it is not subspace.

a. $\left\{ \begin{bmatrix} a-2b \\ 3a+4b \\ -2b \end{bmatrix} \middle| \ a, b \in \mathbb{R} \right\}$ b. $\left\{ \begin{bmatrix} x+y-3z \\ y+2u+v \\ x-v \end{bmatrix} \middle| \ u, v, x, y, z \in \mathbb{R} \right\}$

c. $\left\{ \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} \middle| \ \begin{array}{l} 2a-4c=0 \\ 3b+9d=0 \end{array} \right\}$ d. $\left\{ \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} \middle| \ \begin{array}{l} 2a-4c=1 \\ 3b+9d=0 \end{array} \right\}$

e. $\left\{ \begin{bmatrix} a \\ b \\ c \\ d \\ e \end{bmatrix} \middle| \ \begin{array}{r} 2a=e+4c \\ 7b+e=3d+a \end{array} \right\}$

Answers

4.2.re1. $T(a+bt) = [a \quad b]^T$. 4.2.re2. a. $T(\mathbf{0}_V) = T(\mathbf{0}_V + \mathbf{0}_V) = T(\mathbf{0}_V) + T(\mathbf{0}_V)$. Now subtract $T(\mathbf{0}_V)$ from the left and right sides to obtain $\mathbf{0}_W = T(\mathbf{0}_V)$. b. By a., $\mathbf{0}_V \in \ker T$. To show $\ker T$ is closed under addition, suppose $T(\mathbf{u}) = T(\mathbf{v}) = \mathbf{0}_W$. Then $T(\mathbf{u}+\mathbf{v}) = T(\mathbf{u}) + T(\mathbf{v}) = \mathbf{0}_W + \mathbf{0}_W = \mathbf{0}_W$. To show $\ker T$ is closed under scalar multiplication, suppose $T(\mathbf{u}) = \mathbf{0}_W$ and $c \in \mathbb{R}$. Then $T(c\mathbf{u}) = cT(\mathbf{u}) = c\mathbf{0}_W = \mathbf{0}_W$. c. By a., $\mathbf{0}_W \in \operatorname{ran} T$. To show that $\operatorname{ran} T$ is closed under vector addition, note that if $T(\mathbf{u})$ and $T(\mathbf{v})$ are elements of $\operatorname{ran} T$, then their sum $T(\mathbf{u}) + T(\mathbf{v}) = T(\mathbf{u}+\mathbf{v})$ is also in $\operatorname{ran} T$. To show that $\operatorname{ran} T$ is closed under scalar multiplication, note that if $T(\mathbf{u})$ is any element of $\operatorname{ran} T$ and c is any scalar, then $cT(\mathbf{u}) = T(c\mathbf{u})$ is also in $\operatorname{ran} T$. 4.2.re3. The augmented matrix $\begin{bmatrix} 1 & 0 & 2 & 3 \\ 2 & 1 & 5 & 8 \\ -1 & 1 & -1 & 1 \end{bmatrix}$ reduces to $\begin{bmatrix} 1 & 0 & 2 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$. System is inconsistent because there's pivot in the augmented column.

4.2.re.e5a. rref $= \begin{bmatrix} 1 & 0 & 5 & 0 & 4 \\ 0 & 1 & -3 & 0 & -2 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix}$. null space $= \operatorname{span}\left\{ \begin{bmatrix} -5 \\ 3 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -4 \\ 2 \\ 0 \\ 0 \\ 1 \end{bmatrix} \right\}$. 4.2.re.e5b. null space $= \operatorname{span}\left\{ \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ -2 \\ 0 \\ 1 \end{bmatrix} \right\}$. 4.2.re.e5c. null space $= \operatorname{span}\left\{ \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ -2 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -3 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \right\}$.

4.2.re.e6a. span $\left\{ \begin{bmatrix} 1 \\ 3 \\ 0 \end{bmatrix}, \begin{bmatrix} -2 \\ 4 \\ -2 \end{bmatrix} \right\} = \text{Col} \begin{bmatrix} 1 & -2 \\ 3 & 4 \\ 0 & -2 \end{bmatrix}$. 4.2.re.e6b. Col $\begin{bmatrix} 0 & 0 & 1 & 1 & -3 \\ 2 & 1 & 0 & 1 & 0 \\ 0 & -1 & 1 & 0 & 0 \end{bmatrix}$.

4.2.re.e6c. Nul $\begin{bmatrix} 2 & 0 & -4 & 0 \\ 0 & 3 & 0 & 9 \end{bmatrix}$. 4.2.re.e6d. The given set does not include $\mathbf{0}$ and so is not a subspace.

4.2.re.e6e. Nul $\begin{bmatrix} 2 & 0 & -4 & 0 & -1 \\ -1 & 7 & 0 & -3 & 1 \end{bmatrix}$.

4.3: Linear independence, bases.

Definition 4.3.1. *A set of vectors* $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ *in a vector space* V *is* **linearly independent** *if the only solution to*

$$x_1\mathbf{v}_1 + x_2\mathbf{v}_2 + \cdots + x_n\mathbf{v}_n = \mathbf{0} \quad (\text{in } V) \tag{4.3.2}$$

is $\mathbf{x} = \mathbf{0}$ *(in* $\mathbb{R}^n$*). A set of vectors is* **linearly dependent** *if it is not linearly independent.*

If (4.3.2) is true for some nonzero $\mathbf{x}$, that equation is called a **dependence relation** for $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$.

Fact 4.3.3. *A set of two or more vectors is linearly dependent iff some vector in that set is a linear combination of the others.*

See 1.7.re1, where we proved this statement for vectors in $\mathbb{R}^n$.

When $V \neq \mathbb{R}^n$, deciding the linear dependence or independence of a set of vectors requires us to determine if (4.3.2) has a nontrivial solution $\mathbf{x}$, often by finding an equivalent matrix system (and sometimes by some clever analysis).

4.3.re1. Determine whether the given set of vectors is linearly independent.

a. $\left\{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 1 \\ 0 & -1 & 2 \end{bmatrix}, \begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}\right\} \subset M_{2\times 3}$.

b. $\{1, t, t^2\} \subset \mathbb{P}$.

c. $\{t-1, t+1, t\} \subset \mathbb{P}$.

d. $\{1, t-1, (t-1)(t-2)\} \subset \mathbb{P}$.

e. $\{1, \sin^2 t, \cos^2 t, \cos^3 t\} \subset C(-\infty, \infty)$.

Definition 4.3.4. *A set* $B = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\} \subset V$ *is a* **basis** *for* V *if*

1. *B spans V*
2. *B is linearly independent.*

4.3.re2. $\{1, t, t^2\}$ spans $\mathbb{P}_2$, since every element of $\mathbb{P}_2$ has the form $at^2 + bt + c$. As seen in 4.3.re1, $\{1, t, t^2\}$ is linearly independent. Therefore $\{1, t, t^2\}$ is a basis for $\mathbb{P}_2$.

4.3.re3. Since the $n \times n$ identity matrix I has a pivot in every row and every column, its columns $\{\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_n\}$ are linearly independent and span $\mathbb{R}^n$. Therefore $\{\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_n\}$ is a basis for $\mathbb{R}^n$.

Fact 4.3.5. *An $n \times n$ matrix A is invertible iff its columns form a basis for $\mathbb{R}^n$.*

Bases for the null, column, and row spaces of a matrix

Fact 4.3.6. *If A and B are row equivalent, then* $\operatorname{Row} A = \operatorname{Row} B$ *and* $\operatorname{Nul} A = \operatorname{Nul} B$.

4.3.re4. It is important to note that row operations may change the column space of a matrix. For example, $\begin{bmatrix} 1 \\ 1 \end{bmatrix} \sim \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ but

$$\operatorname{Col} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \left\{ \begin{bmatrix} x \\ x \end{bmatrix} \,\middle|\, x \in \mathbb{R} \right\},$$

the set of vectors in $\mathbb{R}^2$ whose two components are equal, while

$$\operatorname{Col} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \left\{ \begin{bmatrix} x \\ 0 \end{bmatrix} \,\middle|\, x \in \mathbb{R} \right\},$$

the set of vectors in $\mathbb{R}^2$ whose second component are zero.

4.2.re4, continued. The spanning set for $\operatorname{Nul} B$ found earlier,

$$\left\{ \begin{bmatrix} 1 \\ -2 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -3 \\ -1 \\ 0 \\ 1 \\ 0 \end{bmatrix} \right\} \tag{4.3.7}$$

is linearly independent. To see this, recall that if $\mathbf{x}$ is in this span, then

$$\mathbf{x} = x_3 \begin{bmatrix} 1 \\ -2 \\ 1 \\ 0 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} -3 \\ -1 \\ 0 \\ 1 \\ 0 \end{bmatrix}.$$

If $\mathbf{x}$ equals 0, then x_3 and x_4, being components of $\mathbf{x}$, must both be zero. Therefore, the set (4.3.7) is a basis for the null space of B.

Fact 4.3.8. *Suppose $A \sim B$, where B is in row echelon form.*

1. *The pivot columns of A are a basis for* $\operatorname{Col} A$.
2. *The pivot rows of B are a basis for* $\operatorname{Row} A$.
3. *The spanning set for* $\operatorname{Nul} A$ *produced by writing the solutions to $A\mathbf{x} = \mathbf{0}$ in parametric vector form is a basis for* $\operatorname{Nul} A$.

4.3.re5. Find bases for the null, column, and row spaces of the given matrix. For which of these $m \times n$ matrices is the column space all of $\mathbb{R}^m$?

a. $\begin{bmatrix} 1 & 2 & 5 & 3 \\ 1 & 2 & 5 & 3 \\ -1 & -1 & -3 & -6 \end{bmatrix}$ b. $\begin{bmatrix} 1 & 0 & -5 & 3 \\ 2 & 1 & -9 & 8 \\ -1 & 1 & 6 & -1 \end{bmatrix}$ c. $\begin{bmatrix} 1 & 2 & -3 & 1 & 7 \\ 1 & 4 & -7 & 4 & 13 \\ 1 & 4 & -7 & 5 & 13 \end{bmatrix}$

d. $\begin{bmatrix} 1 & 2 & -1 & -1 & -2 \\ 2 & 5 & -4 & 0 & -3 \\ -1 & -1 & -1 & 5 & 3 \end{bmatrix}$ e. $\begin{bmatrix} 1 & -1 & 2 & 15 & 3 \\ 2 & -2 & 5 & 33 & 6 \\ 0 & 0 & -1 & -3 & 1 \\ 1 & -1 & 2 & 15 & 4 \end{bmatrix}$

Spanning set theorem 4.3.9. *If S is a subset (other than $\{\mathbf{0}\}$) of a vector space, then some subset of S is a basis for* $\operatorname{span} S$.

For example, if S were a finite set of vectors in $\mathbb{R}^m$, one could find a basis for their span by forming the matrix with the elements of S for its columns and choosing the pivot columns of that matrix.

4.3.re6. Find a basis for $\operatorname{span}\left\{ \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}, \begin{bmatrix} 2 \\ 2 \\ -1 \end{bmatrix}, \begin{bmatrix} 5 \\ 5 \\ -3 \end{bmatrix}, \begin{bmatrix} 3 \\ 3 \\ -6 \end{bmatrix} \right\}$.

4.3.re7. Find a basis for $\operatorname{span}\{1, t-1, t+1, (t-1)(t-2)\}$.

