No notes, books, electronic devices, or outside materials of any kind.

Read each problem carefully and simplify your answers.

Unless otherwise indicated, supporting work will be required on every problem worth more than 3 points.

You are expected to know the values of all trig functions at multiples of $\pi/4$ and of $\pi/6$.

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1a(6 pts). Fill in the blanks to complete this statement of the Mean Value Theorem (MVT).

If the function $f(x)$ is ______________________________________________ ,

then there is a number $c$ in ______________

with the property that ______________________________________________ .

1b(7 pts). Find all $c$ as promised by the MVT for $f(x) = x^3 - 2x$ on the interval $[-1, 1]$.

2(9 pts). Find the linearization $L(x)$ of the function $\cos x$ at $a = \frac{\pi}{6}$.

3(10 pts). Sketch the graph of a function $h(x)$ satisfying all of the following.

- $h(x)$ is defined for all $x$ other than 0.
- $h(x)$ is continuous on $(-\infty, 0)$ and on $(0, \infty)$.
- $x = 0$ is an asymptote of the graph of $h(x)$
- $h'(x) > 0$ on $(-1, 0)$
- $h'(x) < 0$ on $(-\infty, -1)$ and on $(0, \infty)$
- $h''(x) > 0$ on $(-3, 0)$ and on $(0, \infty)$
- $h''(x) < 0$ on $(-\infty, -3)$

4(10 pts). Evaluate the limit: $\displaystyle\lim_{x\to 1} \frac{x - 1 - \ln x}{(x-1)^2}$

5(14 pts). Find the absolute maximum and minimum of $x^2 + \frac{16}{x}$ on the interval $[1, 10]$.

6(18 pts). A jet flying horizontally at altitude 1 km with constant speed 200 km/hr passes directly over an observer on the ground. How fast is the angle of elevation from the observer to the jet decreasing when the two are 2 km apart?
("angle of elevation" is the angle between the line from observer to jet and the horizontal.)

7(20 pts). The function $s(x) = (6 - x)\sqrt{x}$ has for its domain the interval $[0, \infty)$. Find the following, if they exist.

a. The interval(s) on which $s(x)$ is increasing.

b. The interval(s) on which $s(x)$ is concave up.

c. The $x$-value(s) at which $s(x)$ has a local maximum.

d. The $x$-value(s) at which $s(x)$ has a local minimum.

e. The $x$-value(s) at which $s(x)$ has an inflection point.

8(6 pts). A particle traveling on an axis is at position $s(t) = (6 - t)\sqrt{t}$ for all $t \geq 0$.
Answer the following. You are not required to repeat work you already did in problem 7.

a. On what interval(s) of time $t$ is the particle moving in the positive direction?

b. Find the total distance traveled by the particle from time $t = 0$ to $t = 6$.

1a(6 pts).(Source: Students were told in class to prepare for this question from 4.2.) If $f(x)$ is continuous on $[a, b]$ and differentiable on $(a, b)$, then there is a number $c$ in $(a, b)$ with the property that

$$f'(c) = \frac{f(b) - f(a)}{b - a}$$

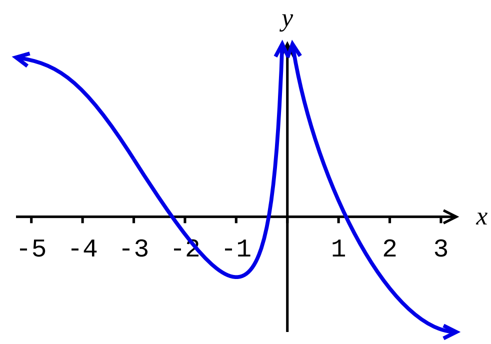
1b(7 pts).(Source: 4.2.12) Set $f'(c) = 3c^2 - 2$ equal to $\frac{f(1)-f(-1)}{1-(-1)} = \frac{-1-1}{2} = -1$ and solve:

$$3c^2 - 2 = -1 \implies 3c^2 = 1 \implies c^2 = \frac{1}{3} \implies c = \pm\sqrt{\frac{1}{3}}$$

2(9 pts).(Source: 3.10.1-4) Generally, $L(x) = f(a) = f'(a)(x - a)$. Since $(\cos x)' = \sin x$,

$$L(x) = \cos(\pi/6) - \sin(\pi/6)\left(x - \frac{\pi}{6}\right) = \frac{\sqrt{3}}{2} - \frac{1}{2}\left(x - \frac{\pi}{6}\right)$$

3(10 pts).(Source: 4.3.24-31) Here is one possible graph. In a correct solution, $h(x)$ is decreasing on $(-\infty, -1)$ and on $(0, \infty)$, and increasing on $(-1, 0)$. The graph is concave up on $(-3, 0)$ and on $(0, \infty)$, and concave down on $(-\infty, -3)$.




