

# USING QUASI-INTERPOLANTS IN A RESULT OF FAVARD

Thomas Kunkle\*

**Abstract.** A proof of Favard can be restructured using quasi-interpolants of the type discussed in these proceedings [6] and his result strengthened.

## 1 INTRODUCTION

In a paper in this volume [6], Lei and Cheney discuss the quasi-interpolant; that is, a linear operator of the form

$$L : f \mapsto \sum_i f(y_i)g_i$$

for some collection  $\{g_i\}$  of  $d$ -variate functions and a set  $\{y_i\}$  of points in  $\mathbb{R}^d$ . Of particular interest are those  $L$  satisfying  $L^2 = L$  and  $Lp = p$  for each  $p$  in the space  $\Pi_n$  of all  $d$ -variate polynomials of total degree  $n$  or less.

One such  $L$  is obtained as follows. Fix  $n$  a natural number, and suppose that, for every  $i$  in the index set,  $X_i$  is a finite set of points in  $\mathbb{R}^d$  with the property that, for any function  $f$ , there is a unique polynomial  $P_i f$  in  $\Pi_n$  agreeing with  $f$  on  $X_i$ . Let  $\{\theta_i\}$  be a partition of unity on  $\mathbb{R}^d$ . Then the quasi-interpolant

$$L : f \mapsto \sum_i \theta_i P_i f \tag{1.1}$$

is a projector and its restriction to  $\Pi_n$  is the identity. One can think of  $Lf$  as a moving average of polynomials, since the weights  $\{\theta_i\}$  are not constant.

Just such an operator occurs implicitly in the original proof of a univariate extension theorem by Favard [4,2]. The purpose of this note is to demonstrate that, when this proof is modified to make explicit use of quasi-interpolants, a stronger result is immediately obtained. Furthermore, quasi-interpolants allow one to prove similar multivariate extension theorems [5].

Favard's result is the following, except that originally, the  $n$ th derivative of the extension  $Lf$  was only piecewise continuous.

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\*Mathematics Dept., College of Charleston, Charleston, SC 29424, kunkle@math.cofc.edu

**Theorem 1.2** Let  $\mathbf{m} := \{m_i : i \in \mathbb{Z}\}$  be a strictly increasing, bi-infinite sequence of real numbers diverging to  $\pm\infty$ , and let  $f$  be a real-valued function defined on  $\mathbf{m}$ . Then there exists an infinitely differentiable extension  $Lf$  of  $f$  to  $\mathbb{R}$  with the property that, on each interval  $[m_i, m_{i+1}]$ ,

$$|D^n Lf| \leq C \max\left\{ |[m_s, \dots, m_{s+n}]f| : i - n < s \leq i \right\} \quad (1.3)$$

where  $C$  is a constant independent of both  $f$  and  $\mathbf{m}$ .

Here  $[m_s, \dots, m_{s+n}]f$  denotes the divided difference of  $f$  at the real numbers  $\{m_s, \dots, m_{s+n}\}$ . By Rolle's Theorem, any smooth function agreeing with  $f$  on  $\mathbf{m}$  would have an  $n$ th derivative at least as large in the uniform norm as the numbers

$$n!|[m_s, \dots, m_{s+n}]f|.$$

The extension promised by Theorem 1.2 has an  $n$ th derivative that is no more than a constant times this necessary size. In several papers [1,2,3], de Boor has estimated this constant as well as

$$\sup_{f, \mathbf{m}} \frac{\inf\{\|D^n F\|_\infty : F \in L_\infty^{(n)}, F = f \text{ on } \mathbf{m}\}}{\max\{n!|[m_i, \dots, m_{i+n}]f| : i \in \mathbb{Z}\}},$$

and investigated the optimization problem:

$$\text{Minimize the norm of } F \in L_p^{(n)} \text{ such that } F = f \text{ on } \mathbf{m}.$$

## 2 PROOF OF THEOREM 1.2

Define  $P_i f$  to be the polynomial of degree less than  $n$  agreeing with  $f$  at the real numbers  $\{m_i, \dots, m_{i+n-1}\}$ . We will take for  $L$  a quasi-interpolant of the form (1.1), with functions  $\{\theta_i\}$  which are to be constructed.

Choose a function  $\theta \in C^\infty[0, 1]$  that satisfies

$$\begin{aligned} \theta(0) = 1, \quad \theta(1) = 0, \\ \forall k \in \mathbb{N}, \quad D^k \theta(0) = D^k \theta(1) = 0, \end{aligned}$$

and

$$\theta + \theta(1 - \cdot) \equiv 1. \quad (2.1)$$

(Here,  $\theta(1 - \cdot)$  represents the function given by the rule

$$\theta(1 - \cdot) : [0, 1] \rightarrow \mathbb{R} : x \mapsto \theta(1 - x).$$

The dot “.” is used similarly below, enabling us to make a clear distinction between a function such as  $\theta$  and its output  $\theta(x)$  at some point  $x$  in its domain.)