Answers

4.3.re.e1a. $a\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} + b\begin{bmatrix} 0 & 0 & 1 \\ 0 & -1 & 2 \end{bmatrix} + c\begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} a & 0 & 0 \\ 0 & a & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 & b \\ 0 & -b & 2b \end{bmatrix} + \begin{bmatrix} c & c & 0 \\ 0 & 0 & 0 \end{bmatrix}$ $= \begin{bmatrix} a+c & c & b \\ 0 & a-b & 2b \end{bmatrix}$. If this $= \mathbf{0}$, row1, columns 2,3, and 1 imply $a = b = c = 0$. The matrices are linearly dependent. 4.3.re.e1b. Suppose $a + bt + ct^2 = 0$. Evaluating at $t = 0$, 1, and 2 yields the system of equations $a = 0, a + b + c = 0, a + 2b + 4c = 0$; in aug'd matrix form, $\begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 \\ 1 & 2 & 4 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \Longrightarrow$ $a = b = c = 0$. The polynomials are linearly independent. 4.3.re.e1c. Note $\frac{1}{2}(t-1) + \frac{1}{2}(t+1) = t$. Since one polynomial is a linear combination of the other two, these polynomials are linearly dependent. 4.3.re.e1d. Suppose $a + b(t-1) + c(t-1)(t-2) = 0$. Evaluating at $t = 1$, 2, and 3 yields the system of equations $a = 0, a + b = 0, a + 2b + 2c = 0$. The matrix of coefficients is lower triangular with a nonzero diagonal, hence invertible. Therefore the only solution to this homogeneous system is $a = b = c = 0$. Polys are linearly independent. 4.3.re.e1e. $\sin^2 t+\cos^2 t-1=0$ is true for all t, so the functions are linearly dependent. 4.3.re.e5a. null sp bs $=\left\{\begin{bmatrix}-1\\-2\\1\\0\end{bmatrix},\begin{bmatrix}-9\\3\\0\\1\end{bmatrix}\right\}$. col sp bs $=\left\{\begin{bmatrix}1\\1\\-1\end{bmatrix},\begin{bmatrix}2\\2\\-1\end{bmatrix}\right\}$. row sp bs $=\left\{\begin{bmatrix}1\\0\\1\\9\end{bmatrix}^T,\begin{bmatrix}0\\1\\2\\-3\end{bmatrix}^T\right\}$. col sp $\neq \mathbb{R}^3$. 4.3.re.e5b. null sp bs $=\left\{\begin{bmatrix}5\\-1\\1\\0\end{bmatrix},\begin{bmatrix}-3\\-2\\0\\1\end{bmatrix}\right\}$. col sp bs $=\left\{\begin{bmatrix}1\\2\\-1\end{bmatrix},\begin{bmatrix}0\\1\\1\end{bmatrix}\right\}$. row sp bs $=\left\{\begin{bmatrix}1\\0\\-5\\3\end{bmatrix}^T,\begin{bmatrix}0\\1\\1\\2\end{bmatrix}^T\right\}$. col sp $\neq \mathbb{R}^3$. 4.3.re.e5c. null sp bs $=\left\{\begin{bmatrix}-1\\2\\1\\0\\0\end{bmatrix},\begin{bmatrix}-1\\-3\\0\\0\\1\end{bmatrix}\right\}$. col sp bs $=\left\{\begin{bmatrix}1\\1\\1\end{bmatrix},\begin{bmatrix}2\\4\\4\end{bmatrix},\begin{bmatrix}1\\4\\5\end{bmatrix}\right\}$. row sp bs $=\left\{\begin{bmatrix}1\\0\\1\\0\\1\end{bmatrix}^T,\begin{bmatrix}0\\1\\-2\\0\\3\end{bmatrix}^T,\begin{bmatrix}0\\0\\0\\1\\0\end{bmatrix}^T\right\}$. col sp $=\mathbb{R}^3$. 4.3.re.e5d. null sp bs $=\left\{\begin{bmatrix}-3\\2\\1\\0\\0\end{bmatrix},\begin{bmatrix}4\\-1\\0\\0\\1\end{bmatrix}\right\}$. col sp bs $=\left\{\begin{bmatrix}1\\2\\-1\end{bmatrix},\begin{bmatrix}2\\5\\-1\end{bmatrix},\begin{bmatrix}-1\\0\\5\end{bmatrix}\right\}$. row sp bs $=\left\{\begin{bmatrix}1\\0\\3\\0\\-4\end{bmatrix}^T,\begin{bmatrix}0\\1\\-2\\0\\1\end{bmatrix}^T,\begin{bmatrix}0\\0\\0\\1\\0\end{bmatrix}^T\right\}$. col sp $=\mathbb{R}^3$. 4.3.re.e5e. null sp bs $=\left\{\begin{bmatrix}1\\1\\0\\0\\0\end{bmatrix},\begin{bmatrix}-9\\-3\\0\\1\\0\end{bmatrix}\right\}$. col sp bs $=\left\{\begin{bmatrix}1\\2\\0\\1\end{bmatrix},\begin{bmatrix}2\\5\\-1\\2\end{bmatrix},\begin{bmatrix}3\\6\\1\\4\end{bmatrix}\right\}$. row sp bs $=\left\{\begin{bmatrix}1\\-1\\0\\9\\0\end{bmatrix}^T,\begin{bmatrix}0\\0\\1\\3\\0\end{bmatrix}^T,\begin{bmatrix}0\\0\\0\\0\\1\end{bmatrix}^T\right\}$. col sp $\neq \mathbb{R}^4$.

4.3.re6. Same as the column space basis found in 4.3.re5a. 4.3.re7. Omit $t+1=(t-1)+2(1)$. Remaining polynomials $\{1,t-1,(t-1)(t-2)\}$ are linearly independent—see 4.3.re1d.

4.4: The coordinate map

Fact 4.4.1. *If $B = \{\mathbf{b}_1, \mathbf{b}_2, \ldots, \mathbf{b}_n\}$ is a basis for the vector space V, then each $\mathbf{v} \in V$ can be written uniquely as a linear combination of the elements of B:*

$$\mathbf{v} = x_1\mathbf{b}_1 + x_2\mathbf{b}_2 + \cdots x_n\mathbf{b}_n$$

The scalars $x_1, x_2, \ldots, x_n$ are called the **coordinates** *of $\mathbf{v}$ relative to B. The vector of these coordinates is called the* **coordinate vector** *of $\mathbf{v}$ and is denoted $[\mathbf{v}]_B$:*

$$[\mathbf{v}]_B = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$$

The **coordinate map**

$$V \to \mathbb{R}^n : \mathbf{v} \mapsto [\mathbf{v}]_B$$

is a linear, one-to-one map from V onto $\mathbb{R}^n$.

4.4.re1. As seen in 4.3.re2, $B = \{1, t, t^2\}$ is a basis for $\mathbb{P}_2$. The coordinate vector of the polynomial $(t-3)^2$ relative to B is $[\,9 \quad -6 \quad 1\,]^T$, since $(t-3)^2$ can be written $9 - 6t + t^2$. Observe that this is the only linear combination of $\{1, t, t^2\}$ which adds up to $(t-3)^2$, as promised by 4.4.1.

4.4.re2. Let $B = \{1, t-1, (t-1)^2\}$.

a.* Prove that B is a basis for $\mathbb{P}_2$.

b. Find $p(t) \in \mathbb{P}_2$ is $[p(t)]_B = [\,2 \quad -1 \quad 3\,]^T$.

c. Find $[q(t)]_B \in \mathbb{R}^3$ if $q(t) = (t-2)(t+3)$.

If $B = \{\mathbf{b}_1, \mathbf{b}_2, \ldots, \mathbf{b}_p\}$ is a basis for $\mathbb{R}^n$, then the matrix $A = [\mathbf{b}_1 \;\; \mathbf{b}_2 \;\; \cdots \;\; \mathbf{b}_p]$ has a pivot in every row and column. Consequently, p must equal n and A is invertible. The coordinates of a vector $\mathbf{x}$ in $\mathbb{R}^n$ relative to B is the vector $[\mathbf{x}]_B$ that satisfies $A[\mathbf{x}]_B = \mathbf{x}$, that is, $[\mathbf{x}]_B = A^{-1}\mathbf{x}$. Since this coordinate map from $\mathbb{R}^n$ into $\mathbb{R}^n$ is obtained by matrix multiplication, 1.8.1 tells us that it is a linear transformation, as promised by 4.4.1. Fact 4.4.1 is also consistent with the invertible matrix theorem 2.3.1, which tells us that, since A^{-1} is invertible , the transformation $\mathbf{x} \mapsto A^{-1}\mathbf{x}$ is one-to-one and onto.

4.4.re3. Determine whether the given set is a basis for $\mathbb{R}^n$; if it is, find the coordinates of the given vector relative to B

a. $\left\{ \begin{bmatrix} 1 \\ 4 \\ 3 \end{bmatrix}, \begin{bmatrix} 2 \\ 9 \\ 6 \end{bmatrix}, \begin{bmatrix} -1 \\ -3 \\ -4 \end{bmatrix} \right\}, \begin{bmatrix} 3 \\ 11 \\ 11 \end{bmatrix}$ b. $\left\{ \begin{bmatrix} 1 \\ 0 \\ -1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \\ -2 \\ -1 \end{bmatrix}, \begin{bmatrix} 3 \\ 0 \\ -4 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 2 \\ 1 \end{bmatrix} \right\}, \begin{bmatrix} 1 \\ -1 \\ -2 \\ -1 \end{bmatrix}$

c. $\left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} \right\}, \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$ d. $\left\{ \begin{bmatrix} 1 \\ 0 \\ -1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ -1 \\ -2 \\ -1 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ -4 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \right\}, \begin{bmatrix} -2 \\ 0 \\ -10 \\ -1 \end{bmatrix}$

4.4.re4. Use problems 4.4.29 and 4.4.30 in the text to answer the following.
a. Determine whether the polynomials $1+t+t^2$, $1+2t+3t^2$, and $-1+t^2$ are linearly independent by considering their coordinate vectors relative to the standard basis for $\mathbb{P}_2$.
b. Determine whether $5t+8t^2-3t^3$ is in span$\{1-t^3,\ 2+t+t^3,\ 3t+4t^2\}$ by considering coordinate vectors.

Answers

4.4.re.e2a. The three monomials $\{1, t, t^2\}$ are in the span of B because

$$1 \in \operatorname{span} B \qquad t = (t-1)+1 \in \operatorname{span} B \qquad t^2 = (t-1)^2 + 2(t-1) + 1 \in \operatorname{span} B.$$

Therefore any linear combination of $\{1, t, t^2\}$ is also in span B, and so B is a spanning set for $\mathbb{P}_2$. Now suppose that

$$a + b(t-1) + c(t-1)^2 = \mathbf{0} \tag{4.4.2}$$

(that is, $a + b(t-1) + c(t-1)^2 = 0$ for all values of t). Evaluating (4.4.2) at $t = 1$ implies $a = 0$. Now take the derivative of (4.4.2) and evaluate it at $t = 1$:

$$b + 2c(t-1) = \mathbf{0} \quad \Longrightarrow \quad b + 2c \cdot 0 = 0 \quad \Longrightarrow \quad b = 0.$$

Now take the second derivative of (4.4.2):

$$2c = \mathbf{0} \quad \Longrightarrow \quad c = 0.$$

B is linearly independent, since the only solution to (4.4.2) is $a = b = c = 0$. Therefore B is a basis, i.e., a linearly independent spanning set, for $\mathbb{P}_2$. 4.4.re.e2b. $p(t) = 2 \cdot 1 - 1(t-1) + 3(t-1)^2$. 4.4.re.e2c. The coordinates of $q(t)$ are the scalars a, b, c satisfying

$$(t-2)(t+3) = a + b(t-1) + c(t-1)^2. \tag{4.4.3}$$

Evaluate (4.4.3) at $t = 1$ to obtain $-4 = a$. Now evaluate (4.4.3) at two other t-values, say $t = 0$ and $t = 2$:

$$\begin{aligned} -6 &= -4 - b + c \\ 0 &= -4 + b + c \end{aligned}, \quad \text{or} \quad \begin{aligned} -2 &= -b + c \\ 4 &= b + c \end{aligned}$$

The solution of this 2×2 system is $b = 3$, $c = 1$. Therefore $[q(t)]_B = \begin{bmatrix} -4 \\ 3 \\ 1 \end{bmatrix}$. 4.4.re.e3a. basis for $\mathbb{R}^3$; coord. $= \begin{bmatrix} -1 & 1 & -2 \end{bmatrix}^T$. 4.4.re.e3b. basis for $\mathbb{R}^4$; coord. $= \begin{bmatrix} 0 & 1 & 0 & 0 \end{bmatrix}^T$. 4.4.re.e3c. not a basis; coordinates are not defined. 4.4.re.e3d. basis for $\mathbb{R}^4$; coord. $= \begin{bmatrix} -2 & 0 & 3 & -1 \end{bmatrix}^T$. 4.4.re.e4a. Their coordinate vectors are the same three seen in 4.4.re3c. Since these vectors are linearly dependent, so are the polynomials, by problem 4.4.29. 4.4.re.e4b. The coordinate vectors (relative to the standard basis for $\mathbb{P}_4$) of the four polynomials are $[0\ \ 5\ \ 8\ \ -3]^T$, $[1\ \ 0\ \ 0\ \ -1]^T$, $[2\ \ 1\ \ 0\ \ 1]^T$, and $[0\ \ 3\ \ 4\ \ 0]^T$. Row reduce the augmented matrix $\begin{bmatrix} 1 & 2 & 0 & 0 \\ 0 & 1 & 3 & 5 \\ 0 & 0 & 4 & 8 \\ -1 & 1 & 0 & -3 \end{bmatrix}$ to determine that the corresponding linear system is consistent. Since $[0\ \ 5\ \ 8\ \ -3]^T$ is in the span of the other three vectors, problem 4.4.30 implies that $5t+8t^2-3t^3$ lies in span$\{1-t^3,\ 2+t+t^3,\ 3t+4t^2\}$.

4.5: Dimension

Fact 4.5.1. *Any two bases of a vector space V must have the same number of elements, called the* **dimension** *of V, written* $\dim V$.

4.5.re1. $\dim \mathbb{R}^n = n$, since any basis for $\mathbb{R}^n$ must contain a pivot in every row and column and therefore must have exactly n elements.

4.5.re2. Find the dimension of the vector space

$$\operatorname{span}\left\{\begin{bmatrix}1\\1\\-2\\7\end{bmatrix}\begin{bmatrix}2\\3\\-4\\18\end{bmatrix}\begin{bmatrix}-1\\0\\2\\-3\end{bmatrix}\right\}$$

The vector space in question is the same as the column space of

$$F=\begin{bmatrix}1&2&-1\\1&3&0\\-2&-4&2\\7&18&-3\end{bmatrix}\sim\begin{bmatrix}1&2&-1\\0&1&1\\0&0&0\\0&0&0\end{bmatrix}.$$

Since the pivot columns of F form a basis for $\operatorname{Col} F$, the dimension of $\operatorname{Col} F = 2$.

