No notes, books, electronic devices, or outside materials of any kind.

Read each problem carefully and simplify your answers.

Unless otherwise indicated, supporting work will be required on every problem worth more than 2 points.

Solve or **find the solution** always means to find the general solution, if it exists.

1(16 pts). Suppose $A = \begin{bmatrix} 1 & 2 \\ -3 & 1 \end{bmatrix}$, $B = \begin{bmatrix} 1 & 2 & 3 \\ -1 & 2 & 0 \end{bmatrix}$, and $C = \begin{bmatrix} 2 & 4 \\ 1 & -3 \\ 3 & -5 \end{bmatrix}$.

Calculate each of the following, if it exists.

a. AB b. BA c. AC d. $C^T + B$

2(10 pts). Suppose E is a square matrix whose first and third columns are the same. Could E be invertible? Why or why not?

3(6 pts). Suppose $G^{-1} = \begin{bmatrix} 1 & 2 \\ -3 & 1 \end{bmatrix}$. Find the solution $\mathbf{x}$ to $G\mathbf{x} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$, if it exists.

4(17 pts). Find the inverse of $F = \begin{bmatrix} 1 & 0 & 3 \\ 2 & 1 & 6 \\ -1 & 1 & -4 \end{bmatrix}$.

5a(17 pts). Find the determinant of $M = \begin{bmatrix} 1 & -1 & 9 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 1 & -1 & 2 \\ -2 & 1 & 0 & 3 \end{bmatrix}$.

5b(4 pts). What must be the determinant of M^3?

6(15 pts). Choose **one** of the following sets and prove whether or it is a subspace of $\mathbb{R}^3$.

a. $H = \left\{ \begin{bmatrix} a \\ b \\ c \end{bmatrix} \middle| \; a - 2b + 3c = 0 \right\}$ b. $J = \left\{ \begin{bmatrix} a+b \\ -2b \\ 3a \end{bmatrix} \middle| \; a, b \text{ real} \right\}$

7(15 pts). Answer **one** of the following parts. Clearly indicate which part you're answering.

a. Let $T : \mathbb{P} \to \mathbb{R} : \mathbf{p}(t) \mapsto \mathbf{p}(1)$. That is, T is the transformation from $\mathbb{P}$ to $\mathbb{R}$ defined by $T(\mathbf{p}) = \mathbf{p}(1)$. Determine whether T is linear and then prove your conclusion. Include in your solution the definition of it means for a transformation from one vector space to another to be linear.

b. Determine whether the polynomials $\{2t, 1-t, 1+t\}$ are linearly independent and then prove your conclusion. Include in your solution the definition of what it means for three vectors in a vector space to be linearly independent.

1(16 pts).(Source: 2.1.1-2) To multiply two matrices, the number of columns of the first must equal the number of rows of the second. The product of an $m \times n$ and an $n \times p$ matrix is an $m \times p$ matrix.

a. $$AB = \begin{bmatrix} 1 & 2 \\ -3 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 & 3 \\ -1 & 2 & 0 \end{bmatrix} = \begin{bmatrix} 1 \cdot 1 + 2(-1) & 1 \cdot 2 + 2 \cdot 2 & 1 \cdot 3 + 2 \cdot 0 \\ -3 \cdot 1 + 1(-1) & -3 \cdot 2 + 1 \cdot 2 & -3 \cdot 3 + 1 \cdot 0 \end{bmatrix}$$
$$= \begin{bmatrix} -1 & 6 & 3 \\ -4 & -4 & -9 \end{bmatrix}.$$

b. BA does not exist. c. AC does not exist.

d. $$C^T + B = \begin{bmatrix} 2 & 1 & 3 \\ 4 & -3 & -5 \end{bmatrix} + \begin{bmatrix} 1 & 2 & 3 \\ -1 & 2 & 0 \end{bmatrix} = \begin{bmatrix} 3 & 3 & 6 \\ 3 & -1 & -5 \end{bmatrix}.$$

2(10 pts).(Source: 2.3.23) Solution one: Since the first and third columns are identical, the columns of E are linearly dependent. By the invertible matrix theorem, E cannot be invertible.

