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Complex Arithmetic

Math 111, College of Charleston

A **complex number** is an expression of the form

$$x + iy$$

where x and y are real numbers and i is the "imaginary" square root of -1. For example, $2+3i$ is a complex number. Just as we use the symbol $\mathbb{R}$ to stand for the set of real numbers, we use $\mathbb{C}$ to denote the set of all complex numbers. Any real number x is also a complex number, $x + 0i$; in set notation, $\mathbb{R} \subset \mathbb{C}$.

Assume for this paragraph that

$$z = x + iy.$$

Then x is called the **real part** of z and y is called the **imaginary part** of z. This is written

$$x = \mathrm{Re}(z) \qquad \text{and} \qquad y = \mathrm{Im}(z).$$

Two complex numbers are equal if and only if their real parts are equal and their imaginary parts are equal. The **conjugate** of z is the complex number

$$\overline{z} = x - iy$$

and the **absolute value** of z is

$$|z| = \sqrt{x^2 + y^2}.$$

Note that when $y = 0$, this is the same as the absolute value formula for real numbers x. Note also that, since $(x + iy)(x - iy) = x^2 - i^2y^2 = x^2 + y^2$,

$$z\overline{z} = |z|^2.$$

You can add, subtract, multiply, and divide complex numbers using the usual rules of algebra, keeping in mind that $i^2 = -1$.

Example 1: *Write the sum in $x + iy$ form:*

$$(2 + 3i) + (4 - i) = 6 + 2i$$

end Example 1

Example 2: *Write the sum in $x + iy$ form:*

$$(1 + 5i) + \overline{(2 - 3i)} = 1 + 5i + 2 + 3i = 3 + 8i$$

end Example 2

Example 3: *Write the number in $x + iy$ form:*

$$2(1 - 7i) - \frac{1}{2}(5 + 6i) = 2 - 14i - \frac{5}{2} - 3i = \frac{-1}{2} - 17i.$$

end Example 3

Example 4: *Find the product. Write your answer in $x + iy$ form:*

$$\begin{aligned}(4 - 7i)(2 + 3i) &= 8 - 14i + 12i - 21i^2 \\ &= 8 - (-1)21 - 2i \\ &= 29 - 2i\end{aligned}$$

end Example 4

Writing a quotient in $x + iy$ form requires the use of the conjugate, as the next example demonstrates.

Example 5: *Find the quotient. Write your answer in $x + iy$ form:*

$$\frac{4-7i}{2-3i} = \left(\frac{4-7i}{2-3i}\right)\left(\frac{2+3i}{2+3i}\right) = \frac{29-2i}{4+9} = \frac{29}{13} - \frac{2}{13}i$$

end Example 5

The interesting thing about the act of conjugation is that it commutes with the arithmetic operations $+ - \times \div$:

$$\begin{aligned}
\overline{u+v} &= \overline{u} + \overline{v} \\
\overline{u-v} &= \overline{u} - \overline{v} \\
\overline{u \times v} &= \overline{u} \times \overline{v} \\
\overline{u \div v} &= \overline{u} \div \overline{v}
\end{aligned}$$

Complex number arise naturally in the study of the solutions of polynomial equations. If we evaluate a polynomial p with real coefficients at a complex number z,

$$\overline{p(z)} = p(\overline{z}).$$

Consequently, if z is a zero of the p, so $\overline{z}$ must also be a zero of p. This is the basis for the following fact.

Lemma. *The complex zeros of a polynomial with real coefficients come in conjugate pairs.*

Example 6: *Expand the polynomial:*

$$\begin{aligned}
(x-2+i)(x-2-i) &= x^2 - 2x - ix \\
&\qquad - 2x + 4 + 2i \\
&\qquad\qquad + ix - 2i - i^2 \\
&= x^2 - 4x + 4 - i^2 = x^2 - 4x + 4 + 1 \\
&= x^2 - 4x + 5
\end{aligned}$$

Note that, despite its having real coefficients, the polynomial equation $x^2 - 4x + 5 = 0$ has only non-real solutions $x = 2 \pm i$, and these solutions are conjugates of one another, as promised by the Lemma above.

end Example 6

By the way, there's an easier solution to the above example using the difference of squares:

Example 7: *Expand the polynomial:*

$$\begin{aligned}
\big(x-3+4i\big)\big(x-3-4i\big) &= \big((x-3)+4i\big)\big((x-3)-4i\big) \\
&= (x-3)^2 - (4i)^2 \\
&= x^2 - 6x + 9 - 16i^2 = x^2 - 6x + 9 + 16 \\
&= x^2 - 6x + 25
\end{aligned}$$

end Example 7

Exercises

1. Write in $x + iy$ form:

a. $3 + 2i + 2(1 - i)$

b. $3(4 - 5i) - \overline{(2 + 4i)}$

c. $-2(2 - i) + \frac{1}{3}(1 + 4i)$

d. $\frac{2}{3}(1 + 8i) + \frac{3}{2}(2 - 7i)$

e. $\frac{1}{5}(7 - 4i) - \frac{2}{3}\overline{(6 - 5i)}$

f. $(i + 1)(i - 1)$

g. $(2 - 3i)(2 + 3i)$

h. $(4 - i)\overline{(5 + 2i)}$

i. $(3 + \frac{1}{2}i)(\frac{1}{2} - \frac{1}{3}i)$

j. $(4 + i) \div (1 - 8i)$

k. $\overline{(3 - 2i)} \div 2(1 - i)$

l. $(1 + 2i) \div (1 - 2i)$

m. $(3 + 4i) \div (5 + 6i)$

2. Expand the polynomial:

a. $(x + 2i)(x - 2i)$

b. $(x - 3i)(x + 3i)$

c. $(x + i\sqrt{5})(x - i\sqrt{5})$

d. $(x - 2 + 4i)(x - 2 - 4i)$

e. $(x - 3 + i)(x - 3 - i)$

f. $(x + 1 - 2i)(x + 1 + 2i)$

g. $(x + \frac{1}{2} - i)(x + \frac{1}{2} + i)$

h. $(2x + 1 - i\sqrt{3})(2x + 1 + i\sqrt{3})$

i. $(x + 3 + i\sqrt{5})(x + 3 - i\sqrt{5})$

Answers

1a. 5 1b. $10 - 11i$ 1c. $\frac{-11}{3} + \frac{10}{3}i$ 1d. $\frac{11}{3} - \frac{31}{6}i$ 1e. $\frac{-13}{5} - \frac{62}{15}i$ 1f. -2 1g. 13 1h. $18 - 13i$ 1i. $\frac{5}{3} - \frac{3}{4}i$ 1j. $\frac{-4+33i}{65}$ 1k. $\frac{1}{4} + \frac{5}{4}i$ 1l. $\frac{-3}{5} + \frac{4}{5}i$ 1m. $\frac{39+2i}{61}$ 2a. $x^2 + 4$ 2b. $x^2 + 9$ 2c. $x^2 + 5$ 2d. $x^2 - 4x + 20$ 2e. $x^2 - 6x + 10$ 2f. $x^2 + 2x + 5$ 2g. $x^2 + x + \frac{5}{4}$ 2h. $4x^2 + 4x + 4$ 2i. $x^2 + 6x + 14$