4.5.re3. Find the dimension of the given vector space. See 4.3.re5.

a. $\operatorname{span}\left\{\begin{bmatrix}1\\2\\-1\end{bmatrix},\begin{bmatrix}2\\5\\-1\end{bmatrix},\begin{bmatrix}1\\4\\1\end{bmatrix},\begin{bmatrix}-2\\-3\\3\end{bmatrix},\begin{bmatrix}-1\\0\\5\end{bmatrix}\right\}$ b. $\operatorname{span}\left\{\begin{bmatrix}1\\1\\1\end{bmatrix},\begin{bmatrix}2\\4\\4\end{bmatrix},\begin{bmatrix}-3\\-7\\-7\end{bmatrix},\begin{bmatrix}1\\4\\5\end{bmatrix},\begin{bmatrix}7\\13\\13\end{bmatrix}\right\}$

c. $\operatorname{span}\left\{\begin{bmatrix}1\\2\\0\\1\end{bmatrix},\begin{bmatrix}-1\\-2\\0\\-1\end{bmatrix},\begin{bmatrix}2\\5\\-1\\2\end{bmatrix},\begin{bmatrix}15\\33\\-3\\15\end{bmatrix},\begin{bmatrix}3\\6\\1\\4\end{bmatrix}\right\}$ d. $\operatorname{span}\left\{\begin{bmatrix}1\\1\\-1\end{bmatrix},\begin{bmatrix}2\\2\\-1\end{bmatrix},\begin{bmatrix}5\\5\\-3\end{bmatrix},\begin{bmatrix}3\\3\\-6\end{bmatrix}\right\}$

The next result follows from 4.3.8. The first part is illustrated in 4.5.re2 and 4.5.re3.

Fact 4.5.2. *If A is any matrix, then*

1. $\dim \operatorname{Col} A = \dim \operatorname{Row} A =$ *the number of pivot columns in A, and*
2. $\dim \operatorname{Nul} A =$ *the number of non-pivot columns in A.*

The **rank** of a matrix is the dimension of its column space, and the **nullity** of a matrix is the dimension of its null space.

Rank-Nullity theorem 4.5.3. *For any matrix A,*

$$\operatorname{rank} A + \operatorname{nullity} A = \text{the number of columns of } A$$

Although it has no basis, the vector space $\{\mathbf{0}\}$ is said to be 0-dimensional. If V has a basis consisting of some finite number of vectors, then V is said to be **finite-dimensional**; otherwise, V is **infinite-dimensional**.

4.5.re4. The set of all monomial functions $\{1, t, t^2, t^3, t^4, \ldots\}$ is a basis for the vector space $\mathbb{P}$ of all polynomials. Therefore $\mathbb{P}$ is infinite-dimensional.

Fact 4.5.4. *If H is a subspace of the vector space V, then $\dim H \leq \dim V$.*

4.5.re5. Since $\dim \mathbb{R}^n = n$, all subspaces of $\mathbb{R}^n$ must have dimension $\leq n$.
In $\mathbb{R}^2$, the only 0-dimensional subspace is $\{(0,0)\}$, the one-dimensional subspaces (the spans of one nonzero vector) are lines through the origin, and the only 2-dimensional subspace is $\mathbb{R}^2$ itself.
In $\mathbb{R}^3$, the only 0-dimensional subspace is $\{(0,0,0)\}$, the one-dimensional subspaces (the spans of one nonzero vector) are lines through the origin, the 2-dimensional subspaces (spans of two linearly independent vectors) are planes through the origin, and the only 3-dimensional subspace is $\mathbb{R}^3$ itself.
(See also Examples 2 and 4 of section 4.5 in the text.)

Fact 4.5.5. *Suppose V is a finite-dimensional vector space.*

1. *Any spanning set for V can, by the removal of some elements, be reduced to a basis for V.*
2. *Any linearly independent set in V can, with some additional elements, be enlarged to a basis for V.*

Part 1 of 4.5.5 is a restatement of the Spanning Set Theorem 4.3.9.

4.5.re6. The vectors

$$\left\{ \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix} \right\}$$

are linearly independent, since neither is a scalar multiple of the other. To expand this set to a basis for $\mathbb{R}^3$, we just have to add a vector not in the span of the two original vectors. Either pick a third vector at random and test if the three are linearly independent, or add the three standard basis vectors for $\mathbb{R}^3$ and row-reduce to find a basis for the column space $\mathbb{R}^3$ of

$$A = \begin{bmatrix} 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ -1 & -1 & 0 & 0 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 & 1 \end{bmatrix}.$$

The pivots columns of A

$$\left\{ \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \right\}$$

are a basis for $\mathbb{R}^3$.

4.5.re7. Find a basis for $\mathbb{R}^4$ that contains the two (linearly independent) vectors

$$\begin{bmatrix}1\\1\\0\\0\end{bmatrix}, \begin{bmatrix}1\\-1\\0\\1\end{bmatrix}$$

See also examples 4.3.re6 and 4.3.re7.

The next fact is a useful consequence of 4.5.5.

Basis Theorem 4.5.6. *If B is a subset of a finite-dimensional vector space V, if $\#B = \dim V$, and if B either spans V or is linearly independent, then B must be a basis for V.*

That is, if B has the correct number of elements to be a basis for V, then in order to test if B is a basis for V, we need only check that it spans V or is linearly independent.

4.5.re8. Use 4.5.6 to determine whether the given sets are bases for $\mathbb{P}_3$. See also 4.4.re4.

a. $\{1,\ 2+t,\ 3+2t+t^2,\ 4+3t+2t+t^3\}$ b. $\{1+t^2,\ 1-t^2,\ t+t^3,\ t-t^3\}$

Answers

4.5.re.e3a. 3. 4.5.re.e3b. 3. 4.5.re.e3c. 3. 4.5.re.e3d. 2. 4.5.re7. There are many correct answers. Here's one: $\{[1\ \ 1\ \ 0\ \ 0]^T, [1\ \ -1\ \ 0\ \ 1]^T, [1\ \ 0\ \ 0\ \ 0]^T, [0\ \ 0\ \ 1\ \ 0]^T\}$. 4.5.re.e8a. The matrix formed by the coordinate vectors

$$\begin{bmatrix}1&2&3&4\\0&1&2&3\\0&0&1&2\\0&0&0&1\end{bmatrix}$$

has a pivot in every row, so the polynomials span $\mathbb{P}_3$. Since $\dim \mathbb{P}_3 = 4$, these four polynomials must be a basis for $\mathbb{P}_3$. 4.5.re.e8b. Row-reduce the matrix formed by the coordinate vectors:

$$\begin{bmatrix}1&1&0&0\\0&0&1&1\\1&-1&0&0\\0&0&1&-1\end{bmatrix} \sim \begin{bmatrix}1&1&0&0\\0&-2&0&0\\0&0&1&1\\0&0&0&-2\end{bmatrix}$$

There's a pivot in every column, so the four polynomials are linearly independent. Since $\dim \mathbb{P}_3 = 4$, these four polynomials must be a basis for $\mathbb{P}_3$.

5.1: Eigenvalues and eigenvectors

Definition 5.1.1. *If A is a square matrix, and λ is an* **eigenvalue** *of A if any of the following equivalent statements are true.*

1. $A\mathbf{x} = \lambda\mathbf{x}$ *for some vector* $\mathbf{x} \neq \mathbf{0}$.
2. $(A - \lambda I)\mathbf{x} = \mathbf{0}$ *for some vector* $\mathbf{x} \neq \mathbf{0}$.
3. *The matrix $(A - \lambda I)$ is singular, that is, non-invertible.*
4. $\det(A - \lambda I) = 0$.

In case of 1 or 2, $\mathbf{x}$ is said to be an **eigenvector** of A corresponding to λ.
Equation 4 is called the **characteristic equation** of A.
For any given eigenvalue λ, the **eigenspace** of A corresp. to λ is $\mathrm{Nul}(A - \lambda I)$. This eigenspace consists of all eigenvectors of A corresp. to λ (together with $\mathbf{0}$, which doesn't count as an eigenvector).
Confirming that a scalar λ is an eigenvalue of a matrix A requires showing that $A - \lambda I$ is singular. Confirming that a given vector is an eigenvector is simply a matter of matrix multiplication.

5.1.re1. Determine whether the given scalar is an eigenvalue of $B = \begin{bmatrix} 4 & 1 & -6 \\ 2 & -7 & -8 \\ 1 & 1 & -3 \end{bmatrix}$

a. 1 b. 3

a. To find out if 1 is an eigenvalue, row-reduce $B - I$:

$$\begin{bmatrix} 4 & 1 & -6 \\ 2 & -7 & -8 \\ 1 & 1 & -3 \end{bmatrix} - \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 3 & 1 & -6 \\ 2 & -8 & -8 \\ 1 & 1 & -4 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & -4 \\ 3 & 1 & -6 \\ 2 & -8 & -8 \end{bmatrix}$$
$$\sim \begin{bmatrix} 1 & 1 & -4 \\ 0 & -2 & 6 \\ 0 & -10 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & -4 \\ 0 & -2 & 6 \\ 0 & 0 & -30 \end{bmatrix}$$

Because $B - I$ is invertible, 1 is not an eigenvalue of B.
b. The scalar 3 is an eigenvalue of B since

$$B - 3I = \begin{bmatrix} 4 & 1 & -6 \\ 2 & -7 & -8 \\ 1 & 1 & -3 \end{bmatrix} - \begin{bmatrix} 3 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{bmatrix} = \begin{bmatrix} 1 & 1 & -6 \\ 2 & 10 & -8 \\ 1 & 1 & -6 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & -6 \\ 2 & 10 & -8 \\ 0 & 0 & 0 \end{bmatrix} \tag{5.1.2}$$

is singular.

5.1.re1, continued. Determine whether the given vector is an eigenvector of B.
c. $[1 \quad -2 \quad 1]^T$ d. $[2 \quad 1 \quad 0]^T$

c. Multiply

$$B\mathbf{x} = \begin{bmatrix} 4 & 1 & -6 \\ 2 & -7 & -8 \\ 1 & 1 & -3 \end{bmatrix} \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix} = \begin{bmatrix} -4 \\ 8 \\ -4 \end{bmatrix} = -4 \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}$$

and we see that $[1 \quad -2 \quad 1]^T$ is an eigenvector of B corresp. to -4.
d. Since the product

$$B\mathbf{x} = \begin{bmatrix} 4 & 1 & -6 \\ 2 & -7 & -8 \\ 1 & 1 & -3 \end{bmatrix} \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 9 \\ -3 \\ 3 \end{bmatrix}$$

is not a scalar multiple of $[2 \quad 1 \quad 0]^T$, this vector is not an eigenvector of B.

The basis of an eigenspace

Since an eigenspace (of one matrix) is a nullspace (of another), we can find a basis for an eigenspace of a given matrix and a given eigenvalue we did as in section 4.3.

5.1.re1, continued. The eigenspace of B corresp. to 3 is the null space of

$$(B-3I) = \begin{bmatrix} 1 & 1 & -6 \\ 2 & 10 & -8 \\ 1 & 1 & -6 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & -6 \\ 2 & 10 & -8 \\ 0 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & -6 \\ 0 & 8 & 4 \\ 0 & 0 & 0 \end{bmatrix}$$
$$\sim \begin{bmatrix} 1 & 1 & -6 \\ 0 & 1 & \frac{1}{2} \\ 0 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & -\frac{13}{2} \\ 0 & 1 & \frac{1}{2} \\ 0 & 0 & 0 \end{bmatrix}.$$

For $\mathbf{x}$ to belong to the null space of $B-3I$, we need $x_1 = \frac{13}{2}x_3$ and $x_2 = -\frac{1}{2}x_3$, where x_3 is free. Therefore the null space is the span of the vector $[\frac{13}{2} \quad -\frac{1}{2} \quad 1]^T$, and this single vector forms a basis for the eigenspace of B corresp. to 3. So does $[13 \quad -1 \quad 2]^T$, or any nonzero scalar multiple of $[\frac{13}{2} \quad -\frac{1}{2} \quad 1]^T$.

5.1.re2. Find a basis for the eigenspace for $U = \begin{bmatrix} -7 & -9 & -6 \\ 14 & 16 & 10 \\ -12 & -12 & -7 \end{bmatrix}$ corresponding to the given eigenvalue.

a. 2 b. 1 c. -1

5.1.re2, continued. Find a basis for the eigenspace for $W = \begin{bmatrix} 0 & 0 & 3 \\ -9 & 3 & 9 \\ 0 & 0 & 3 \end{bmatrix}$ corresponding to the given eigenvalue.

d. 3 e. 0

Answers

5.1.re.e2a. $\{[1 \quad -1 \quad 0]^T\}$. 5.1.re.e2b. $\{[0 \quad 2 \quad -3]^T\}$. 5.1.re.e2c. $\{[-1 \quad 2 \quad -2]^T\}$.
5.1.re.e2d. $\{[1 \quad 1 \quad 1]^T, [1 \quad 0 \quad 1]^T\}$. 5.1.re.e2e. $\{[1 \quad 3 \quad 0]^T\}$.

5.2: The characteristic polynomial and equation

If A is an $n \times n$ matrix, then $\det(A - \lambda I)$ is a polynomial of degree n called its **characteristic polynomial**. The eigenvalues of A are the solutions to the **characteristic equation**

$$\det(A - \lambda I) = 0.$$

5.2.re1. Find the characteristic polynomial and eigenvalues of the given matrix.

a. $\begin{bmatrix} -9 & 4 \\ -20 & 9 \end{bmatrix}$ b. $\begin{bmatrix} 2 & 3 \\ 12 & 2 \end{bmatrix}$ c. $\begin{bmatrix} 12 & 2 \\ 2 & 3 \end{bmatrix}$

d. $\begin{bmatrix} 1 & 1 & -2 \\ -2 & -2 & 2 \\ 0 & 0 & -1 \end{bmatrix}$ e. $\begin{bmatrix} 2 & 1 & -2 \\ 0 & 3 & 2 \\ 0 & 0 & 4 \end{bmatrix}$

The examples seen in 5.2.re1 illustrate these important facts.

