A complex number is an expression of the form

\[ x + iy \]

where \( x \) and \( y \) are real numbers and \( i \) is the “imaginary” square root of \(-1\). For example, \( 2 + 3i \) is a complex number. Just as we use the symbol \( \mathbb{R} \) to stand for the set of real numbers, we use \( \mathbb{C} \) to denote the set of all complex numbers. Any real number \( x \) is also a complex number, \( x + 0i \); in set notation, \( \mathbb{R} \subset \mathbb{C} \).

Assume for this paragraph that \( z = x + iy \).

Then \( x \) is called the real part of \( z \) and \( y \) is called the imaginary part of \( z \). This is written

\[ x = \text{Re}(z) \quad \text{and} \quad y = \text{Im}(z). \]

Two complex numbers are equal if and only if their real parts are equal and their imaginary parts are equal. The conjugate of \( z \) is the complex number

\[ \overline{z} = x - iy \]

and the absolute value of \( z \) is

\[ |z| = \sqrt{x^2 + y^2}. \]

Note that when \( y = 0 \), this is the same as the absolute value formula for real numbers \( x \). Note also that, since \((x + iy)(x - iy) = x^2 - i^2y^2 = x^2 + y^2\),

\[ z\overline{z} = |z|^2. \]

You can add, subtract, multiply, and divide complex numbers using the usual rules of algebra, keeping in mind that \( i^2 = -1 \).

**Example 1:** Write the sum in \( x + iy \) form:

\[ (2 + 3i) + (4 - i) = 6 + 2i \]

**end Example 1**

**Example 2:** Write the sum in \( x + iy \) form:

\[ (1 + 5i) + (2 - 3i) = 1 + 5i + 2 + 3i = 3 + 8i \]

**end Example 2**

**Example 3:** Write the number in \( x + iy \) form:

\[ 2(1 - 7i) - \frac{1}{2}(5 + 6i) = 2 - 14i - \frac{5}{2} - 3i = \frac{-1}{2} - 17i. \]

**end Example 3**

**Example 4:** Find the product. Write your answer in \( x + iy \) form:

\[ (4 - 7i)(2 + 3i) = 8 - 14i + 12i - 21i^2 \]
\[ = 8 - (-1)21 - 2i \]
\[ = 29 - 2i \]

**end Example 4**
Writing a quotient in \( x + iy \) form requires the use of the conjugate, as the next example demonstrates.

**Example 5:** Find the quotient. Write your answer in \( x + iy \) form:

\[
\frac{4 - 7i}{2 - 3i} = \left( \frac{4 - 7i}{2 - 3i} \right) \left( \frac{2 + 3i}{2 + 3i} \right) = \frac{29 - 2i}{13} = 2 - \frac{2}{13}i
\]

*end Example 5*

The interesting thing about the act of conjugation is that it commutes with the arithmetic operations \(+, -, \times, \div\):

\[
\begin{align*}
\bar{u + v} &= \bar{u} + \bar{v} \\
\bar{u - v} &= \bar{u} - \bar{v} \\
\bar{u \times v} &= \bar{u} \times \bar{v} \\
\bar{u \div v} &= \bar{u} \div \bar{v}
\end{align*}
\]

Complex number arise naturally in the study of the solutions of polynomial equations. If we evaluate a polynomial \( p \) with real coefficients at a complex number \( z \),

\[\bar{p}(z) = p(\bar{z}).\]

Consequently, if \( z \) is a zero of the \( p \), so \( \bar{z} \) must also be a zero of \( p \). This is the basis for the following fact.

**Lemma.** The complex zeros of a polynomial with real coefficients come in conjugate pairs.

**Example 6:** Expand the polynomial:

\[
(x - 2 + i)(x - 2 - i) = x^2 - 2x - ix - 2x + 4 + 2i + ix - 2i - i^2 = x^2 - 4x + 4 + 1 = x^2 - 4x + 5
\]

Note that, despite its having real coefficients, the polynomial equation \( x^2 - 4x + 5 = 0 \) has only non-real solutions \( x = 2 \pm i \), and these solutions are conjugates of one another, as promised by the Lemma above.

*end Example 6*

By the way, there’s an easier solution to the above example using the difference of squares:

**Example 7:** Expand the polynomial:

\[
(x - 3 + 4i)(x - 3 - 4i) = (x - 3 + 4i)(x - 3 - 4i) \\
= (x - 3)^2 - (4i)^2 \\
= x^2 - 6x + 9 - 16i^2 = x^2 - 6x + 9 + 16 = x^2 - 6x + 25
\]

*end Example 7*
Exercises

1. Write in $x + iy$ form:

   a. $3 + 2i + 2(1 - i)$
   b. $3(4 - 5i) - (2 + 4i)$
   c. $-2(2 - i) + \frac{1}{3}(1 + 4i)$
   d. $\frac{2}{3}(1 + 8i) + \frac{3}{2}(2 - 7i)$
   e. $\frac{1}{5}(7 - 4i) - \frac{2}{5}(6 - 5i)$
   f. $(i + 1)(i - 1)$
   g. $(2 - 3i)(2 + 3i)$
   h. $(4 - i)(5 + 2i)$
   i. $(3 + \frac{1}{2}i)(\frac{3}{2} - \frac{1}{3}i)$
   j. $(4 + i) \div (1 - 8i)$
   k. $(3 - 2i) \div 2(1 - i)$
   l. $(1 + 2i) \div (1 - 2i)$
   m. $(3 + 4i) \div (5 + 6i)$

2. Expand the polynomial:

   a. $(x + 2i)(x - 2i)$
   b. $(x - 3i)(x + 3i)$
   c. $(x + i\sqrt{5})(x - i\sqrt{5})$
   d. $(x - 2 + 4i)(x - 2 - 4i)$
   e. $(x - 3 + i)(x - 3 - i)$
   f. $(x + 1 - 2i)(x + 1 + 2i)$
   g. $(x + \frac{1}{2} - i)(x + \frac{1}{2} + i)$
   h. $(2x + 1 - i\sqrt{3})(2x + 1 + i\sqrt{3})$
   i. $(x + 3 + i\sqrt{5})(x + 3 - i\sqrt{5})$

Answers

1a. 5  1b. 10 - 11i  1c. $\frac{-11 + 10}{2}i$  1d. $\frac{11 - 11i}{3}$  1e. $\frac{-11 - 22i}{3}$  1f. -2  1g. 13  1h. 18 - 13i  1i. $\frac{1}{2} - \frac{3}{2}i$  1j. $\frac{1 + 3i\sqrt{3}}{6}$  1k. $\frac{1}{2} + \frac{5}{2}i$
11. $\frac{-1}{2} + \frac{1}{2}i$  1m. $\frac{-20 - 26i}{5}$  2a. $x^2 + 4$  2b. $x^2 + 9$  2c. $x^2 + 5$  2d. $x^2 - 4x + 20$  2e. $x^2 - 6x + 10$  2f. $x^2 + 2x + 5$  2g. $x^2 + x + \frac{5}{4}$
2h. $4x^2 + 4x + 1$  2i. $x^2 + 6x + 14$