No notes, books, electronic devices, or outside materials of any kind.

Read each problem carefully and simplify your answers.

Unless otherwise indicated, supporting work will be required on every problem worth more than 2 points.

You are expected to know the values of all trig functions at multiples of $\pi/4$ and of $\pi/6$.

You will not receive credit for using l’Hospital’s Rule (a technique learned later in a calculus course) on any problem on this exam.

When asked to find $f'(x)$, you are expected to do so using the definition of derivative as a limit, and not the shortcut methods learned later in a calculus course.

1a (11 pts) State the precise, $\varepsilon$-$\delta$ definition of what it means for $\lim_{x \to a} f(x)$ to equal $L$.

1b (10 pts) Write an $\varepsilon$-$\delta$ proof of the fact that $\lim_{x \to -2} \frac{2x + 1}{x + 2} = -4$.

2 (37 pts) Evaluate the limit or briefly explain why it does not exist.

   a. $\lim_{x \to -2^-} \frac{2x + 1}{x + 2}$
   b. $\lim_{x \to -2^+} \frac{2x + 1}{x + 2}$
   c. $\lim_{x \to -3} \frac{2x^2 + 7x + 3}{x^2 + 5x + 6}$
   d. $\lim_{x \to \infty} \frac{2x^2 + 7x + 3}{x^2 + 5x + 6}$
   e. $\lim_{x \to \infty} \left(\sqrt{x + 3} - \sqrt{x - 2}\right)$
   f. $\lim_{x \to \infty} \left[\ln(2x + 1) - \ln(x + 2)\right]$

2g (4 pts) Use (some of) your answers to 2a-f to find equations of all asymptotes to $y = \frac{2x^2 + 7x + 3}{x^2 + 5x + 6}$.

3 (12 pts) On the axes provided, sketch the graph of a function $\xi(x)$ for which all of the following conditions are true.

   $\lim_{x \to -1^+} \xi(x) = -\infty$
   $\lim_{x \to -1^-} \xi(x) = \infty$
   $\lim_{x \to 0} \xi(x) = \infty$
   $\lim_{x \to 1^-} \xi(x) = 2$

   $\lim_{x \to 1^+} \xi(x)$ does not exist.

   $\xi(x)$ is continuous from the left at $x = 1$.

   $\lim_{x \to 2} \xi(x) = 4$
   $\xi(2) = 3$
   $\lim_{x \to -\infty} \xi(x) = -\infty$
   $\lim_{x \to \infty} \xi(x) = -1$

4 (6 pts) On the axes provided, sketch the graph of a continuous function $\ell(x)$ for which

   $\ell(-2) = 0$
   $\ell'(x) = 1$
   $\ell(1) = 0$
   $\ell'(1) = -1$

5a (16 pts) Use the definition of the derivative to find $f'(a)$ if $f(x) = x^3 + 3x$.

You can use shortcut methods from Chapter 3 to check your work, but to receive credit on this problem, you must calculate the derivative from its definition as a limit.

5b (4 pts) Suppose that the distance traveled by an object moving in a straight line is $t^3 + 3t$ meters at time $t$ seconds. Use your answer to 5a to find the object’s velocity at time $t = 2$. 
1a(11 pts). (Source: 2.4. Definition 2. Students were told in class to be prepared to answer this question from 2.4.) 
\[ \lim_{x \to a} f(x) = L \] means that, for any positive number \( \varepsilon \), there’s a corresponding positive number \( \delta \) so that \( |f(x) - L| < \varepsilon \) whenever \( 0 < |x - a| < \delta \).

In this definition and the proof below, wording matters. Small changes in wording can significantly alter the meaning of your writing.

1b(10 pts). (Source: 2.4.17) Here’s the thinking I did before writing my proof:

\[ |2 + 3x - (-4)| = |6 + 3x| = |3||x + 2| = 3|x + 2|. \]

To achieve \( 3|x + 2| < \varepsilon \), just make sure that \( |x + 2| < \frac{1}{3} \varepsilon = \delta \).

**Proof:** Suppose that \( \varepsilon > 0 \). Choose \( \delta = \frac{1}{3} \varepsilon \). Then

\[ |2 + 3x - (-4)| = |6 + 3x| = |3||x + 2| = 3|x + 2| < 3\delta = \varepsilon \]

whenever \( 0 < |x + 2| < \delta \), as desired.

2a,b(7 pts). (Source: 2.2.31,32) As \( x \to -2 \), \( \frac{2x+1}{x+2} \to \frac{-3}{0} \), which indicates that \( \frac{2x+1}{x+2} \) is blowing up to one of \( \pm \infty \). Look at the signs to decide. The numerator \( 2x + 1 \to -3 \), so \( 2x + 1 < 0 \) (for \( x \) sufficiently close to \(-2\)). The denominator \( x + 2 \) is positive if \( x > -2 \) and negative if \( x < -2 \). Therefore

\[ \frac{2x + 1}{x + 2} = \frac{-}{+} = + \text{ if } x < -2 \quad \frac{2x + 1}{x + 2} = \frac{-}{-} = - \text{ if } x > -2 \]

and the limit is \( \infty \) as \( x \to -2^- \) (a) and the limit is \(-\infty \) as \( x \to -2^+ \) (b).