Solution two: The IMT states that for E to be invertible, it must have a pivot in every column. Remember that a "pivot" is by definition the lead entry of a nonzero row in a matrix in row echelon form, and the "lead entry" of a row is its leftmost nonzero entry, if there is one.

If the first column is zero then E has no pivot in column one and can't be invertible.

If the first column is nonzero, then it can be row-reduced to $\mathbf{e}_1$. But the row operations that reduce column 1 to $\mathbf{e}_1$ must also reduce column 3 to $\mathbf{e}_1$:

$$E \sim \begin{bmatrix} 1 & \# & 1 & \# & \cdots \\ 0 & \# & 0 & \# & \cdots \\ 0 & \# & 0 & \# & \cdots \\ & \vdots & & & \\ 0 & \# & 0 & \# & \cdots \end{bmatrix}$$

Consequently, column 3 has no pivot, since its only nonzero entry is not the lead entry of its row. Since E has a column with no pivot, it is not invertible.

3(6 pts).(Source: 2.2.7-8) The unique solution is
$$\mathbf{x} = G^{-1} \begin{bmatrix} 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ -3 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 1 \cdot 1 + 2(-1) \\ -3 \cdot 1 + 1(-1) \end{bmatrix} = \begin{bmatrix} -1 \\ -4 \end{bmatrix}.$$

4(17 pts).(Source: 2.2.41-42, 3.3.11-16)
Solution one: Augment F with the identity and row-reduce.

row operation	**result**
(beginning matrix)	$\begin{matrix} 1 & 0 & 3 & 1 & 0 & 0 \\ 2 & 1 & 6 & 0 & 1 & 0 \\ -1 & 1 & -4 & 0 & 0 & 1 \end{matrix}$
$\mathbf{r}_2 \leftarrow \mathbf{r}_2 - 2\,\mathbf{r}_1$ $\mathbf{r}_3 \leftarrow \mathbf{r}_3 + \mathbf{r}_1$	$\begin{matrix} 1 & 0 & 3 & 1 & 0 & 0 \\ 0 & 1 & 0 & -2 & 1 & 0 \\ 0 & 1 & -1 & 1 & 0 & 1 \end{matrix}$
$\mathbf{r}_3 \leftarrow \mathbf{r}_3 - \mathbf{r}_2$	$\begin{matrix} 1 & 0 & 3 & 1 & 0 & 0 \\ 0 & 1 & 0 & -2 & 1 & 0 \\ 0 & 0 & -1 & 3 & -1 & 1 \end{matrix}$
$\mathbf{r}_3 \leftarrow -\,\mathbf{r}_3$ $\mathbf{r}_1 \leftarrow \mathbf{r}_1 - 3\,\mathbf{r}_3$	$\begin{matrix} 1 & 0 & 0 & 10 & -3 & 3 \\ 0 & 1 & 0 & -2 & 1 & 0 \\ 0 & 0 & 1 & -3 & 1 & -1 \end{matrix}$

Therefore $F^{-1} = \begin{bmatrix} 10 & -3 & 3 \\ -2 & 1 & 0 \\ -3 & 1 & -1 \end{bmatrix}$.

Solution two: The matrix of cofactors of F are

$$\begin{bmatrix} \begin{vmatrix} 1 & 6 \\ 1 & -4 \end{vmatrix} & -\begin{vmatrix} 2 & 6 \\ -1 & -4 \end{vmatrix} & \begin{vmatrix} 2 & 1 \\ -1 & 1 \end{vmatrix} \\ -\begin{vmatrix} 0 & 3 \\ 1 & -4 \end{vmatrix} & \begin{vmatrix} 1 & 3 \\ -1 & -4 \end{vmatrix} & -\begin{vmatrix} 1 & 0 \\ -1 & 1 \end{vmatrix} \\ \begin{vmatrix} 0 & 3 \\ 1 & 6 \end{vmatrix} & -\begin{vmatrix} 1 & 3 \\ 2 & 6 \end{vmatrix} & \begin{vmatrix} 1 & 0 \\ 2 & 1 \end{vmatrix} \end{bmatrix} = \begin{bmatrix} -10 & 2 & 3 \\ 3 & -1 & -1 \\ -3 & 0 & 1 \end{bmatrix}.$$