Fact 5.2.1.

1. *The eigenvalues of a triangular matrix are its diagonal elements.*
2. *A (square) matrix is singular iff zero is one of its eigenvalues.*
3. *Row operations can change the eigenvalues of a matrix.*

That is, if B is obtained by a row operation on A, you should not expect A and B to have the same eigenvalues.

The multiplicities of an eigenvalue

The **algebraic multiplicity** of an eigenvalue is its multiplicity as a zero of the characteristic polynomial. The **geometric multiplicity** of an eigenvalue is the dimension of the associated eigenspace. Our text does not use the term "geometric multiplicity."

5.2.re2. Let $E = \begin{bmatrix} 3 & 0 & 0 \\ 7 & 10 & -7 \\ 14 & 14 & -11 \end{bmatrix}$. The characteristic polynomial of E is $(3-\lambda)^2(-4-\lambda)$, so $\lambda = 3$ is an eigenvalue of algebraic multiplicity 2, and $\lambda = -4$ is an eigenvalue of algebraic multiplicity 1. The geometric multiplicity of each is found by row-reducing $E - \lambda I$ until its pivot locations become clear.

The geometric multiplicity of $\lambda = 3$ is 2, that is, the number of non-pivot columns of

$$E - 3I = \begin{bmatrix} 0 & 0 & 0 \\ 7 & 7 & -7 \\ 14 & 14 & -14 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & -1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

$\lambda = -4$ has geometric multiplicity 1, since

$$E + 4I = \begin{bmatrix} 7 & 0 & 0 \\ 7 & 14 & -7 \\ 14 & 14 & -7 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & -1 \\ 0 & 0 & 0 \end{bmatrix}$$

has only one non-pivot column.

5.2.re3. The characteristic polynomial of $F = \begin{bmatrix} 5 & -1 \\ 0 & 5 \end{bmatrix}$ is $(5-\lambda)^2$, so $\lambda = 5$ has algebraic multiplicity 2. Since

$$F - 5I = \begin{bmatrix} 0 & -1 \\ 0 & 0 \end{bmatrix}$$

has only non-pivot column, $\lambda = 5$ has geometric multiplicity 1.

Fact 5.2.2. *If λ is an eigenvalue of a matrix, then*

$$1 \leq \textit{geometric multiplicity of } \lambda \leq \textit{algebraic multiplicity of } \lambda$$

Similarity

Definition 5.2.3. *The $n \times n$ matrices A and B are said to be* **similar** *if*

$$A = PBP^{-1}$$

for some invertible matrix P.

Fact 5.2.4.
If A is similar to B, then

1. *A^n and B^n are similar for all integers n, and*
2. *A and B have the same*
   a. *determinant,*
   b. *characteristic polynomial, and, therefore,*
   c. *eigenvalues.*

Furthermore,

3. *if A is similar to B and B is similar to C, then A is similar to C.*

The converse to 5.2.4.2 is false; if two matrices has the same characteristic polynomial, it is not necessary that the matrices be similar.

5.2.re4. $(1-\lambda)^2$ is the characteristic polynomial of both $\begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}$ and $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, but these two matrices are not similar, since the identity matrix is similar only to itself.

Answers

5.2.re.e1a. $1-\lambda^2$, $\lambda = \pm 1$. 5.2.re.e1b. $(2-\lambda)^2 - 36$, $\lambda = -4, 8$. 5.2.re.e1c. $\lambda^2 - 15\lambda + 32$, $\lambda = \frac{15}{2} \pm \frac{\sqrt{97}}{2}$. 5.2.re.e1d. $-\lambda(\lambda+1)^2$, $\lambda = 0, -1$. 5.2.re.e1e. $(2-\lambda)(3-\lambda)(4-\lambda)$, $\lambda = 2, 3, 4$.

5.3: Diagonalization

Definition 5.3.1. *A square matrix A is* **diagonalizable** *if it is similar to a diagonal matrix D:*

$$A = PDP^{-1} \tag{5.3.2}$$

Fact 5.3.3. *The $n \times n$ matrix A is diagonalizable iff $\mathbb{R}^n$ has a basis consisting of eigenvectors of A. In that case, the columns of P in (5.3.2) are eigenvectors of A and the diagonal elements of D are the associated eigenvalues.*

5.3.re1. The matrix $U = \begin{bmatrix} -7 & -9 & -6 \\ 14 & 16 & 10 \\ -12 & -12 & -7 \end{bmatrix}$ seen in 5.1.re2 satisfies

$$U \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} = 2 \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} \qquad U \begin{bmatrix} 0 \\ 2 \\ -3 \end{bmatrix} = 1 \begin{bmatrix} 0 \\ 2 \\ -3 \end{bmatrix} \qquad U \begin{bmatrix} -1 \\ 2 \\ -2 \end{bmatrix} = -1 \begin{bmatrix} -1 \\ 2 \\ -2 \end{bmatrix}$$

and therefore

$$U \begin{bmatrix} 1 & 0 & -1 \\ -1 & 2 & 2 \\ 0 & -3 & -2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & -1 \\ -1 & 2 & 2 \\ 0 & -3 & -2 \end{bmatrix} \begin{bmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix} \tag{5.3.4}$$

The matrix formed by these three eigenvectors is invertible, since its row echelon form has a pivot in every row and column:

$$\begin{bmatrix} 1 & 0 & -1 \\ -1 & 2 & 2 \\ 0 & -3 & -2 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & -1 \\ 0 & 2 & 1 \\ 0 & 0 & -\frac{1}{2} \end{bmatrix}$$

Therefore (5.3.4) implies

$$U = \begin{bmatrix} 1 & 0 & -1 \\ -1 & 2 & 2 \\ 0 & -3 & -2 \end{bmatrix} \begin{bmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} 1 & 0 & -1 \\ -1 & 2 & 2 \\ 0 & -3 & -2 \end{bmatrix}^{-1}$$

Fact 5.3.5. *If $\lambda_1, \lambda_2, \ldots, \lambda_p$ are p different eigenvalues of A, and if $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_p$ are eigenvectors of A corresponding to these eigenvalues, than $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_p\}$ is linearly independent.*
More generally, if $B_1, B_2, \ldots B_p$ are bases for the eigenspaces of $\lambda_1, \lambda_2, \ldots, \lambda_p$, then $B_1 \cup B_2 \cup \cdots \cup B_p$ (that is, the collection of all the vectors in all the bases) is linearly independent.

Fact 5.3.6. *The $n \times n$ matrix A is diagonalizable iff the geometric multiplicity of every eigenvalue equals its algebraic multiplicity.*

5.3.re2. As seen in 5.1.re2, $\begin{bmatrix}1\\1\\1\end{bmatrix}$ and $\begin{bmatrix}1\\3\\0\end{bmatrix}$ are eigenvectors of $W = \begin{bmatrix}0&0&3\\-9&3&9\\0&0&3\end{bmatrix}$ corresponding to the eigenvalues 3 an 0. Fact 5.3.5 says that these two vectors must linearly independent. Furthermore, since

$$\left\{\begin{bmatrix}1\\1\\1\end{bmatrix}, \begin{bmatrix}1\\0\\1\end{bmatrix}\right\} \text{ and } \left\{\begin{bmatrix}1\\3\\0\end{bmatrix}\right\}$$

are bases for the eigenspaces of W corresponding to 3 and 0, their union

$$\left\{\begin{bmatrix}1\\1\\1\end{bmatrix}, \begin{bmatrix}1\\0\\1\end{bmatrix}, \begin{bmatrix}1\\3\\0\end{bmatrix}\right\}$$

must also be linearly independent.
Find matrices P and D for which $W = PDP^{-1}$.

5.3.re3. Find matrices P and D for which the given matrix equals PDP^{-1}, or explain why none exist.

a. $\begin{bmatrix}-3&-5\\0&2\end{bmatrix}$ b. $\begin{bmatrix}1&-4\\-2&-1\end{bmatrix}$ c. $\begin{bmatrix}5&-1\\1&3\end{bmatrix}$

d. $\begin{bmatrix}1&2&-1\\-4&7&-3\\0&0&3\end{bmatrix}$ e. $\begin{bmatrix}2&-3&3\\0&4&-2\\0&1&1\end{bmatrix}$ f. $\begin{bmatrix}-1&-18&18\\0&5&-3\\0&0&2\end{bmatrix}$

Answers

5.3.re2. $P = \begin{bmatrix}1&1&1\\1&0&3\\1&1&0\end{bmatrix}$, $D = \begin{bmatrix}3&0&0\\0&3&0\\0&0&0\end{bmatrix}$. 5.3.re.e3a. $P = \begin{bmatrix}-1&1\\1&0\end{bmatrix}$, $D = \begin{bmatrix}2&0\\0&-3\end{bmatrix}$.
5.3.re.e3b. $P = \begin{bmatrix}2&1\\-1&1\end{bmatrix}$, $D = \begin{bmatrix}3&0\\0&-3\end{bmatrix}$. 5.3.re.e3c. Not diag'bl. $\lambda = 4$ has alg.mult. 2 but geo.mult. 1.
5.3.re.e3d. Not diag'bl. $\lambda = 3$ has alg.mult. 2 but geo.mult. 1.
5.3.re.e3e. $P = \begin{bmatrix}-1&4&-3\\0&1&2\\0&1&1\end{bmatrix}$, $D = \begin{bmatrix}2&0&0\\0&2&0\\0&0&3\end{bmatrix}$. 5.3.re.e3f. $P = \begin{bmatrix}1&0&-3\\0&1&1\\0&1&0\end{bmatrix}$, $D = \begin{bmatrix}-1&0&0\\0&2&0\\0&0&5\end{bmatrix}$.

5.4: The matrix of a transformation

Recall that if V is an n-dimensional vector space and $\mathcal{B}$ a basis for V, then the coordinate map

$$\mathbf{v} \mapsto [\mathbf{v}]_{\mathcal{B}}$$

is a one-to-one linear map from V onto $\mathbb{R}^n$ which allows us to think of V and $\mathbb{R}^n$ as being the "same."
If the vector space W is m-dimensional with a basis $\mathcal{C}$, and if T is a linear transformation from V into W, then there's an $m \times n$ matrix M (called the matrix of T relative to $\mathcal{B}$ an $\mathcal{C}$) with the property that

$$M[\mathbf{v}]_{\mathcal{B}} = [T(\mathbf{v})]_{\mathcal{C}}$$

for every $\mathbf{v}$ in V. That is, as T maps $\mathbf{v}$ in V to $T(\mathbf{v})$ in W, so M maps the coordinates of $\mathbf{v}$ in $\mathbb{R}^n$ to the coordinates of $T(\mathbf{v})$ in $\mathbb{R}^m$, as indicated in this diagram:

$$\begin{array}{ccc} V & \xrightarrow{T} & W \\ {\scriptstyle [\ ]_{\mathcal{B}}}\big\downarrow & & \big\uparrow{\scriptstyle [\ ]_{\mathcal{C}}^{-1}} \\ \mathbb{R}^n & \xrightarrow{M} & \mathbb{R}^m \end{array}$$

If $\mathbf{b}_j$ stands for the jth element of $\mathcal{B}$, then the columns of M can be found by the rule

$$j\text{th column of } M = M\mathbf{e}_j = [T(\mathbf{b}_j)]_{\mathcal{C}}.$$

5.4.re1. Suppose $\{\mathbf{b}_1, \mathbf{b}_2, \mathbf{b}_3, \mathbf{b}_4\}$ is a basis for V and $\{\mathbf{c}_1, \mathbf{c}_2, \mathbf{c}_3\}$ a basis for W, and

$$\begin{aligned} T(\mathbf{b}_1) &= \mathbf{c}_1 - \mathbf{c}_3 \\ T(\mathbf{b}_2) &= \mathbf{c}_2 \\ T(\mathbf{b}_3) &= \mathbf{c}_1 + 2\mathbf{c}_2 + \mathbf{c}_3 \\ T(\mathbf{b}_4) &= -\mathbf{c}_1 - \mathbf{c}_3 \end{aligned}$$

a. Find the matrix M for T relative to $\mathcal{B}$ and $\mathcal{C}$
b. Use M compute $T(\mathbf{b}_1 - \mathbf{b}_2 + \mathbf{b}_3 - 2\mathbf{b}_4)$.
c. Solve $M\mathbf{x} = \mathbf{e}_1$, $\mathbf{e}_2$, and $\mathbf{e}_3$. Then use your answers to find three elements of V which T send to $\mathbf{c}_1$, $\mathbf{c}_2$, and $\mathbf{c}_3$.
d. Does T map V onto W? Why or why not?
e. Find a nonzero $\mathbf{x} \in \mathbb{R}^4$ for which $M\mathbf{x} = \mathbf{0}$, and use your answer to find find a nonzero element of V which T sends to $\mathbf{0}$ in W.

