1. Suppose \( A = \begin{bmatrix} 1 & 2 \\ -3 & 1 \end{bmatrix}, \ B = \begin{bmatrix} 1 & 2 & 3 \\ -1 & 2 & 0 \end{bmatrix}, \ \text{and} \ C = \begin{bmatrix} 2 & 4 \\ 1 & -3 \\ 3 & -5 \end{bmatrix}. \)

Calculate each of the following, if it exists.

a. \( AB \)

b. \( BA \)

c. \( AC \)

d. \( C^T + B \)

2. Suppose \( E \) is a square matrix whose first and third columns are the same. Could \( E \) be invertible? Why or why not?

3. Suppose \( G^{-1} = \begin{bmatrix} 1 & 2 \\ -3 & 1 \end{bmatrix} \). Find the solution \( x \) to \( Gx = \begin{bmatrix} 1 \\ -1 \end{bmatrix} \), if it exists.

4. Find the inverse of \( F = \begin{bmatrix} 1 & 0 & 3 \\ 2 & 1 & 6 \\ -1 & 1 & -4 \end{bmatrix} \).

5a. Find the determinant of \( M = \begin{bmatrix} 1 & -1 & 9 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 1 & -1 & 2 \\ -2 & 1 & 0 & 3 \end{bmatrix} \).

5b. What must be the determinant of \( M^3 \)?

6. Choose one of the following sets and prove whether or it is a subspace of \( \mathbb{R}^3 \).

a. \( H = \left\{ \begin{bmatrix} a \\ b \\ c \end{bmatrix} \mid a - 2b + 3c = 0 \right\} \)

b. \( J = \left\{ \begin{bmatrix} a + b \\ -2b \\ 3a \end{bmatrix} \mid a, b \ \text{real} \right\} \)

7. Answer one of the following parts. Clearly indicate which part you’re answering.

a. Let \( T : \mathbb{P} \rightarrow \mathbb{R} : p(t) \mapsto p(1) \). That is, \( T \) is the transformation from \( \mathbb{P} \) to \( \mathbb{R} \) defined by \( T(p) = p(1) \). Determine whether \( T \) is linear and then prove your conclusion. Include in your solution the definition of it means for a transformation from one vector space to another to be linear.

b. Determine whether the polynomials \( \{2t, 1-t, 1+t\} \) are linearly independent and then prove your conclusion. Include in your solution the definition of what it means for three vectors in a vector space to be linearly independent.
1. (16 pts) (Source: 2.1.1-2) To multiply two matrices, the number of columns of the first must equal the number of rows of the second. The product of an $m \times n$ and an $n \times p$ matrix is an $m \times p$ matrix.

a. $AB = \begin{bmatrix} 1 & 2 \\ -3 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 & 3 \\ -1 & 2 & 0 \end{bmatrix} = \begin{bmatrix} 1 \cdot 1 + 2(-1) & 1 \cdot 2 + 2 \cdot 2 & 1 \cdot 3 + 2 \cdot 0 \\ -3 \cdot 1 + 1(-1) & -3 \cdot 2 + 1 \cdot 2 & -3 \cdot 3 + 1 \cdot 0 \end{bmatrix} = \begin{bmatrix} -1 & 6 & 3 \\ -4 & -4 & -9 \end{bmatrix}$.

b. $BA$ does not exist. c. $AC$ does not exist.

d. $C^T + B = \begin{bmatrix} 2 & 1 & 3 \\ 4 & -3 & -5 \end{bmatrix} + \begin{bmatrix} 1 & 2 & 3 \\ -1 & 2 & 0 \end{bmatrix} = \begin{bmatrix} 3 & 3 & 6 \\ 3 & -1 & -5 \end{bmatrix}$.

2. (10 pts) (Source: 2.3.23) Solution one: Since the first and third columns are identical, the columns of $E$ are linearly dependent. By the invertible matrix theorem, $E$ cannot be invertible.