4(10 pts).(Source: 4.4.27,28) Either apply l'Hospital's Rule twice:

$$\frac{x - 1 - \ln x}{(x-1)^2} = \text{“}\frac{0}{0}\text{”} \overset{HR}{\hookrightarrow} \frac{1 - x^{-1}}{2(x-1)} = \text{“}\frac{0}{0}\text{”} \overset{HR}{\hookrightarrow} \frac{x^{-2}}{2} \longrightarrow \frac{1}{2}$$

Or, use l'Hôpital's Rule once and simplify the result by multiplying top and bottom by $x$:

$$\frac{x - 1 - \ln x}{(x-1)^2} = \text{“}\frac{0}{0}\text{”} \overset{HR}{\hookrightarrow} \frac{1 - x^{-1}}{2(x-1)} \cdot \frac{x}{x} = \frac{x - 1}{2x(x-1)} = \frac{1}{2x} \longrightarrow \frac{1}{2}$$

Either way, l'Hospital's Rule tells us that the original limit also equals $\frac{1}{2}$.

5(14 pts).(Source: 4.1.53) The absolute extrema of $k(x) = x^2 + 16x^{-1}$ on $[1, 10]$ can occur only at the endpoints or critical points inside $(1, 10)$.
$k'(x) = 2x - 16x^{-2} = 2x - \frac{16}{x^2}$ is defined everywhere on $(1, 10)$, so the only critical points are where $k'(x) = 0$:

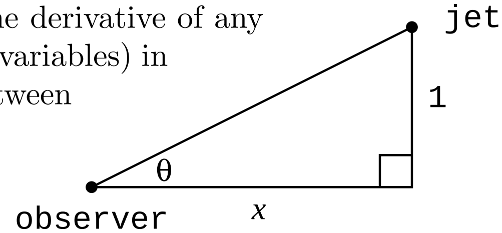
$$2x - \frac{16}{x^2} = 0 \implies 2x = \frac{16}{x^2} \implies x^3 = 8 \implies x = 2$$

Now compare values of $k(x)$ at the endpoints and the critical point:

| $x$ | 1 | 2 | 10 |
|---|---|---|---|
| $x^2+\frac{16}{x}$ | $1+16=17$ | $4+\frac{16}{4}=8$ | $100+\frac{16}{10}=101.6$ |

On $[1,10]$, the absolute maximum of $k$ is 101.6 and its absolute minimum is 8.

6(18 pts).(Source: 3.9.43,45) See figure at right. The question asks for $\frac{d\theta}{dt}$, given that $\frac{dx}{dt}=200$. It does *not* ask for or give information about the derivative of any other function. Relate $x$ and $\theta$ (and, importantly, no other variables) in a trig equation, then differentiate to get a relationship between their derivatives:




$$\frac{x}{1}=\cot\theta \quad\Longrightarrow\quad \frac{dx}{dt}=-\csc^2\theta\frac{d\theta}{dt}$$

At the moment in question, the hypotenuse is 2 (which means $\theta$ must be $\pi/6$) and $\csc\theta=\frac{1}{\sin\theta}=\frac{1}{1/2}=2$. Now solve:

$$200=-2^2\cdot\frac{d\theta}{dt} \quad\Longrightarrow\quad -50=\frac{d\theta}{dt}.$$

7(20 pts).(Source: 4.3.43-45) $s(x)=6x^{1/2}-x^{3/2}=6\sqrt{x}-(\sqrt{x})^3$.

a. $s'(x)=3x^{-1/2}-\frac{3}{2}x^{1/2}=\frac{3}{2}x^{-1/2}(2-x)$. Here's a sign chart for $s'$:

$$\begin{array}{rl}
\frac{3}{2}x^{-1/2}=\frac{3}{2\sqrt{x}}: & \text{DNE}++++++++++++++++ \\
2-x: & \quad+++++0----------- \\
\frac{3}{2}x^{-1/2}(2-x): & \text{DNE}++++0----------- \\
\hline
x: & \quad 0 \qquad\qquad\quad 2
\end{array}$$

$s(x)$ increases on $[0,2]$.

b. $s''=-\frac{3}{2}x^{-3/2}-\frac{3}{4}x^{-1/2}=-\frac{3}{2}\frac{1}{(\sqrt{x})^3}-\frac{3}{4}\frac{1}{\sqrt{x}}<0$ for all $x>0$. The graph of $s$ is never concave up.

c. By the first derivative test, $s$ has a local maximum at $x=2$, and ...

d. ... no local minimum.

e. Since its graph never changes concavity, the graph of $s$ has no inflection points.

8.(Source: 3.7.1-4)

a(2 pts). The particle is moving forward when $s(t)$ is increasing. In problem 7, we saw that this is when $0\le t\le 2$.

b(4 pts). At any time, we can calculate the particle's position using $s(t)=(6-t)\sqrt{t}$. From time $t=0$ to $t=2$, the particle moves forward from position $s(0)=0$ to $s(2)=4\sqrt{2}$, and from $t=2$ to $t=6$, it moves backward from position $s(2)=4\sqrt{2}$ to $s(6)=0$. That's $4\sqrt{2}$ units forward and $4\sqrt{2}$ units backward for a total of $8\sqrt{2}$ units.