5.4.re2. Suppose $S : \mathbb{P}_2 \to \mathbb{R}^3 : p(t) \mapsto \begin{bmatrix} p(-1) \\ p(0) \\ p(1) \end{bmatrix}$

a. Find the matrix M for S relative to the standard bases for $\mathbb{P}_2$ and $\mathbb{R}^3$.
b. Solve $M\mathbf{x} = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$ and use your answer to find a quadratic polynomial $p(t)$ for which $p(-1) = 1$, $p(0) = 1$ and $p(1) = 0$.

Eigenvalues and eigenvectors

If T is a linear transformation from a vector space V to itself, and if λ is a scalar and $\mathbf{v}$ a nonzero vector in V satisfying

$$T(\mathbf{v}) = \lambda\mathbf{v}$$

then λ is an **eigenvalue** and $\mathbf{v}$ an **eigenvector** of T.

5.4.re3. We let $C^\infty(\mathbb{R})$ stand for the vector space of all functions possessing derivatives of all orders on $\mathbb{R}$. The differential operator $\frac{d}{dt}$ is a linear transformation from $C^\infty(\mathbb{R})$ into (in fact, onto) itself. Since $\frac{d}{dt}e^t = e^t$, the function e^t is an eigenvector of $\frac{d}{dt}$, and the scalar 1 is an eigenvalue. In fact, every real number λ is an eigenvalue of $\frac{d}{dt}$. Find a corresponding eigenvector, that is, a nonzero function $f(t)$ for which $f'(t) = \lambda f(t)$.

Fact 5.4.1. *If V is finite-dimensional and $\mathcal{B}$ a basis for V and $T : V \to V$ a linear transformation, then the eigenvalues of T are the eigenvalues of its matrix relative to $\mathcal{B}$, and the eigenvectors of T are those vectors in V whose $\mathcal{B}$-coordinates are eigenvectors of that matrix.*

5.4.re4. Suppose T is a linear transformation from V into itself, and $\mathcal{B} = \{\mathbf{b}_1, \mathbf{b}_2\}$ is a basis for V, and

$$T(\mathbf{b}_1) = 4\mathbf{b}_1 + \mathbf{b}_2 \quad T(\mathbf{b}_2) = -2\mathbf{b}_1 + \mathbf{b}_2.$$

a. Find the matrix M for T relative to $\mathcal{B}$ and its eigenvalues.

b. Find an eigenvector in $\mathbb{R}^2$ for each eigenvalue for M and the corresponding eigenvectors in V for T.

Similarity revisited

Two $n\times\ n$ matrices are similar exactly when one is the matrix of the other relative to a certain basis for $\mathbb{R}^n$. Specifically, if A and B are similar

$$A = PBP^{-1}$$

then the columns of P form a matrix $\mathcal{B}$ for $\mathbb{R}^n$, and

$$\begin{array}{ccc} \mathbf{x} & \xrightarrow{A} & A\mathbf{x} \\ P^{-1}\big\downarrow & & \big\uparrow P \\ [\mathbf{x}]_{\mathcal{B}} & \xrightarrow{B} & B[\mathbf{x}]_{\mathcal{B}} = [A\mathbf{x}]_{\mathcal{B}} \end{array}$$

An $n\times n$ matrix is diagonalizable exactly when its matrix relative to a particular basis for $\mathbb{R}^n$ is diagonal.

5.4.re5. Let $A = \begin{bmatrix} 4 & -2 \\ 1 & 1 \end{bmatrix}$ and let $T : \mathbb{R}^2 \to \mathbb{R}^2 : \mathbf{x} \mapsto A\mathbf{x}$.

a. Find a basis $\mathcal{B}$ for $\mathbb{R}^2$ so that the matrix of T relative to $\mathcal{B}$ is diagonal.

b. Find the matrix N for T relative to the basis $\left\{ \begin{bmatrix} 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 5 \\ 3 \end{bmatrix} \right\}$

Answers

5.4.re.e1a. $M = \begin{bmatrix} 1 & 0 & 1 & -1 \\ 0 & 1 & 2 & 0 \\ -1 & 0 & 1 & -1 \end{bmatrix}$. 5.4.re.e1b. $\begin{bmatrix} 1 & 0 & 1 & -1 \\ 0 & 1 & 2 & 0 \\ -1 & 0 & 1 & -1 \end{bmatrix} \begin{bmatrix} 1 \\ -1 \\ 1 \\ -2 \end{bmatrix} = \begin{bmatrix} 4 \\ 1 \\ 2 \end{bmatrix}$ and so $T(\mathbf{b}_1 - \mathbf{b}_2 + \mathbf{b}_3 - 2\mathbf{b}_4) = 4\mathbf{c}_1 + \mathbf{c}_2 + 2\mathbf{c}_3$. 5.4.re.e1c. Augment M with the 3×3 identity and reduce to rref. If we set the free variable x_4 equal zero, solutions are $\mathbf{x} = \begin{bmatrix} \frac{1}{2} \\ -1 \\ \frac{1}{2} \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$, and $\begin{bmatrix} -\frac{1}{2} \\ -1 \\ \frac{1}{2} \end{bmatrix}$. Therefore $\mathbf{c}_1 = T(\frac{1}{2}\mathbf{b}_1 - \mathbf{b}_2 + \frac{1}{2}\mathbf{b}_3)$, $\mathbf{c}_2 = T(\mathbf{b}_2)$, and $\mathbf{c}_3 = T(-\frac{1}{2}\mathbf{b}_1 - \mathbf{b}_2 + \frac{1}{2}\mathbf{b}_3)$. 5.4.re.e1d. T is onto: since $\mathcal{C}$ is in the range of T, so is anything in $\operatorname{span}\mathcal{C} = W$. 5.4.re.e1e. Augment M with $\mathbf{0}$ and row reduce to rref. When the free var is 1, $\mathbf{x} = [0 \quad -2 \quad 1 \quad 1]^T$ and therefore $T(-2\mathbf{b}_2 + \mathbf{b}_3 + \mathbf{b}_4) = \mathbf{0}_W$. 5.4.re.e2a. S sends $\{1, t, t^2\}$ to $\left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \right\}$, and so $M = \begin{bmatrix} 1 & -1 & 1 \\ 1 & 0 & 0 \\ 1 & 1 & 1 \end{bmatrix}$. 5.4.re.e2b. Solution is $\mathbf{x} = \begin{bmatrix} 1 \\ -\frac{1}{2} \\ -\frac{1}{2} \end{bmatrix}$, and so the desired polynomial is $p(t) = 1 - \frac{1}{2}t - \frac{1}{2}t^2$. 5.4.re3. $e^{\lambda t}$. 5.4.re.e4a. $M = \begin{bmatrix} 4 & -2 \\ 1 & 1 \end{bmatrix}$. $\lambda = 2, 3$. 5.4.re.e4b. $\lambda = 2$: $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ and $\mathbf{b}_1 + \mathbf{b}_2$. $\lambda = 3$: $\begin{bmatrix} 2 \\ 1 \end{bmatrix}$ and $2\mathbf{b}_1 + \mathbf{b}_2$. 5.4.re.e5a. See 5.4.re4. $\left\{ \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \end{bmatrix} \right\}$. 5.4.re.e5b. Let $P = \begin{bmatrix} 2 & 5 \\ 1 & 3 \end{bmatrix}$. Since $A = PMP^{-1}$, can solve for $N = P^{-1}AP = \begin{bmatrix} 3 & 2 \\ 0 & 2 \end{bmatrix}$.

5.5: Complex eigenvalues of a real matrix

For a review of complex numbers, see Appendix B in our text, or the notes at https://kunklet.people.cofc.edu/MATH111/Carith.pdf

If the real $n \times n$ matrix A has a nonreal eigenvalue λ, then any eigenvector of A corresp. to λ must also be nonreal. Consequently, there exist real numbers a and b and real linearly independent vectors $\mathbf{v}$ and $\mathbf{w}$ with the property that

$$A(\mathbf{v} + i\mathbf{w}) = (a - bi)(\mathbf{v} + i\mathbf{w})$$

As a consequence, if we let P be the $n \times 2$ matrix whose columns are $\mathbf{v}$ and $\mathbf{w}$,

$$P = [\mathbf{v} \quad \mathbf{w}]$$

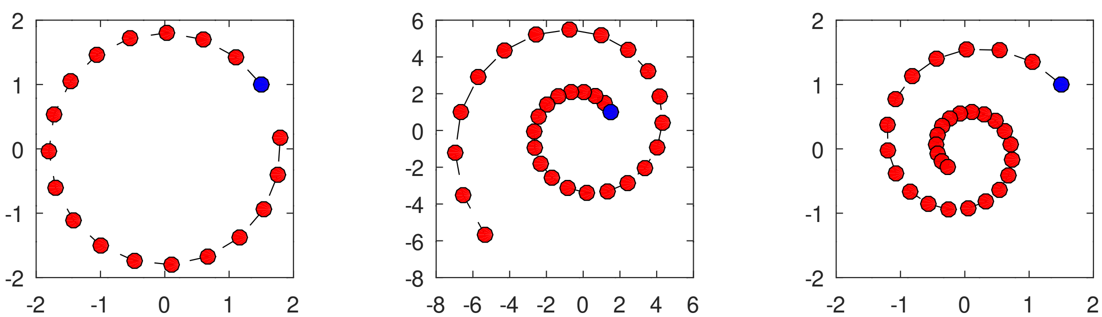
then

$$AP = P \begin{bmatrix} a & -b \\ b & a \end{bmatrix} \tag{5.5.1}$$

In the special case that $a^2 + b^2 = 1$, the matrix

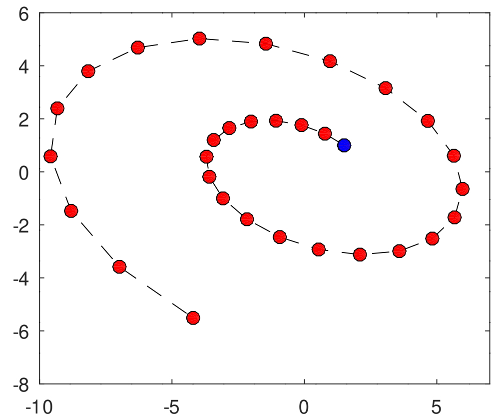
$$C = \begin{bmatrix} a & -b \\ b & a \end{bmatrix}$$

is the standard matrix for a rotation in $\mathbb{R}^2$ (§1.9); if $a^2 + b^2 \neq 1$, then C is the standard matrix for a rotation and scaling in $\mathbb{R}^2$. See plots below of $\mathbf{x}$ (blue) in $\mathbb{R}^2$ and $C\mathbf{x}, C^2\mathbf{x}, C^3\mathbf{x}, \ldots$ (red) for three different (a, b) pairs.



$a^2 + b^2 = 1$ $\qquad$ $a^2 + b^2 > 1$ $\qquad$ $a^2 + b^2 < 1$

When A sends the vector $P\mathbf{x}$ to $PC\mathbf{x}$, the effect is a rotation (and possibly scaling) of coordinates in the two dimensional column space of P.



When $n = 2$, the matrix P invertible, and

$$A = PCP^{-1},$$

That is, A is similar to a rotation-and-scaling matrix in $\mathbb{R}^2$. The figure is a plot of points $\mathbf{x}$ (blue) and $A\mathbf{x}, A^2\mathbf{x}, A^3\mathbf{x}, \ldots$ (red) for one such A.

5.5.re1. To find the eigenvalues of $A = \begin{bmatrix} 0 & 5 \\ -2 & 6 \end{bmatrix}$, solve the characteristic equation

$$\begin{vmatrix} -\lambda & 5 \\ -2 & 6-\lambda \end{vmatrix} = -\lambda(6-\lambda) + 10 = \lambda^2 - 6\lambda + 10 = 0.$$

Completing the square,

$$\lambda^2 - 6\lambda + 9 = -10 + 9 \quad \Longrightarrow \quad (\lambda - 3)^2 = -1 \quad \Longrightarrow \quad \lambda = 3 \pm i.$$

As expected, the complex eigenvalues come in a conjugate pair, as will their eigenvectors. To find a basis for the eigenspace corresponding to $3 + i$, row-reduce

$$A - (3+i)I = \begin{bmatrix} -3-i & 5 \\ -2 & 3-i \end{bmatrix}.$$

Since this 2×2 is singular, the second row must be a multiple of the first. (You can confirm this by showing $\frac{5}{-3-i} = \frac{3-i}{-2}$.) Therefore, $A - (3+i)$ must row reduce to

$$\begin{bmatrix} 1 & -\frac{3}{2} + \frac{1}{2}i \\ 0 & 0 \end{bmatrix}$$

and so the eigenspace is spanned by the vector $\begin{bmatrix} \frac{3}{2} - \frac{1}{2}i \\ 1 \end{bmatrix}$. In case you prefer to eliminate the fractions, the eigenspace is also spanned by $\begin{bmatrix} 3-i \\ 2 \end{bmatrix}$. Using this, we find

$$P = \begin{bmatrix} 3 & -1 \\ 2 & 0 \end{bmatrix} \text{ and } C = \begin{bmatrix} 3 & 1 \\ -1 & 3 \end{bmatrix}.$$

The eigenspace corresponding to the conjugate eigenvalue $3 - i$ must be spanned by the conjugate vector $\begin{bmatrix} 3+i \\ 2 \end{bmatrix}$, which gives

$$P = \begin{bmatrix} 3 & 1 \\ 2 & 0 \end{bmatrix} \text{ and } C = \begin{bmatrix} 3 & -1 \\ 1 & 3 \end{bmatrix}.$$