Use these cofactors to compute $\det F$, For instance, by expanding along the bottom row of F, we find $\det F = [\,-1 \quad 1 \quad -4\,]\,[-3 \quad 0 \quad 1\,]^T = (-1)(-3) + (1)(0) + (-4)(1) = -1$. Therefore, by Cramer's rule,

$$F^{-1} = \frac{1}{\det F}\,\mathrm{adj}\,F = \frac{1}{-1}\begin{bmatrix} -10 & 2 & 3 \\ 3 & -1 & -1 \\ -3 & 0 & 1 \end{bmatrix}^T = \begin{bmatrix} 10 & -3 & 3 \\ -2 & 1 & 0 \\ -3 & 1 & -1 \end{bmatrix}$$

5a(17 pts).(Source: 3.1.9-10, 3.2.7-9) Solution one: by cofactor expansion along the second row of M, and then along the first row of $M_{2,3}$ (the submatrix obtained form M by deleting row 2, column 3)

$$\begin{aligned} |M| = -2\begin{vmatrix} 1 & -1 & 0 \\ 0 & 1 & 2 \\ -2 & 1 & 3 \end{vmatrix} &= -2\left(1\begin{vmatrix} 1 & 2 \\ 1 & 3 \end{vmatrix} - (-1)\begin{vmatrix} 0 & 2 \\ -2 & 3 \end{vmatrix}\right) \\ &= -2\,(1(1\cdot 3 - 2\cdot 1) - (-1)(0\cdot 3 - 2(-2))) \\ &= -2(1+4) = -10 \end{aligned}$$

Soution two: by row reduction,

$$\begin{vmatrix} 1 & -1 & 9 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 1 & -1 & 2 \\ -2 & 1 & 0 & 3 \end{vmatrix} \overset{1}{=} \begin{vmatrix} 1 & -1 & 9 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 1 & -1 & 2 \\ 0 & -1 & 18 & 3 \end{vmatrix} \overset{2}{=} -\begin{vmatrix} 1 & -1 & 9 & 0 \\ 0 & 1 & -1 & 2 \\ 0 & 0 & 2 & 0 \\ 0 & -1 & 18 & 3 \end{vmatrix}$$

$$\overset{3}{=} -2\begin{vmatrix} 1 & -1 & 9 & 0 \\ 0 & 1 & -1 & 2 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 18 & 3 \end{vmatrix} \overset{1}{=} -2\begin{vmatrix} 1 & -1 & 9 & 0 \\ 0 & 1 & -1 & 2 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 17 & 5 \end{vmatrix} \overset{1}{=} -2\begin{vmatrix} 1 & -1 & 9 & 0 \\ 0 & 1 & -1 & 2 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 5 \end{vmatrix}$$

$$\overset{4}{=} -2\cdot 5 = -10$$

Notes:
[1] Row replacements; no change to determinant.
[2] Row interchange; determinant changes by a sign.
[3] Factor 2 out of row 3.
[4] The determinant of a triangular matrix equals the product of its main diagonal.

5b(4 pts).(Source: 3.2.35) Because the determinant of a product is the product of the determinants, $|M^3| = |MMM| = |M||M||M| = |M|^3 = (-10)^3 = -1000$.

6(15 pts).
a.(Source: 4.2.9-10) Solution one: H is a subspace, since it equals the null space of the 1×3 matrix $[\,1 \quad -2 \quad 3\,]$.

