A Flat Extension Theorem via Ideal Projection
Tom Kunkle
Mathematics Department, College of Charleston

Abstract

Any linear functional on the space of polynomials $\Pi$ is determined by its values at the monomials, or its moments, and the flat extension problem has to do with whether a functional with finitely many prescribed moments can be extended to $\Pi$ without increasing the rank of the moment matrix. We obtain a generalization of the flat extension theorems of Curtos and Fialkow and Laurent and Mourrain by looking at the problem as one of ideal projection.

Definitions

Let $\Pi := \mathbb{K}[x_1, x_2, \ldots, x_k]$ be the set of polynomials in $k$ indeterminates over the field $\mathbb{K}$ and $\Pi_n$ be the set of polynomials of total degree $\leq n$. The typical element $p$ of $\Pi$ is

$$p = \sum a_i x^n$$

For any $a \in \mathbb{K}^k$, let $\delta_a \in \Pi$ denote the functional $\delta_a(p) = a \cdot p$. A functional $L \in \Pi^*$ given by the rule

$$Lp := \sum a_i \delta_a(p)$$

is completely determined by its moments $\{\delta_a(L)\}_{a \in \mathbb{K}^k}$. The moment matrix of $L$

$$M := \begin{bmatrix} \delta_a(L) \end{bmatrix}_{a \in \mathbb{K}^k}$$

satisfies $L(pq) = \langle pq \rangle = \langle p \rangle \langle q \rangle$.

Flat Extension & Ideal projection

In order that the moment matrix $M$ be a flat extension of $M_B$, $M = P M_B P$ and $L = LP$ for a projector $P$ onto $B$. The dependence relations among the columns of $M$ are simultaneously recurrence relations on the moments of $L$:

$$M = \{ p \in \Pi : L(p\Pi) = 0 \} = \{ p \in \Pi : \delta_a(L) = 0 \}$$

If $M_B$ is invertible, then ker $M = \ker P$, and so $P$ is an ideal projector, a finite-rank projector whose kernel is an ideal.

For an introduction to ideal projection see de Boor [1]. A fundamental property of any ideal projector $P$ is that

$$Pp = p(X)(P^\dagger) \quad \forall p \in \Pi$$

where $X^n := \prod_{i=1}^n x_i^{n_i}$, a product of the commuting operators $X_i : B \rightarrow B : b \mapsto P^n b$. Consequently, an ideal projector is determined by its restriction $N$ to $\Pi + \Pi B = B^*$, and any flat extension $M_B \rightarrow M$ is uniquely determined by the extension $M_B \rightarrow M_B$.

Examples

Because the moment matrix must be a Hanks Matrix, flat extensions sometimes isn't possible:

<table>
<thead>
<tr>
<th>$1$ $x$ $x^2$ $x^3$ $\ldots$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$1$ $1$ $a$ $a^2$ $a^3$ $\ldots$</td>
</tr>
<tr>
<td>$1$ $a$ $a^2$ $a^3$ $\ldots$</td>
</tr>
<tr>
<td>$1$ $a^2$ $a^3$ $\ldots$</td>
</tr>
<tr>
<td>$1$ $a^3$ $\ldots$</td>
</tr>
<tr>
<td>$\vdots$ $\vdots$ $\vdots$</td>
</tr>
</tbody>
</table>

$M_B = 1 (1) \rightarrow x^2 a^3 a^4 a^5 \ldots = M$

(in which case $L = \delta_a$) it's determined entirely by the initial extension

$$M_B \rightarrow M_B = 1 x \begin{bmatrix} 1 & a \\ a & a^2 \end{bmatrix}.$$ 

One of the main results of Curtos and Fialkow and later Laurent and Mourrain has to do with whether a flat extension $M_B \rightarrow M_C$ is the restriction of a flat extension $M_C \rightarrow M$. Our main result has been to replace earlier sufficient conditions with a necessary and sufficient condition.

Flat Extension Theorems:

In the field $\mathbb{R}$, if $M_B \rightarrow M_C \rightarrow M_C$, where $M_C$ is invertible, then $M_C$ has a flat extension $M_C \rightarrow M$. If $\mathbb{C}$ is a monomial space connected to $1$.

References


Interpretation in terms of recurrences

To give a simple example, suppose that

$$C = B = \text{span}\{x^{31}, x^{32}, x^{33}\},$$

so that

$$B^* = \text{span}\{x^{21}, x^{22}, x^{23}, x^{24}, x^{25}\},$$

and suppose $M_B \rightarrow M_B$. The dependence relations of the columns of $M_B$, mean that

$$\begin{bmatrix} y_1 \notag \\
 y_1 + y_2 \notag \\
 y_1 + y_2 + y_3 
\end{bmatrix} = A \begin{bmatrix} y_1 \\
 y_2 \notag \\
 y_2 + y_3 
\end{bmatrix}$$

for some matrix $A$. If we can extend $M_B \rightarrow M$, it can only be by the same relations,

$$\begin{bmatrix} y_i \\
 y_i + y_{i+1} \\
 y_i + y_{i+1} + y_{i+2} 
\end{bmatrix} = A \begin{bmatrix} y_i \\
 y_{i+1} \\
 y_{i+2} 
\end{bmatrix}$$

for all $i,j$. But are there recurrence relations consistent? The flat extension theorem promises that they are consistent for all $y_i,j$ if and only if they are consistent in the values they prescribe for

$$\begin{bmatrix} y_0 \\
 y_1 \\
 y_1 + y_2 \\
 y_1 + y_2 + y_3 
\end{bmatrix}.$$