MATH 203–03 (Kunkle), Quiz 5 | Name: ______________________

10 pts, 10 minutes | Feb 24, 2023

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1 (10 pts). Compute the determinant by cofactor expansion.

$$\begin{vmatrix} 2 & -1 & 5 & 1 \\ 0 & 0 & 3 & 0 \\ 6 & -2 & 1 & 0 \\ -3 & 0 & 1 & 2 \end{vmatrix}$$

Solution:

1.(Source: 3.1.9-12) The i, j cofactor of a matrix is $(-1)^{i+j}$ times the determinant of the submatrix obtained by deleting the ith and jth column of the matrix. You can picture the $(-1)^{i+j}$ like this:

$$\begin{matrix} + & - & + & - & \cdots \\ - & + & - & + & \cdots \\ + & - & + & - & \cdots \\ - & + & - & + & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{matrix} \tag{0}$$

Expanding along the second row,

$$\begin{vmatrix} 2 & -1 & 5 & 1 \\ 0 & 0 & 3 & 0 \\ 6 & -2 & 1 & 0 \\ -3 & 0 & 1 & 2 \end{vmatrix} = -3 \begin{vmatrix} 2 & -1 & 1 \\ 6 & -2 & 0 \\ -3 & 0 & 2 \end{vmatrix}$$

When we expand along the second row of the 3×3, this becomes

$$-3 \left(-6 \begin{vmatrix} -1 & 1 \\ 0 & 2 \end{vmatrix} + (-2) \begin{vmatrix} 2 & 1 \\ -3 & 2 \end{vmatrix} \right)$$

When applying the signs (0), don't confuse the second row of the 3×3 with the third row of the original 4×4.

Now calculate the 2×2's:

$$-3\left(-6(-1 \cdot 2) + (-2)(2 \cdot 2 - 1(-3))\right) = 6.$$

(done)

Note that the formula $\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$ is equivalent to cofactor expansion along the top row of the 2×2 (or, for that matter, along any of its rows or columns).

MATH 203–01 (Kunkle), Quiz 5
10 pts, 10 minutes

Name: ______________________
Feb 24, 2023

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1 (10 pts). Explain why the columns of the $n \times n$ matrix M must span $\mathbb{R}^n$ whenever $M^2\mathbf{x} = \mathbf{0}$ has only the trivial solution.

1.(Source: 2.3.34) The solution uses the Invertible Matrix Theorem, or IMT (Theorem 8 §2.3 of out text) as well as this fact we saw in class: if A and B are square matrices of the same size, then A and B are invertible if and only if AB is invertible. This is a consequence of Theorem 6 §2.2 and the IMT; see exercises 2.3.35&36. You can also remember it by using Theorems 4 and 6 from §3.2 of our text.

Solution:

By the IMT (parts a and d, as it appears our text), for $M^2\mathbf{x} = \mathbf{0}$ to have only the trivial solution, $M^2 = MM$ must be invertible. But this implies M must be invertible, and so its columns must span $\mathbb{R}^n$ (IMT, a and h).