Solution two: $\mathbf{0} \in H$, since $0 - 2\cdot 0 + 3\cdot 0 = 0$.
Suppose $\begin{bmatrix} a \\ b \\ c \end{bmatrix}$ and $\begin{bmatrix} a' \\ b' \\ c' \end{bmatrix}$ are in H, meaning $a-2b+3c = 0$ and $a'-2b'+3c' = 0$. Then their sum, $\begin{bmatrix} a+a' \\ b+b' \\ c+c' \end{bmatrix}$ is also in H, since $(a+a')-2(b+b')+3(c+c') = a+a'-2b-2b'+3c+3c' = (a-2b+3c)+(a'-2b'+3c') = 0+0 = 0$.
Finally, if $\begin{bmatrix} a \\ b \\ c \end{bmatrix}$ is in H, meaning $a-2b+3c = 0$, and k is any scalar, the $k\begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} ka \\ kb \\ kc \end{bmatrix}$ is in H, because $ka-2kb+3kc = k(a-2b+3c) = k\cdot 0 = 0$.
Since H containts $\mathbf{0}$ and is closed under vector addition and scalar multiplication, H is a subspace.

b.(Source: 4.1.11-12,17-18, 4.2.13-14) Solution one: J is a subspace since it equals $\left\{ a \begin{bmatrix} 1 \\ 0 \\ 3 \end{bmatrix} + b \begin{bmatrix} 1 \\ -2 \\ 0 \end{bmatrix} \;\middle|\; a, b \text{ real} \right\}$, the span of $\left\{ \begin{bmatrix} 1 \\ 0 \\ 3 \end{bmatrix}, \begin{bmatrix} 1 \\ -2 \\ 0 \end{bmatrix} \right\}$.

Solution two: when $a = b = 0$, the vector $\begin{bmatrix} a+b \\ -2b \\ 3a \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \in J$.

For any two vectors $\begin{bmatrix} a+b \\ -2b \\ 3a \end{bmatrix}$ and $\begin{bmatrix} a'+b' \\ -2b' \\ 3a' \end{bmatrix}$ in J, their sum $\begin{bmatrix} a+b+a'+b' \\ -2b-2b' \\ 3a+3a' \end{bmatrix}$ $= \begin{bmatrix} (a+a')+(b+b') \\ -2(b+b') \\ 3(a+a') \end{bmatrix}$ is in J (since $a + a'$ and $b + b'$ are real numbers).

Finally, if $\begin{bmatrix} a+b \\ -2b \\ 3a \end{bmatrix}$ is any element in J and c is a scalar, then $c \begin{bmatrix} a+b \\ -2b \\ 3a \end{bmatrix} = \begin{bmatrix} c(a+b) \\ c(-2b) \\ c(3a) \end{bmatrix} =$ $\begin{bmatrix} ca+cb \\ -2cb \\ 3ca \end{bmatrix}$ is in J (since ca and cb are real).

Since J containts $\mathbf{0}$ and is closed under vector addition and scalar multiplication, J is a subspace.

7(15 pts).

a.(Source: 4.2.43) A transformation $T : V \to W$ is linear if, for all $\mathbf{u}$ and $\mathbf{v}$ in V and all $c \in \mathbb{R}$,

$$T(\mathbf{u}+\mathbf{v}) = T(\mathbf{u}) + T(\mathbf{v})$$
$$T(c\mathbf{u}) = c\,T(\mathbf{u})$$

T is this case is linear, since for any two polynomials $\mathbf{p}$ and $\mathbf{q}$ and for any scalar c,

$$T(\mathbf{p}+\mathbf{q}) = (\mathbf{p}+\mathbf{q})(1) = \mathbf{p}(1) + \mathbf{q}(1) = T(\mathbf{p}) + T(\mathbf{q})$$
$$T(c\mathbf{p}) = (c\mathbf{p})(1) = c(\mathbf{p}(1)) = c\,T(\mathbf{p})$$

b.(Source: 4.3.44) Three vectors $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$ are linearly independent if the only scalars a, b, c for which $a\mathbf{u} + b\mathbf{v} + c\mathbf{w} = \mathbf{0}$ are $a = b = c = 0$.

In this case, the polynomials $\{2t, 1-t, 1+t\}$ are not linearly independent, since

$$2t + (1-t) - (1+t) = 0$$

for all t; that is, $1 \cdot 2t + 1 \cdot (1-t) + (-1)(1+t)$ equals the zero function $\mathbf{0}$.