Either choice gives a correct factorization of A:

$$\begin{bmatrix} 0 & 5 \\ -2 & 6 \end{bmatrix} = \begin{bmatrix} 3 & -1 \\ 2 & 0 \end{bmatrix} \begin{bmatrix} 3 & 1 \\ -1 & 3 \end{bmatrix} \begin{bmatrix} 3 & -1 \\ 2 & 0 \end{bmatrix}^{-1} = \begin{bmatrix} 3 & 1 \\ 2 & 0 \end{bmatrix} \begin{bmatrix} 3 & -1 \\ 1 & 3 \end{bmatrix} \begin{bmatrix} 3 & 1 \\ 2 & 0 \end{bmatrix}^{-1}$$

5.5.re2. Find the eigenvalues of the given matrix.

a. $\begin{bmatrix} 3 & -4 \\ 2 & -1 \end{bmatrix}$ b. $\begin{bmatrix} -4 & -15 \\ 3 & 8 \end{bmatrix}$ c. $\begin{bmatrix} 5 & 1 \\ -10 & -1 \end{bmatrix}$

d. $\begin{bmatrix} 3 & -10 \\ 4 & -9 \end{bmatrix}$ e. $\begin{bmatrix} -1 & 0 & 0 \\ 0 & -7 & 13 \\ 0 & -5 & 9 \end{bmatrix}$ f. $\begin{bmatrix} -4 & 10 & 0 \\ -2 & 4 & 1 \\ 0 & 0 & 3 \end{bmatrix}$

5.5.re3. Find P and C so that the given matrix equals PCP^{-1}. Answers are not unique: If

$$P = [\mathbf{v}\ \ \mathbf{w}] \qquad C = \begin{bmatrix} a & -b \\ b & a \end{bmatrix}$$

is a solution, so is

$$P = [\mathbf{v}\ \ -\mathbf{w}] \qquad C = \begin{bmatrix} a & b \\ -b & a \end{bmatrix}$$

a. $\begin{bmatrix} 3 & -4 \\ 2 & -1 \end{bmatrix}$ b. $\begin{bmatrix} -4 & -15 \\ 3 & 8 \end{bmatrix}$ c. $\begin{bmatrix} 5 & 1 \\ -10 & -1 \end{bmatrix}$

d. $\begin{bmatrix} 3 & -10 \\ 4 & -9 \end{bmatrix}$ e. $\begin{bmatrix} -7 & 13 \\ -5 & 9 \end{bmatrix}$ f. $\begin{bmatrix} -4 & 10 \\ -2 & 4 \end{bmatrix}$

Answers

5.5.re.e2a. $1 \pm 2i$. 5.5.re.e2b. $2 \pm 3i$. 5.5.re.e2c. $2 \pm i$. 5.5.re.e2d. $-3 \pm 2i$. 5.5.re.e2e. -1, $1 \pm i$.
5.5.re.e2f. 3, $\pm 2i$. 5.5.re.e3a. $P = \begin{bmatrix} 1 & -1 \\ 1 & 0 \end{bmatrix}$, $C = \begin{bmatrix} 1 & -2 \\ 2 & 1 \end{bmatrix}$. 5.5.re.e3b. $P = \begin{bmatrix} 1 & 2 \\ 0 & -1 \end{bmatrix}$, $C = \begin{bmatrix} 2 & 3 \\ -3 & 2 \end{bmatrix}$.
5.5.re.e3c. $P = \begin{bmatrix} 0 & 1 \\ 1 & -3 \end{bmatrix}$, $C = \begin{bmatrix} 2 & -1 \\ 1 & 2 \end{bmatrix}$. 5.5.re.e3d. $P = \begin{bmatrix} 1 & 2 \\ 1 & 1 \end{bmatrix}$, $C = \begin{bmatrix} -3 & 2 \\ -2 & -3 \end{bmatrix}$.
5.5.re.e3e. $P = \begin{bmatrix} 2 & 3 \\ 1 & 2 \end{bmatrix}$, $C = \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}$. 5.5.re.e3f. $P = \begin{bmatrix} 1 & 3 \\ 1 & 1 \end{bmatrix}$, $C = \begin{bmatrix} 0 & -2 \\ 2 & 0 \end{bmatrix}$.

6.1: Inner Product, Length, and Orthogonality

Remember that $\mathbb{R}^n$ stands for the vector space of all $n \times 1$ *column* vectors.

Definition 6.1.1. *If* $\mathbf{u}$ *and* $\mathbf{v}$ *are in* $\mathbb{R}^n$*, their* **inner product** *is the scalar*

$$\mathbf{u} \cdot \mathbf{v} = \mathbf{u}^T \mathbf{v} = u_1 v_1 + u_2 v_2 + \cdots + u_n v_n.$$

The inner product is also known as the "dot product" or "scalar product."

Properties 6.1.2. *For any* $\mathbf{u}$, $\mathbf{v}$, *and* $\mathbf{w}$ *in* $\mathbb{R}^n$*, and any scalar* c,

1. $\mathbf{u} \cdot \mathbf{v} = \mathbf{v} \cdot \mathbf{u}$
2. $(\mathbf{u} + \mathbf{v}) \cdot \mathbf{w} = \mathbf{u} \cdot \mathbf{w} + \mathbf{v} \cdot \mathbf{w}$.
3. $(c\mathbf{u}) \cdot \mathbf{v} = c(\mathbf{u} \cdot \mathbf{v}) = \mathbf{u} \cdot (c\mathbf{v})$.
4. $\mathbf{u} \cdot \mathbf{u} \geq 0$*; furthermore,* $\mathbf{u} \cdot \mathbf{u} = 0$ *iff* $\mathbf{u} = \mathbf{0}$.

Definition 6.1.3. *The* **length** *of* $\mathbf{u}$ *in* $\mathbb{R}^n$ *is*

$$\|\mathbf{u}\| = \sqrt{\mathbf{u} \cdot \mathbf{u}} = \sqrt{u_1^2 + u_2^2 + \cdots + u_n^2}$$

The length is also known as the "norm" or "magnitude."

The **distance between** $\mathbf{u}$ *and* $\mathbf{v}$ *in* $\mathbb{R}^n$ *is*

$$\|\mathbf{u} - \mathbf{v}\|.$$

A **unit vector** *is a vector in* $\mathbb{R}^n$ *having length 1. Two vectors are* **parallel** *if one is a scalar multiple of the other, and in the* **same direction** *if one is a positive scalar multiple of the other. To* **normalize** *a vector is to produce a unit vector in the same direction.*

More properties 6.1.4.

5. $\|c\mathbf{u}\| = |c|\|\mathbf{u}\|$
6. $\|\mathbf{u} - \mathbf{v}\| = \|\mathbf{v} - \mathbf{u}\|$.

6.1.re1. Let

$$\mathbf{u} = \begin{bmatrix} 1 \\ -1 \\ -3 \\ 5 \end{bmatrix} \qquad \mathbf{v} = \begin{bmatrix} 2 \\ 0 \\ 4 \\ -2 \end{bmatrix} \qquad \mathbf{w} = \begin{bmatrix} 1 \\ 0 \\ 2 \\ -1 \end{bmatrix}$$

Find the following.

a. $\mathbf{u} \cdot \mathbf{v}$ b. $\mathbf{v} \cdot \mathbf{u}$ c. $\mathbf{u} \cdot \mathbf{u}$

d. $\|\mathbf{u}\|$ e. $\|\mathbf{v}\|$ f. $\|\mathbf{w}\|$

g. $\mathbf{u} \cdot \mathbf{v} + \mathbf{w} \cdot \mathbf{v}$ h. $(\mathbf{u} + \mathbf{w}) \cdot \mathbf{v}$ i. $(\mathbf{u} - \mathbf{v}) \cdot \mathbf{w}$

j. $\|\mathbf{u} - \mathbf{v}\|$

k. A unit vector in the same direction as $\mathbf{u}$.

l. A unit vector in the opposite direction of $\mathbf{u}$.

Orthogonality

Definition 6.1.5. *We say the vectors* $\mathbf{u}$ *and* $\mathbf{v}$ *are* **orthogonal** *and write* $\mathbf{u} \perp \mathbf{v}$ *if* $\mathbf{u} \cdot \mathbf{v} = 0$.

6.1.re2. $\begin{bmatrix} -1 \\ 1 \\ 2 \end{bmatrix} \perp \begin{bmatrix} -1 \\ 1 \\ -1 \end{bmatrix}$ because $\begin{bmatrix} -1 \\ 1 \\ 2 \end{bmatrix} \cdot \begin{bmatrix} -1 \\ 1 \\ -1 \end{bmatrix} = \begin{bmatrix} -1 & 1 & 2 \end{bmatrix} \begin{bmatrix} -1 \\ 1 \\ -1 \end{bmatrix} = 0.$

Definition 6.1.6. *The* **orthogonal complement** *of a set* $W \subset \mathbb{R}^n$ *is the set*

$$W^{\perp} = \left\{ \mathbf{x} \in \mathbb{R}^n \;\middle|\; \mathbf{x} \perp \mathbf{w} \textit{for every } \mathbf{w} \in W \right\}$$

Fact 6.1.7. *If* W *is a subset of* $\mathbb{R}^n$, *then* $W^{\perp}$ *is a subspace of* $\mathbb{R}^n$.

If W is a collection of p vectors in $\mathbb{R}^n$, and P is the $p \times n$ matrix whose columns are the elements of W, then $\mathbf{x} \in W^{\perp}$ iff $P\mathbf{x} = \mathbf{0}$. That is, $W^{\perp} = \operatorname{Nul} P$.

6.1.re3. Find a basis for the vector space

$$\left\{ \begin{bmatrix} 2 \\ -3 \\ -7 \\ -4 \end{bmatrix}, \begin{bmatrix} 1 \\ -2 \\ -4 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 3 \\ 5 \\ -3 \end{bmatrix} \right\}^{T}$$

This orthogonal complement is the null space of the matrix

$$\begin{bmatrix} 2 & -3 & -7 & -4 \\ 1 & -2 & -4 & 0 \\ -1 & 3 & 5 & -3 \end{bmatrix}$$

The reduced row echelon form of this matrix is

$$\begin{bmatrix} 1 & 0 & -2 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

and so its null space is one-dimensional and has the basis $\left\{ \begin{bmatrix} 2 \\ -1 \\ 1 \\ 0 \end{bmatrix} \right\}$.

6.1.re4. Find a basis for the given orthogonal complement.

a. $\left\{ \begin{bmatrix} 1 \\ 0 \\ -5 \\ 3 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ -9 \\ 8 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \\ 6 \\ -1 \end{bmatrix} \right\}^{\perp}$

b. $\left\{ \begin{bmatrix} 1 \\ 2 \\ -1 \\ -1 \\ -2 \end{bmatrix}, \begin{bmatrix} 2 \\ 5 \\ -4 \\ 0 \\ -3 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \\ -5 \\ -3 \end{bmatrix} \right\}^{\perp}$

Answers

6.1.re.e1a. -20. 6.1.re.e1b. -20. 6.1.re.e1c. 36. 6.1.re.e1d. 6. 6.1.re.e1e. $\sqrt{24} = 2\sqrt{6}$. 6.1.re.e1f. $\sqrt{6}$. 6.1.re.e1g. -8. 6.1.re.e1h. -8. 6.1.re.e1i. -22. 6.1.re.e1j. 10. 6.1.re.e1k. $\frac{1}{6}\begin{bmatrix}1\\-1\\-3\\5\end{bmatrix} = \begin{bmatrix}1/6\\-1/6\\-3/6\\5/6\end{bmatrix}$.

6.1.re.e1l. $-\frac{1}{6}\begin{bmatrix}1\\-1\\-3\\5\end{bmatrix} = \begin{bmatrix}-1/6\\1/6\\3/6\\-5/6\end{bmatrix}$. 6.1.re.e4a. $\left\{\begin{bmatrix}5\\-1\\1\\0\end{bmatrix}, \begin{bmatrix}-3\\-2\\0\\1\end{bmatrix}\right\}$. 6.1.re.e4b. $\left\{\begin{bmatrix}-3\\2\\1\\0\\0\end{bmatrix}, \begin{bmatrix}4\\-1\\0\\0\\1\end{bmatrix}\right\}$.

