USING QUASI-INTERPOLANTS IN A RESULT OF FAVARD

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Abstract. A proof of Favard can be restructured using quasi-interpolants of the type discussed in these proceedings [6] and his result strengthened.

1 INTRODUCTION

In a paper in this volume [6], Lei and Cheney discuss the quasi-interpolant; that is, a linear operator of the form

$$L : f \mapsto \sum_i f(y_i) g_i$$

for some collection $\{g_i\}$ of d-variate functions and a set $\{y_i\}$ of points in $\mathbb{R}^d$. Of particular interest are those L satisfying $L^2 = L$ and $Lp = p$ for each p in the space Π_n of all d-variate polynomials of total degree n or less.

One such L is obtained as follows. Fix n a natural number, and suppose that, for every i in the index set, X_i is a finite set of points in $\mathbb{R}^d$ with the property that, for any function f, there is a unique polynomial $P_i f$ in Π_n agreeing with f on X_i. Let $\{\theta_i\}$ be a partition of unity on $\mathbb{R}^d$. Then the quasi-interpolant

$$L : f \mapsto \sum_i \theta_i P_i f \tag{1.1}$$

is a projector and its restriction to Π_n is the identity. One can think of Lf as a moving average of polynomials, since the weights $\{\theta_i\}$ are not constant.

Just such an operator occurs implicitly in the original proof of a univariate extension theorem by Favard [4,2]. The purpose of this note is to demonstrate that, when this proof is modified to make explicit use of quasi-interpolants, a stronger result is immediately obtained. Furthermore, quasi-interpolants allow one to prove similar multivariate extension theorems [5].

Favard's result is the following, except that originally, the nth derivative of the extension Lf was only piecewise continuous.

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Theorem 1.2 *Let* $\mathbf{m} := \{m_i : i \in \mathbb{Z}\}$ *be a strictly increasing, bi-infinite sequence of real numbers diverging to* $\pm\infty$*, and let* f *be a real-valued function defined on* $\mathbf{m}$*. Then there exists an infinitely differentiable extension* Lf *of* f *to* $\mathbb{R}$ *with the property that, on each interval* $[m_i, m_{i+1}]$*,*

$$|D^n Lf| \leq C \max\Big\{ \big|[m_s, \dots, m_{s+n}]f\big| : i - n < s \leq i \Big\} \tag{1.3}$$

where C *is a constant independent of both* f *and* $\mathbf{m}$*.*

Here $[m_s, \dots, m_{s+n}]f$ denotes the divided difference of f at the real numbers $\{m_s, \dots, m_{s+n}\}$. By Rolle's Theorem, any smooth function agreeing with f on $\mathbf{m}$ would have an nth derivative at least as large in the uniform norm as the numbers

$$n!\big|[m_s, \dots, m_{s+n}]f\big|.$$

The extension promised by Theorem 1.2 has an nth derivative that is no more than a constant times this necessary size. In several papers [1,2,3], de Boor has estimated this constant as well as

$$\sup_{f,\mathbf{m}} \frac{\inf\{\|D^n F\|_\infty : F \in L_\infty^{(n)}, F = f \text{ on } \mathbf{m}\}}{\max\{n!|[m_i, \dots, m_{i+n}]f| : i \in \mathbb{Z}\}},$$

and investigated the optimization problem:

Minimize the norm of $F \in L_p^{(n)}$ *such that* $F = f$ *on* $\mathbf{m}$*.*

2 PROOF OF THEOREM 1.2

Define $P_i f$ to be the polynomial of degree less than n agreeing with f at the real numbers $\{m_i, \dots, m_{i+n-1}\}$. We will take for L a quasi-interpolant of the form (1.1), with functions $\{\theta_i\}$ which are to be constructed.

Choose a function $\theta \in C^\infty[0,1]$ that satisfies

$$\theta(0) = 1, \quad \theta(1) = 0,$$
$$\forall k \in \mathbb{N}, \quad D^k\theta(0) = D^k\theta(1) = 0,$$

and

$$\theta + \theta(1 - \cdot) \equiv 1. \tag{2.1}$$

(Here, $\theta(1 - \cdot)$ represents the function given by the rule

$$\theta(1 - \cdot) : [0,1] \to \mathbb{R} : x \mapsto \theta(1 - x).$$

The dot "$\cdot$" is used similarly below, enabling us to make a clear distinction between a function such as θ and its output $\theta(x)$ at some point x in its domain.)