Solution two: The IMT states that for $E$ to be invertible, it must have a pivot in every column. Remember that a “pivot” is by definition the lead entry of a nonzero row in a matrix in row echelon form, and the “lead entry” of a row is its leftmost nonzero entry, if there is one.

If the first column is zero then $E$ has no pivot in column one and can’t be invertible.

If the first column is nonzero, then it can be row-reduced to $e_1$. But the row operations that reduce column 1 to $e_1$ must also reduce column 3 to $e_1$:

$$E \sim \begin{bmatrix} 1 & \# & 1 & \# & \cdots \\ 0 & \# & 0 & \# & \cdots \\ 0 & \# & 0 & \# & \cdots \\ \vdots & \# & 0 & \# & \cdots \\ 0 & \# & 0 & \# & \cdots \end{bmatrix}$$

Consequently, column 3 has no pivot, since its only nonzero entry is not the lead entry of its row. Since $E$ has a column with no pivot, it is not invertible.

3. (6 pts) (Source: 2.2.7-8) The unique solution is

$$x = G^{-1} \begin{bmatrix} 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ -3 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 1 \cdot 1 + 2(-1) \\ -3 \cdot 1 + 1(-1) \end{bmatrix} = \begin{bmatrix} -1 \\ -4 \end{bmatrix}.$$
4 (17 pts). (Source: 2.2.41-42, 3.3.11-16)
Solution one: Augment $F$ with the identity and row-reduce.

<table>
<thead>
<tr>
<th>row operation</th>
<th>result</th>
</tr>
</thead>
<tbody>
<tr>
<td>(beginning matrix)</td>
<td>$\begin{bmatrix} 1 &amp; 0 &amp; 3 &amp; 1 &amp; 0 &amp; 0 \ 2 &amp; 1 &amp; 6 &amp; 0 &amp; 1 &amp; 0 \ -1 &amp; 1 &amp; -4 &amp; 0 &amp; 0 &amp; 1 \end{bmatrix}$</td>
</tr>
<tr>
<td>$r_2 \leftarrow r_2 - 2r_1$</td>
<td>$\begin{bmatrix} 1 &amp; 0 &amp; 3 &amp; 1 &amp; 0 &amp; 0 \ 0 &amp; 1 &amp; 0 &amp; -2 &amp; 1 &amp; 0 \ 0 &amp; 1 &amp; -1 &amp; 1 &amp; 0 &amp; 1 \end{bmatrix}$</td>
</tr>
<tr>
<td>$r_3 \leftarrow r_3 + r_1$</td>
<td>$\begin{bmatrix} 1 &amp; 0 &amp; 3 &amp; 1 &amp; 0 &amp; 0 \ 0 &amp; 1 &amp; 0 &amp; -2 &amp; 1 &amp; 0 \ 0 &amp; 1 &amp; -3 &amp; 1 &amp; -1 &amp; 1 \end{bmatrix}$</td>
</tr>
<tr>
<td>$r_3 \leftarrow r_3 - r_2$</td>
<td>$\begin{bmatrix} 1 &amp; 0 &amp; 0 &amp; 10 &amp; -3 &amp; 3 \ 0 &amp; 1 &amp; 0 &amp; -2 &amp; 1 &amp; 0 \ 0 &amp; 0 &amp; 1 &amp; -3 &amp; 1 &amp; -1 \end{bmatrix}$</td>
</tr>
</tbody>
</table>

Therefore $F^{-1} = \begin{bmatrix} 10 & -3 & 3 \\ -2 & 1 & 0 \\ -3 & 1 & -1 \end{bmatrix}$.