6.2: Orthogonal and orthonormal sets

Definition 6.2.1. *The set of vectors*

$$\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3, \ldots, \mathbf{u}_p\} \tag{6.2.2}$$

is said to be **orthogonal** *if*

$$\mathbf{u}_i \cdot \mathbf{u}_j = 0 \ \textit{ if } i \neq j$$

and **orthonormal** *if*

$$\mathbf{u}_i \cdot \mathbf{u}_j = \begin{cases} 0 & \textit{if } i \neq j, \textit{ and} \\ 1 & \textit{if } i = j. \end{cases}$$

In other words, an orthonormal set is an orthogonal set of unit vectors.
Let U denote the matrix whose columns are the elements of (6.2.2):

$$U = [\, \mathbf{u}_1 \quad \mathbf{u}_2 \quad \mathbf{u}_3 \quad \ldots \quad \mathbf{u}_p \,]$$

Then the set (6.2.2) is

$$\left\{ \begin{matrix} \text{orthogonal} \\ \text{orthonormal} \end{matrix} \right\} \text{ iff } U^T U = \left\{ \begin{matrix} \text{a diagonal matrix.} \\ \text{the identity matrix.} \end{matrix} \right\}$$

6.2.re1. The set of vectors

$$\left\{ \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \\ 2 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \\ -1 \end{bmatrix} \right\}$$

is an orthogonal set because

$$\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \perp \begin{bmatrix} -1 \\ 1 \\ 2 \end{bmatrix} \qquad \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \perp \begin{bmatrix} -1 \\ 1 \\ -1 \end{bmatrix} \qquad \begin{bmatrix} -1 \\ 1 \\ 2 \end{bmatrix} \perp \begin{bmatrix} -1 \\ 1 \\ -1 \end{bmatrix},$$

meaning

$$[1 \quad 1 \quad 0] \begin{bmatrix} -1 \\ 1 \\ 2 \end{bmatrix} = 0 \qquad [1 \quad 1 \quad 0] \begin{bmatrix} -1 \\ 1 \\ -1 \end{bmatrix} = 0 \qquad [-1 \quad 1 \quad 2] \begin{bmatrix} -1 \\ 1 \\ -1 \end{bmatrix} = 0.$$

These same calculations product the off-diagonal entries of the matrix product

$$\begin{bmatrix} 1 & -1 & -1 \\ 1 & 1 & 1 \\ 0 & 2 & -1 \end{bmatrix}^T \begin{bmatrix} 1 & -1 & -1 \\ 1 & 1 & 1 \\ 0 & 2 & -1 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 0 \\ -1 & 1 & 2 \\ -1 & 1 & -1 \end{bmatrix} \begin{bmatrix} 1 & -1 & -1 \\ 1 & 1 & 1 \\ 0 & 2 & -1 \end{bmatrix} = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 6 & 0 \\ 0 & 0 & 3 \end{bmatrix}.$$

6.2.re2. The set

$$\left\{ \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \\ 2 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \\ -1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} \right\}$$

is not orthogonal because

$$\begin{bmatrix} 1 & 1 & 0 \end{bmatrix} \cdot \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} = 1 \neq 0,$$

(which is enough to guarantee that

$$\begin{bmatrix} 1 & 1 & 0 \\ -1 & 1 & 2 \\ -1 & 1 & -1 \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & -1 & -1 & 0 \\ 1 & 1 & 1 & 1 \\ 0 & 2 & -1 & 1 \end{bmatrix} = \begin{bmatrix} * & * & * & 1 \\ * & * & * & * \\ * & * & * & * \\ 1 & * & * & * \end{bmatrix}$$

is not diagonal).

6.2.re3. The set

$$\left\{ \begin{bmatrix} 1 \\ 0 \\ -1 \\ 1 \end{bmatrix}, \begin{bmatrix} -2 \\ 1 \\ 2 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ -2 \\ 1 \\ 1 \end{bmatrix} \right\}$$

is orthogonal because

$$\begin{bmatrix} 1 & 0 & -1 & -1 \\ -2 & 1 & 2 & 0 \\ 0 & -2 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & -2 & 0 \\ 0 & 1 & -2 \\ -1 & 2 & 1 \\ 1 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 9 & 0 \\ 0 & 0 & 6 \end{bmatrix}$$

If we normalize the vectors in this set, we obtain the orthonormal set

$$\left\{ \begin{bmatrix} 1/\sqrt{3} \\ 0 \\ -1/\sqrt{3} \\ 1/\sqrt{3} \end{bmatrix}, \begin{bmatrix} -2/3 \\ 1/3 \\ 2/3 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ -2/\sqrt{6} \\ 1/\sqrt{6} \\ 1/\sqrt{6} \end{bmatrix} \right\}$$

Orthogonal matrices

Definition 6.2.3. *A square matrix U is said to* **orthogonal** *if its columns are orthonormal, that is, if*

$$U^T U = I. \tag{6.2.4}$$

Because an orthogonal matrix is square, (6.2.4) implies that $U^T = U^{-1}$, that is

$$U^T U = U U^T = I. \tag{6.2.5}$$

Important note: it is impossible for a non-square matrix to satisfy (6.2.5).

6.2.re4. By normalizing the vectors in 6.2.re1 we obtain the orthonormal set

$$\left\{ \begin{bmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \\ 0 \end{bmatrix}, \begin{bmatrix} -1/\sqrt{6} \\ 1/\sqrt{6} \\ 2/\sqrt{6} \end{bmatrix}, \begin{bmatrix} -1/\sqrt{3} \\ 1/\sqrt{3} \\ -1/\sqrt{3} \end{bmatrix} \right\}$$

Therefore

$$\begin{bmatrix} 1\sqrt{2} & -1/\sqrt{6} & -1/\sqrt{3} \\ 1\sqrt{2} & 1/\sqrt{6} & 1/\sqrt{3} \\ 0 & 2/\sqrt{6} & -1/\sqrt{3} \end{bmatrix}$$

is an orthogonal matrix.

Fact 6.2.6. *If U is an orthogonal matrix then*

$$(U\mathbf{u})^T (U\mathbf{v}) = \mathbf{u}^T \mathbf{v} \tag{6.2.7}$$

for all $\mathbf{u}$ *and* $\mathbf{v}$ *in* $\mathbb{R}^n$*. Consequently, for all such* $\mathbf{u}$ *and* $\mathbf{v}$*,*

$$\mathbf{u} \perp \mathbf{v} \textit{ iff } (U\mathbf{u}) \perp (U\mathbf{v}) \tag{6.2.8}$$

and

$$\|U\mathbf{u}\| = \|\mathbf{u}\| \tag{6.2.9}$$

6.2.re5. Explain why (6.2.7) follows from (6.2.4), and (6.2.8) and (6.2.9) follow from (6.2.7).

Coordinates relative to an orthogonal basis

Fact 6.2.10. *If*

$$\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3, \dots, \mathbf{u}_p\}$$

is an orthogonal set of non-zero vectors, and if $\mathbf{u}$ *is in their span, then*

$$\mathbf{u} = \frac{\mathbf{u}\cdot\mathbf{u}_1}{\mathbf{u}_1\cdot\mathbf{u}_1}\mathbf{u}_1 + \frac{\mathbf{u}\cdot\mathbf{u}_2}{\mathbf{u}_2\cdot\mathbf{u}_2}\mathbf{u}_2 + \cdots + \frac{\mathbf{u}\cdot\mathbf{u}_p}{\mathbf{u}_p\cdot\mathbf{u}_p}\mathbf{u}_p \tag{6.2.11}$$

Consequently, any orthogonal set of non-zero vectors is linearly independent and forms a basis for their span, and any set of n *orthogonal, non-zero vectors in* $\mathbb{R}^n$ *is a basis for* $\mathbb{R}^n$*.*

6.2.re1, continued. The set of vectors

$$\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\} = \left\{ \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \\ 2 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \\ -1 \end{bmatrix} \right\}$$

is an orthogonal basis for $\mathbb{R}^3$. The coordinates of the vector $\mathbf{u} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ relative to this basis are

$$\frac{\mathbf{u}\cdot\mathbf{u}_1}{\mathbf{u}_1\cdot\mathbf{u}_1} = \qquad \frac{\mathbf{u}\cdot\mathbf{u}_2}{\mathbf{u}_2\cdot\mathbf{u}_2} = \qquad \frac{\mathbf{u}\cdot\mathbf{u}_3}{\mathbf{u}_3\cdot\mathbf{u}_3} =$$

$$\frac{\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}\cdot\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}}{\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}\cdot\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}} = \frac{1}{2} \qquad \frac{\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}\cdot\begin{bmatrix} -1 \\ 1 \\ 2 \end{bmatrix}}{\begin{bmatrix} -1 \\ 1 \\ 2 \end{bmatrix}\cdot\begin{bmatrix} -1 \\ 1 \\ 2 \end{bmatrix}} = \frac{-1}{6} \qquad \frac{\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}\cdot\begin{bmatrix} -1 \\ 1 \\ -1 \end{bmatrix}}{\begin{bmatrix} -1 \\ 1 \\ -1 \end{bmatrix}\cdot\begin{bmatrix} -1 \\ 1 \\ -1 \end{bmatrix}} = \frac{-1}{3}$$

That is,

$$\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \frac{1}{2}\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} - \frac{1}{6}\begin{bmatrix} -1 \\ 1 \\ 2 \end{bmatrix} - \frac{1}{3}\begin{bmatrix} -1 \\ 1 \\ -1 \end{bmatrix}$$

6.2.re3, continued. The set

$$\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\} = \left\{ \begin{bmatrix} 1 \\ 0 \\ -1 \\ 1 \end{bmatrix}, \begin{bmatrix} -2 \\ 1 \\ 2 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ -2 \\ 1 \\ 1 \end{bmatrix} \right\}$$

is an orthogonal basis for a three-dimensional subspace of $\mathbb{R}^4$. The vector $\mathbf{u} = \begin{bmatrix} 5 \\ -5 \\ -3 \\ -1 \end{bmatrix}^T$ belongs to this subspace, and its coordinates relative to the basis are

$$\frac{\mathbf{u}\cdot\mathbf{v}_1}{\mathbf{v}_1\cdot\mathbf{v}_1} = \qquad \frac{\mathbf{u}\cdot\mathbf{v}_2}{\mathbf{v}_2\cdot\mathbf{v}_2} = \qquad \frac{\mathbf{u}\cdot\mathbf{v}_3}{\mathbf{v}_3\cdot\mathbf{v}_3} =$$

$$\frac{\begin{bmatrix} 5 \\ -5 \\ -3 \\ -1 \end{bmatrix}\cdot\begin{bmatrix} 1 \\ 0 \\ -1 \\ 1 \end{bmatrix}}{\begin{bmatrix} 1 \\ 0 \\ -1 \\ 1 \end{bmatrix}\cdot\begin{bmatrix} 1 \\ 0 \\ -1 \\ 1 \end{bmatrix}} = 3 \qquad \frac{\begin{bmatrix} 5 \\ -5 \\ -3 \\ -1 \end{bmatrix}\cdot\begin{bmatrix} -2 \\ 1 \\ 2 \\ 0 \end{bmatrix}}{\begin{bmatrix} -2 \\ 1 \\ 2 \\ 0 \end{bmatrix}\cdot\begin{bmatrix} -2 \\ 1 \\ 2 \\ 0 \end{bmatrix}} = -1 \qquad \frac{\begin{bmatrix} 5 \\ -5 \\ -3 \\ -1 \end{bmatrix}\cdot\begin{bmatrix} 0 \\ -2 \\ 1 \\ 1 \end{bmatrix}}{\begin{bmatrix} 0 \\ -2 \\ 1 \\ 1 \end{bmatrix}\cdot\begin{bmatrix} 0 \\ -2 \\ 1 \\ 1 \end{bmatrix}} = 2$$

You can confirm that $\mathbf{u}$ is in the span of the $\mathbf{v}_i$'s by checking for yourself that $\mathbf{u} = 3\mathbf{v}_1 - \mathbf{v}_2 + 2\mathbf{v}_3$.

6.2.re3, continued. The vector $\mathbf{x} = \begin{bmatrix} 0 & 1 & 0 & 1 \end{bmatrix}^T$ is not in the span of $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ because

$$\frac{\mathbf{x}\cdot\mathbf{v}_1}{\mathbf{v}_1\cdot\mathbf{v}_1} = -\frac{1}{3} \qquad \frac{\mathbf{x}\cdot\mathbf{v}_2}{\mathbf{v}_2\cdot\mathbf{v}_2} = \frac{1}{9} \qquad \frac{\mathbf{x}\cdot\mathbf{v}_3}{\mathbf{v}_3\cdot\mathbf{v}_3} = -\frac{1}{6}$$

but

$$-\frac{1}{3}\begin{bmatrix} 1 \\ 0 \\ -1 \\ 1 \end{bmatrix} + \frac{1}{9}\begin{bmatrix} -2 \\ 1 \\ 2 \\ 0 \end{bmatrix} - \frac{1}{6}\begin{bmatrix} 0 \\ -2 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} -5/9 \\ 4/9 \\ 7/18 \\ 3/2 \end{bmatrix} \neq \begin{bmatrix} 0 \\ 1 \\ 0 \\ 1 \end{bmatrix} \tag{6.2.12}$$

6.2.re6. Verify that the given set is orthogonal. Then, if possible, express the given vector as a linear combination of these vectors.