For every i, set I_i to be the closed interval $[m_i, m_{i+1}]$, and define $|I_i|$ to be its length. Choose $k = k_i$ so that I_k is the largest of the intervals

$$I_{i-1}, \cdots, I_{i+n-2}.$$

If this leaves more than one possibility for k, take k minimal. Choose $l = l_i$ so that I_l is the largest of

$$I_i, \cdots, I_{i+n-1},$$

with the same rule in case of a tie. Note also that when k and l are viewed as functions of i, we have $k_i \le l_i = k_{i+1}$. Abbreviating k_i and l_i to k and l, construct the weight function θ_i as follows. If $k = l$, define θ_i to be identically zero. If, instead, $k < l$, set

$$\theta_i := \begin{cases} \theta\left(\frac{m_{k+1} - \cdot}{m_{k+1} - m_k}\right) & \text{on } [m_k, m_{k+1}]; \\ 1 & \text{on } [m_{k+1}, m_l]; \\ \theta\left(\frac{\cdot - m_l}{m_{l+1} - m_l}\right) & \text{on } [m_l, m_{l+1}]; \\ 0 & \text{elsewhere.} \end{cases}$$

Consider the resulting sequence $\{\theta_i\}_{-\infty}^{\infty}$. After deleting those terms that are identically zero, for every adjacent pair in the list

$$\ldots, \theta_j, \theta_h, \ldots$$

θ_j decreases to zero and θ_h increases to one on the same interval. From (2.1) it follows that $\{\theta_i\}$ is a partition of unity. By θ_i's construction, $\theta_i(m_j) = 0$ unless $i \le j < i + n$. Thus

$$Lf := \sum_i \theta_i P_i f$$

agrees with f on $\mathbf{m}$. Since each θ_i is infinitely differentiable, so is Lf. It remains only to prove (1.3).

Restricting our attention to the interval I_i, if only one of the functions θ_j is nonzero, then that weight function is identically one and $Lf = P_j f$. Thus $D^n Lf = 0$ and (1.3) is trivially satisfied. If, instead, there are two weight functions, θ_j and θ_h $(j < h)$, with support on this interval, then θ_j decreases to zero and θ_h increases to one on I_i. This means that $l_j = i = k_h$, i.e., that I_i is the largest of the intervals

$$I_j, \ldots, I_{j+n-1}$$

and also the largest of

$$I_{h-1}, \ldots, I_{h+n-2}.$$

Consequently,

$$h - 1 \le i \le j + n - 1, \tag{2.2}$$

and

$$(m_t - m_s) \le (t-s)|I_i| \quad \text{when } j \le s \le t \le h+n-1. \tag{2.3}$$

Set ψ_s to be the polynomial

$$\psi_s := (\cdot - m_s) \cdots (\cdot - m_{s+n-2}).$$

On I_i, Lf can be written

$$\begin{aligned} Lf &= \theta_j P_j f + \theta_h P_h f \\ &= P_j f + \theta_h (P_h - P_j) f \\ &= P_j f + \theta_h \sum_{s=j}^{h-1} (P_{s+1} - P_s) f \\ &= P_j f + \sum_{s=j}^{h-1} (m_{s+n} - m_s) \theta_h \psi_{s+1} [m_s, \cdots, m_{s+n}] f. \end{aligned} \tag{2.4}$$

We now arrive at a bound for $\|D^n(\theta_h \psi_{s+1})\|$, where $\|\cdot\|$ is the max-norm on I_i. Trivially, for some constant c_1 and for $0 \le l \le n$,

$$\|D^l \theta_h\| = c_1 |I_i|^{-l}.$$

Consider $D^{n-l}\psi_{s+1}$. When $l = 0$, this is zero; otherwise it is the sum of products of $l-1$ terms of the form $(\cdot - m_t)$ with $j < t < h+n-1$. On the interval of interest, each of these terms is bounded by the larger of $|m_i - m_t|$ and $|m_{i+1} - m_t|$. By (2.2) and (2.3), this is bounded by $2n|I_i|$. Therefore, with constants c_2 and c_3 depending only on n and θ,

$$\|D^{n-l}\psi_{s+1}\| \le c_2 |I_i|^{l-1}$$

and

$$\|D^n(\theta_h \psi_{s+1})\| \le c_3 |I_i|^{-1}. \tag{2.5}$$

Differentiating (2.4), applying (2.3) to the term $(m_{s+n} - m_s)$, and combining the result with (2.5) gives (1.3), completing the proof of the theorem. ■

In his proof, Favard uses interpolating polynomials of degree n when, as we've seen, $n-1$ suffices. However, the greatest difference the proof of this theorem and Favard's proof is the use of an operator of the form (1.1). Favard writes his extension on each interval I_i as the sum of polynomials and piecewise polynomial "smoothing functions," both of degree n.

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