Solution two: The matrix of cofactors of $F$ are

\[
\begin{vmatrix}
1 & 6 \\
1 & -4 \\
0 & 3 \\
1 & -4 \\
0 & 3 \\
1 & 6
\end{vmatrix}
- \begin{vmatrix}
2 & 6 \\
-1 & -4 \\
1 & 3 \\
-1 & -4 \\
1 & 3 \\
2 & 6
\end{vmatrix}
- \begin{vmatrix}
2 & 1 \\
-1 & 1 \\
1 & 0 \\
-1 & 1 \\
1 & 0 \\
2 & 1
\end{vmatrix}
= \begin{vmatrix}
-10 & 2 & 3 \\
3 & -1 & -1 \\
-3 & 0 & 1
\end{vmatrix}.
\]

Use these cofactors to compute $\text{det } F$. For instance, by expanding along the bottom row of $F$, we find $\text{det } F = \begin{vmatrix} -1 & 1 & -4 \end{vmatrix} \begin{vmatrix} -3 & 0 & 1 \end{vmatrix}^T = (-1)(-3) + (1)(0) + (-4)(1) = -1$. Therefore, by Cramer’s rule,

\[
F^{-1} = \frac{1}{\text{det } F} \text{adj } F = \frac{1}{-1} \begin{vmatrix}
-10 & 2 & 3 \\
3 & -1 & -1 \\
-3 & 0 & 1
\end{vmatrix}^T = \begin{vmatrix}
10 & -3 & 3 \\
-2 & 1 & 0 \\
-3 & 1 & -1
\end{vmatrix}.
\]

5a (17 pts). (Source: 3.1.9-10, 3.2.7-9) Solution one: by cofactor expansion along the second row of $M$, and then along the first row of $M_{2,3}$ (the submatrix obtained form $M$ by deleting row 2, column 3)

\[
|M| = -2 \begin{vmatrix}
1 & -1 & 0 \\
0 & 1 & 2 \\
-2 & 1 & 3
\end{vmatrix} = -2 \begin{vmatrix}
1 & 1 & 2 \\
1 & 3 \\
-2 & 3
\end{vmatrix} = -2 (1(1 \cdot 3 - 2 \cdot 1) - (-1)(0 \cdot 3 - 2(-2))) = -2 (1 + 4) = -10
\]
Solution two: by row reduction,

\[
\begin{vmatrix}
1 & -1 & 9 & 0 \\
0 & 0 & 2 & 0 \\
0 & 1 & -1 & 2 \\
-2 & 1 & 0 & 3
\end{vmatrix}
\quad \overset{1}{=} \quad
\begin{vmatrix}
1 & -1 & 9 & 0 \\
0 & 0 & 2 & 0 \\
0 & 1 & -1 & 2 \\
0 & -1 & 18 & 3
\end{vmatrix}
\quad \overset{2}{=} \quad
\begin{vmatrix}
1 & -1 & 9 & 0 \\
0 & 1 & -1 & 2 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 5
\end{vmatrix}
\quad \overset{3}{=} \quad
-2
\begin{vmatrix}
1 & -1 & 9 & 0 \\
0 & 1 & -1 & 2 \\
0 & 0 & 1 & 0 \\
0 & -1 & 18 & 3
\end{vmatrix}
\quad \overset{4}{=} \quad
-2 \cdot 5 = -10
\]

Notes:
1. Row replacements; no change to determinant.
2. Row interchange; determinant changes by a sign.
3. Factor 2 out of row 3.
4. The determinant of a triangular matrix equals the product of its main diagonal.

5b (4 pts) (Source: 3.2.35) Because the determinant of a product is the product of the determinants, \(|M^3| = |MMM| = |M||M||M| = |M|^3 = (-10)^3 = -1000.\

6 (15 pts).
a. (Source: 4.2.9-10) Solution one: \(H\) is a subspace, since it equals the null space of the \(1 \times 3\) matrix \([1 \quad -2 \quad 3]\).

Solution two: \(0 \in H\), since \(0 - 2 \cdot 0 + 3 \cdot 0 = 0\).