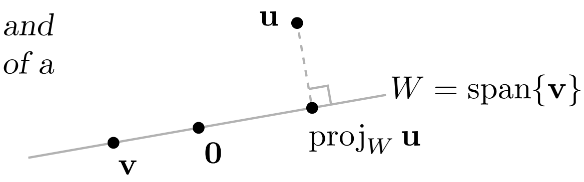
a. $\left\{ \begin{bmatrix} -3 \\ 4 \end{bmatrix}, \begin{bmatrix} 4 \\ 3 \end{bmatrix} \right\}; \begin{bmatrix} 4 \\ 5 \end{bmatrix}$

b. $\left\{ \begin{bmatrix} -2 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \\ 3 \end{bmatrix}, \begin{bmatrix} 4 \\ 7 \\ 1 \end{bmatrix} \right\}; \begin{bmatrix} 2 \\ 0 \\ -1 \end{bmatrix}$

c. $\left\{ \begin{bmatrix} 2 \\ 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \\ 3 \\ -1 \end{bmatrix}, \begin{bmatrix} -1 \\ 2 \\ 1 \\ 0 \end{bmatrix} \right\}; \begin{bmatrix} 1 \\ 0 \\ -1 \\ 0 \end{bmatrix}$

Orthogonal Projection onto a one-dimensional subspace

Definition 6.2.13. *If* $\mathbf{v}$ *is a nonzero vector and* W *its span, then the* **orthogonal projection** *of a vector* $\mathbf{u}$ *onto* W *is*



$$\text{proj}_W\, \mathbf{u} = \frac{\mathbf{u}\cdot\mathbf{v}}{\mathbf{v}\cdot\mathbf{v}}\mathbf{v}$$

The orthogonal projection gets its name from the fact that

$$(\mathbf{u} - \text{proj}_W\, \mathbf{u}) \cdot \mathbf{v} = 0$$

and consequently

$$\mathbf{u} - \text{proj}_W\, \mathbf{u} \in W^\perp. \tag{6.2.14}$$

This means that $\mathbf{u}$ is the sum of a vector in W and a vector in $W^\perp$:

$$\mathbf{u} = \text{proj}_W\, \mathbf{u} + (\mathbf{u} - \text{proj}_W\, \mathbf{u}). \tag{6.2.15}$$

The projection $\text{proj}_W\, \mathbf{u}$ is the vector in W closest to $\mathbf{u}$:

$$\|\mathbf{u} - \text{proj}_W\, \mathbf{u}\| \le \|\mathbf{u} - \mathbf{w}\| \quad \text{for all } \mathbf{w} \in W \tag{6.2.16}$$

The projection $\text{proj}_W\, \mathbf{u}$ is sometimes referred to as the **best approximation** to $\mathbf{u}$ from W and $\|\mathbf{u} - \text{proj}_W\, \mathbf{u}\|$ as the **distance** from $\mathbf{u}$ to W.

6.2.re7. Find the point on the line $y = 3x$ closest to the point $(4, 1)$. What is the distance from $(4, 1)$ to that line? The line $y = 3x$ is the span of the vector $[\,1 \quad 3\,]^T$ in $\mathbb{R}^2$. The closest point on this line to $(4, 1)$ is the projection of $[\,4 \quad 1\,]^T$ onto this span:

$$\frac{\begin{bmatrix}4\\1\end{bmatrix}\cdot\begin{bmatrix}1\\3\end{bmatrix}}{\begin{bmatrix}1\\3\end{bmatrix}\cdot\begin{bmatrix}1\\3\end{bmatrix}}\begin{bmatrix}1\\3\end{bmatrix} = \begin{bmatrix}0.7\\2.1\end{bmatrix}$$

The distance from $(4, 1)$ to the line is the distance from $[\,4 \quad 1\,]^T$ to the projection:

$$\left\|\begin{bmatrix}4\\1\end{bmatrix} - \begin{bmatrix}0.7\\2.1\end{bmatrix}\right\| = \left\|\begin{bmatrix}3.3\\-1.1\end{bmatrix}\right\| = \sqrt{3.3^2 + 1.1^2}.$$

6.2.re7, continued. Write $\begin{bmatrix}4\\1\end{bmatrix}$ as the sum of two vectors, one parallel to $\begin{bmatrix}1\\3\end{bmatrix}$ and one perpendicular to $\begin{bmatrix}1\\3\end{bmatrix}$.

By (6.2.15),

$$\begin{bmatrix}4\\1\end{bmatrix} = \begin{bmatrix}0.7\\2.1\end{bmatrix} + \left(\begin{bmatrix}4\\1\end{bmatrix} - \begin{bmatrix}0.7\\2.1\end{bmatrix}\right)$$
$$= \begin{bmatrix}0.7\\2.1\end{bmatrix} + \begin{bmatrix}3.3\\-1.1\end{bmatrix}$$

6.2.re8. Find the projection of the first vector onto the span of the second. Use this to write the first as a multiple of the second plus a vector orthogonal to the second. Finally, find the distance from the first to the span of the second.

a. $\begin{bmatrix}2\\1\end{bmatrix}$ $\begin{bmatrix}-1\\1\end{bmatrix}$ b. $\begin{bmatrix}-1\\1\end{bmatrix}$ $\begin{bmatrix}-4\\3\end{bmatrix}$ c. $\begin{bmatrix}5\\12\end{bmatrix}$ $\begin{bmatrix}-12\\5\end{bmatrix}$

Answers

6.2.re5. Remember that $\mathbf{u}^T\mathbf{v}$ is the dot product $\mathbf{u}\cdot\mathbf{v}$. (6.2.7): In general, $(AB)^T = B^TA^T$, and so $(U\mathbf{u})^T(U\mathbf{v}) = \mathbf{u}^TU^TU\mathbf{v} = \mathbf{u}^TI\mathbf{v} = \mathbf{u}^T\mathbf{v}$. (6.2.8): since $(U\mathbf{u})\cdot(U\mathbf{v}) = \mathbf{u}\cdot\mathbf{v}$, if either equals zero, both must equal zero. (6.2.9): $\|U\mathbf{u}\|^2 = (U\mathbf{u})^T(U\mathbf{u}) = \mathbf{u}^T\mathbf{u} = \|\mathbf{u}\|^2$. Take square roots to obtain (6.2.9).

6.2.re.e6a. $\begin{bmatrix}4\\5\end{bmatrix} = \frac{8}{25}\begin{bmatrix}-3\\4\end{bmatrix} + \frac{31}{25}\begin{bmatrix}4\\3\end{bmatrix}$. 6.2.re.e6b. $\begin{bmatrix}2\\0\\-1\end{bmatrix} = -\frac{5}{6}\begin{bmatrix}-2\\1\\1\end{bmatrix} - \frac{1}{11}\begin{bmatrix}1\\-1\\3\end{bmatrix} + \frac{7}{66}\begin{bmatrix}4\\7\\1\end{bmatrix}$.

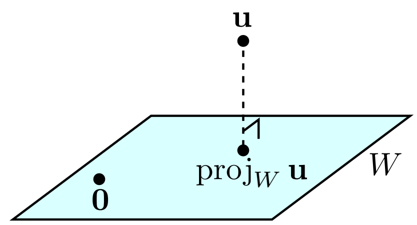
6.2.re.e6c. $\begin{bmatrix}1\\0\\-1\\0\end{bmatrix}$ is not in their span, since it fails to equal $\frac{2}{6}\begin{bmatrix}2\\1\\0\\1\end{bmatrix} - \frac{2}{12}\begin{bmatrix}1\\-1\\3\\-1\end{bmatrix} - \frac{2}{6}\begin{bmatrix}-1\\2\\1\\0\end{bmatrix}$.

6.2.re.e8a. proj $= \begin{bmatrix}1/2\\-1/2\end{bmatrix}$; $\begin{bmatrix}2\\1\end{bmatrix} = \begin{bmatrix}1/2\\-1/2\end{bmatrix} + \begin{bmatrix}3/2\\3/2\end{bmatrix}$. dist $= \left\|\begin{bmatrix}3/2\\3/2\end{bmatrix}\right\| = \frac{3}{2}\sqrt{2}$.

6.2.re.e8b. proj $= \begin{bmatrix}-\frac{28}{25}\\\frac{21}{25}\end{bmatrix}$; $\begin{bmatrix}-1\\1\end{bmatrix} = \begin{bmatrix}-\frac{28}{25}\\\frac{21}{25}\end{bmatrix} + \begin{bmatrix}\frac{3}{25}\\\frac{4}{25}\end{bmatrix}$. dist $= \left\|\begin{bmatrix}\frac{3}{25}\\\frac{4}{25}\end{bmatrix}\right\| = \frac{1}{5}$.

6.2.re.e8c. proj $= \mathbf{0}$; $\begin{bmatrix}5\\12\end{bmatrix} = \begin{bmatrix}0\\0\end{bmatrix} + \begin{bmatrix}5\\12\end{bmatrix}$. dist $= \left\|\begin{bmatrix}5\\12\end{bmatrix}\right\| = 13$.

6.3: Orthogonal projection onto multidimensional subspaces



Definition 6.3.1. *If* $\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3, \ldots, \mathbf{u}_p\}$ *is an orthogonal set of nonzero vectors and* W *their span, then the* **orthogonal projection** *of a vector* $\mathbf{u}$ *onto* W *is*

$$\mathbf{u} = \frac{\mathbf{u}\cdot\mathbf{u}_1}{\mathbf{u}_1\cdot\mathbf{u}_1}\mathbf{u}_1 + \frac{\mathbf{u}\cdot\mathbf{u}_2}{\mathbf{u}_2\cdot\mathbf{u}_2}\mathbf{u}\mathbf{u}_2 + \cdots + \frac{\mathbf{u}\cdot\mathbf{u}_p}{\mathbf{u}_p\cdot\mathbf{u}_p}\mathbf{u}_p \tag{6.3.2}$$

Definition 6.2.13 is a special case of 6.3.1. Just as in section 6.2,

$$\mathbf{u} - \operatorname{proj}_W \mathbf{u} \in W^\perp$$

Therefore $\mathbf{u}$ is the sum of a vector in W and a vector in $W^\perp$:

$$\mathbf{u} = \operatorname{proj}_W \mathbf{u} + (\mathbf{u} - \operatorname{proj}_W \mathbf{u})$$

The projection $\operatorname{proj}_W \mathbf{u}$ is the vector in W closest to $\mathbf{u}$:

$$\|\mathbf{u} - \operatorname{proj}_W \mathbf{u}\| \le \|\mathbf{u} - \mathbf{w}\| \quad \text{for all } \mathbf{w} \in W$$

The projection $\operatorname{proj}_W \mathbf{u}$ is sometimes referred to as the **best approximation** to $\mathbf{u}$ from W and $\|\mathbf{u} - \operatorname{proj}_W \mathbf{u}\|$ as the **distance** from $\mathbf{u}$ to W.
Two notes:
First, the right sides of (6.2.11) and (6.3.2) are the same because $\operatorname{proj}_W \mathbf{u} = \mathbf{u}$ if $\mathbf{u} \in W$.
Second, projection onto W is a linear transformation since (6.3.2) can be written as a matrix product. If U is the matrix whose columns are the basis elements:

$$U = [\,\mathbf{u}_1 \quad \mathbf{u}_2 \quad \mathbf{u}_3 \quad \ldots \quad \mathbf{u}_p\,]$$

then

$$\operatorname{proj}_W \mathbf{u} = U(U^TU)^{-1}U^T\mathbf{u}$$

6.3.re1. Prove that it W is a subspace in $\mathbb{R}^n$ with an orthogonal basis, then the orthogonal projection

$$\mathbb{R}^n \to \mathbb{R}^n : \mathbf{u} \mapsto \operatorname{proj}_W \mathbf{u}$$

maps $\mathbb{R}^n$ onto W. To do this, your must first explain why that $\operatorname{proj}_W \mathbf{u} \in W$ for all $\mathbf{u} \in \mathbb{R}^n$, and then why every $\mathbf{w} \in W$ equals $\operatorname{proj}_W \mathbf{u}$ for some $\mathbf{u} \in \mathbb{R}^n$.

6.3.re2. Verify that the given set is orthogonal. Then find the best approximation to the given vector from the span of the set.

a. $\left\{ \begin{bmatrix} 1 \\ -1 \\ 3 \end{bmatrix}, \begin{bmatrix} 4 \\ 7 \\ 1 \end{bmatrix} \right\}; \begin{bmatrix} 2 \\ 0 \\ -1 \end{bmatrix}$

b. $\left\{ \begin{bmatrix} 2 \\ 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \\ 3 \\ -1 \end{bmatrix}, \begin{bmatrix} -1 \\ 2 \\ 1 \\ 0 \end{bmatrix} \right\}; \begin{bmatrix} 1 \\ 0 \\ -1 \\ 0 \end{bmatrix}$

c. $\left\{ \begin{bmatrix} 1 \\ 0 \\ -1 \\ 1 \end{bmatrix}, \begin{bmatrix} -2 \\ 1 \\ 2 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ -2 \\ 1 \\ 1 \end{bmatrix} \right\}; \begin{bmatrix} 0 \\ 1 \\ 0 \\ 1 \end{bmatrix}$

Answers

6.3.re1. First, (6.3.2) shows $\text{proj}_W \mathbf{u}$ is a linear combination of the basis elements of W and is therefore in W. Second, $\mathbf{w} = \text{proj}_W \mathbf{w}$ for any $\mathbf{w}$ in W. 6.3.re.e2a. $-\frac{1}{11} \begin{bmatrix} 1 \\ -1 \\ 3 \end{bmatrix} + \frac{7}{66} \begin{bmatrix} 4 \\ 7 \\ 1 \end{bmatrix} = \frac{1}{6} \begin{bmatrix} 2 \\ 5 \\ -1 \end{bmatrix}$.

6.3.re.e2b. $\frac{2}{6} \begin{bmatrix} 2 \\ 1 \\ 0 \\ 1 \end{bmatrix} - \frac{2}{12} \begin{bmatrix} 1 \\ -1 \\ 3 \\ -1 \end{bmatrix} - \frac{2}{6} \begin{bmatrix} -1 \\ 2 \\ 1 \\ 0 \end{bmatrix} = \frac{1}{6} \begin{bmatrix} 5 \\ -1 \\ -5 \\ 3 \end{bmatrix}$. 6.3.re.e2c. $\begin{bmatrix} -5/9 \\ 4/9 \\ 7/18 \\ 3/2 \end{bmatrix}$ (See 6.2.re3).