2c(6 pts). (Source: 2.2.15,16) \( \frac{0}{0} \) at \( x = -3 \) indicates that the polynomials in the numerator and denominator have a common factor causing the zero. Find and cancel, and then take the limit:

\[ \lim_{x \to -3} \frac{2x^2 + 7x + 3}{x^2 + 5x + 6} = \lim_{x \to -3} \frac{(2x + 1)(x + 3)}{(x + 2)(x + 3)} = \lim_{x \to -3} \frac{2x + 1}{x + 2} = \frac{-5}{-1} = 5. \]

2d(5 pts). (Source: 2.6.18) The limit of this rational function as \( x \to \infty \) is the same as the limit of the ratio of its lead terms:

\[ \lim_{x \to \infty} \frac{2x^2 + 7x + 3}{x^2 + 5x + 6} = \lim_{x \to \infty} \frac{2x^2}{x^2} = \lim_{x \to \infty} 2 = 2. \]

2e(10 pts). (Source: 2.2.27) When we try to take the limit as it is written we get the inconclusive answer \( \infty - \infty \). Rewrite by rationalizing the numerator:

\[ \frac{(\sqrt{x + 3} - \sqrt{x - 2})}{1} \frac{(\sqrt{x + 3} + \sqrt{x - 2})}{(\sqrt{x + 3} + \sqrt{x - 2})} = \frac{(x + 3) - (x - 2)}{\sqrt{x + 3} + \sqrt{x - 2}} = \frac{5}{\sqrt{x + 3} + \sqrt{x - 2}} \]

The limit of this as \( x \to \infty \) looks like \( \frac{5}{\infty + \infty} \), which tells us that the limit equals 0.
2f (9 pts). (Source: 2.6.42) This limit also starts of as \( \infty - \infty \). Use a property of logs to rewrite as

\[
\lim_{x \to \infty} [\ln(2x + 1) - \ln(x + 2)] = \lim_{x \to \infty} \ln \left( \frac{2x + 1}{x + 2} \right)
\]

Since \( \ln \) is a continuous function, this equals \( \ln \left( \lim_{x \to \infty} \frac{2x + 1}{x + 2} \right) \). Now calculate the limit as in 2d:

\[
\ln \left( \lim_{x \to \infty} \frac{2x}{x} \right) = \ln \left( \lim_{x \to \infty} 2 \right) = \ln 2.
\]

2g (4 pts). (Source: 2.6.49) By 2a and b, \( x = -2 \) is a vertical asymptote, since there the one-sided limits are infinite. (By 2c, \( x = -3 \) is not a VA.) 2d tells us that \( y = 2 \) is a horizontal asymptote. (In fact, the limit of \( y \) as \( x \to -\infty \) is also 2, so \( y = 2 \) is a HA at both the left and right ends of the graph.)

3 (12 pts). (Source: 2.3.15-18; 2.5.5,6; 2.6.5,7) See below for one possible graph of \( \xi(x) \). Make sure your function doesn’t violate the vertical line test (VLT).

4 (6 pts). (Source: 2.7.23) The graph of \( \ell(x) \) must pass through the point \((-2, 0)\) with slope 1 and through the point \((1, 0)\) with slope -1. See above for one possible graph.

5a (16 pts). (Source: 2.7.more.1e)

\[
f'(a) = \lim_{h \to 0} \frac{f(a + h) - f(a)}{h} = \lim_{h \to 0} \frac{(a + h)^3 + 3(a + h) - (a^3 + 3a)}{h}
\]

Expand the binomial \((a + h)^3\). To read how to use Pascal’s triangle for this, go to https://kunklet.people.cofc.edu/MATH120/120review.pdf.

\[
\lim_{h \to 0} \frac{a^3 + 3a^2h + 3ah^2 + h^3 + 3a + 3h - a^3 - 3a}{h}
\]
After terms cancel in the numerator, everything remaining is a multiple of $h$. Factor out the $h$ and cancel, and then you can let $h \to 0$.

$$\lim_{h \to 0} \frac{3a^2h + 3ah^2 + h^3 + 3h}{h} = \lim_{h \to 0} \frac{h(3a^2 + 3ah + h^2 + 3)}{h}$$

$$= \lim_{h \to 0} \frac{3a^2 + 3ah + h^2 + 3}{1} = 3a^2 + 3 = f'(a)$$

5b(4 pts).(Source: 2.7.15) The object’s velocity at time $t = 2$ is $f'(2) = 3 \cdot 2^2 + 3 = 15$ (m/sec).