Suppose \[
\begin{bmatrix}
a \\
b \\
c
\end{bmatrix}
\quad \text{and} \quad
\begin{bmatrix}
a' \\
b' \\
c'
\end{bmatrix}
\]
are in \(H\), meaning \(a - 2b + 3c = 0\) and \(a' - 2b' + 3c' = 0\). Then their sum, \[
\begin{bmatrix}
 a + a' \\
 b + b' \\
 c + c'
\end{bmatrix}
\]
is also in \(H\), since \((a + a') - 2(b + b') + 3(c + c') = a + a' - 2b - 2b' + 3c + 3c' = (a - 2b + 3c) + (a' - 2b' + 3c') = 0 + 0 = 0.\)

Finally, if \[
\begin{bmatrix}
a \\
b \\
c
\end{bmatrix}
\]
is in \(H\), meaning \(a - 2b + 3c = 0\), and \(k\) is any scalar, the \[
\begin{bmatrix}
 a \\
 b \\
 c
\end{bmatrix}
= \begin{bmatrix}
 ka \\
 kb \\
 kc
\end{bmatrix}
\]
is in \(H\), because \(ka - 2kb + 3kc = k(a - 2b + 3c) = k \cdot 0 = 0\). Since \(H\) contains \(0\) and is closed under vector addition and scalar multiplication, \(H\) is a subspace.
b. (Source: 4.1.11-12, 17-18, 4.2.13-14) Solution one: $J$ is a subspace since it equals
\[
\begin{bmatrix}
a & 1 \\
0 & -2 \\
3 & 0
\end{bmatrix} + b \begin{bmatrix}
a & 1 \\
0 & -2 \\
3 & 0
\end{bmatrix}, \text{ the span of } \begin{bmatrix}
a & 1 \\
0 & -2 \\
3 & 0
\end{bmatrix}.
\]
Solution two: when $a = b = 0$, the vector \[
\begin{bmatrix}
a + b \\
-2b \\
3a
\end{bmatrix} = \begin{bmatrix} 0 \\
0 \\
0
\end{bmatrix} \in J.
\]
For any two vectors \[
\begin{bmatrix}
a + a' \\
-2b \\
3a
\end{bmatrix} \text{ and } \begin{bmatrix}
a' + b' \\
-2b' \\
3a'
\end{bmatrix} \text{ in } J, \text{ their sum } \begin{bmatrix}
a + b + a' + b' \\
-2b - 2b' \\
3a + 3a'
\end{bmatrix}
\]
is in $J$ (since $a + a'$ and $b + b'$ are real numbers).
Finally, if \[
\begin{bmatrix}
a + b \\
-2b \\
3a
\end{bmatrix} \text{ is any element in } J \text{ and } c \text{ is a scalar, then } c \begin{bmatrix}
a + b \\
-2b \\
3a
\end{bmatrix} = \begin{bmatrix} c(a + b) \\
c(-2b) \\
c(3a)
\end{bmatrix} = \begin{bmatrix}
ca + cb \\
-2cb \\
3ca
\end{bmatrix} \text{ is in } J \text{ (since } ca \text{ and } cb \text{ are real).}
\]
Since $J$ contains 0 and is closed under vector addition and scalar multiplication, $J$ is a subspace.

7 (15 pts).

a. (Source: 4.2.43) A transformation $T : V \to W$ is linear if, for all $u$ and $v$ in $V$ and all $c \in \mathbb{R}$,
\[
T(u + v) = T(u) + T(v)
\]
\[
T(cu) = cT(u)
\]
$T$ is this case is linear, since for any two polynomials $p$ and $q$ and for any scalar $c$,
\[
T(p + q) = (p + q)(1) = p(1) + q(1) = T(p) + T(q)
\]
\[
T(cp) = (cp)(1) = c(p(1)) = cT(p)
\]

b. (Source: 4.3.44) Three vectors $u$, $v$, and $w$ are linearly independent if the only scalars $a$, $b$, $c$ for which $au + bv + cw = 0$ are $a = b = c = 0$.
In this case, the polynomials \{2$t$, 1 $- t$, 1 $+ t$\} are not linearly independent, since
\[
2t + (1 - t) - (1 + t) = 0
\]
for all $t$; that is, $1 \cdot 2t + 1 \cdot (1 - t) + (-1)(1 + t)$ equals the zero function 0.