For every  $i$ , set  $I_i$  to be the closed interval  $[m_i, m_{i+1}]$ , and define  $|I_i|$  to be its length. Choose  $k = k_i$  so that  $I_k$  is the largest of the intervals

$$I_{i-1}, \dots, I_{i+n-2}.$$

If this leaves more than one possibility for  $k$ , take  $k$  minimal. Choose  $l = l_i$  so that  $I_l$  is the largest of

$$I_i, \dots, I_{i+n-1},$$

with the same rule in case of a tie. Note also that when  $k$  and  $l$  are viewed as functions of  $i$ , we have  $k_i \leq l_i = k_{i+1}$ . Abbreviating  $k_i$  and  $l_i$  to  $k$  and  $l$ , construct the weight function  $\theta_i$  as follows. If  $k = l$ , define  $\theta_i$  to be identically zero. If, instead,  $k < l$ , set

$$\theta_i := \begin{cases} \theta \left( \frac{m_{k+1} - \cdot}{m_{k+1} - m_k} \right) & \text{on } [m_k, m_{k+1}]; \\ 1 & \text{on } [m_{k+1}, m_l]; \\ \theta \left( \frac{\cdot - m_l}{m_{l+1} - m_l} \right) & \text{on } [m_l, m_{l+1}]; \\ 0 & \text{elsewhere.} \end{cases}$$

Consider the resulting sequence  $\{\theta_i\}_{-\infty}^{\infty}$ . After deleting those terms that are identically zero, for every adjacent pair in the list

$$\dots, \theta_j, \theta_h, \dots$$

$\theta_j$  decreases to zero and  $\theta_h$  increases to one on the same interval. From (2.1) it follows that  $\{\theta_i\}$  is a partition of unity. By  $\theta_i$ 's construction,  $\theta_i(m_j) = 0$  unless  $i \leq j < i + n$ . Thus

$$Lf := \sum_i \theta_i P_i f$$

agrees with  $f$  on  $\mathbf{m}$ . Since each  $\theta_i$  is infinitely differentiable, so is  $Lf$ . It remains only to prove (1.3).

Restricting our attention to the interval  $I_i$ , if only one of the functions  $\theta_j$  is nonzero, then that weight function is identically one and  $Lf = P_j f$ . Thus  $D^n Lf = 0$  and (1.3) is trivially satisfied. If, instead, there are two weight functions,  $\theta_j$  and  $\theta_h$  ( $j < h$ ), with support on this interval, then  $\theta_j$  decreases to zero and  $\theta_h$  increases to one on  $I_i$ . This means that  $l_j = i = k_h$ , i.e., that  $I_i$  is the largest of the intervals

$$I_j, \dots, I_{j+n-1}$$

and also the largest of

$$I_{h-1}, \dots, I_{h+n-2}.$$

Consequently,

$$h - 1 \leq i \leq j + n - 1, \tag{2.2}$$

and

$$(m_t - m_s) \leq (t - s)|I_i| \quad \text{when } j \leq s \leq t \leq h + n - 1. \quad (2.3)$$

Set  $\psi_s$  to be the polynomial

$$\psi_s := (\cdot - m_s) \cdots (\cdot - m_{s+n-2}).$$

On  $I_i$ ,  $Lf$  can be written

$$\begin{aligned} Lf &= \theta_j P_j f + \theta_h P_h f \\ &= P_j f + \theta_h (P_h - P_j) f \\ &= P_j f + \theta_h \sum_{s=j}^{h-1} (P_{s+1} - P_s) f \\ &= P_j f + \sum_{s=j}^{h-1} (m_{s+n} - m_s) \theta_h \psi_{s+1} [m_s, \dots, m_{s+n}] f. \end{aligned} \quad (2.4)$$

We now arrive at a bound for  $\|D^n(\theta_h \psi_{s+1})\|$ , where  $\|\cdot\|$  is the max-norm on  $I_i$ . Trivially, for some constant  $c_1$  and for  $0 \leq l \leq n$ ,

$$\|D^l \theta_h\| = c_1 |I_i|^{-l}.$$

Consider  $D^{n-l} \psi_{s+1}$ . When  $l = 0$ , this is zero; otherwise it is the sum of products of  $l - 1$  terms of the form  $(\cdot - m_t)$  with  $j < t < h + n - 1$ . On the interval of interest, each of these terms is bounded by the larger of  $|m_i - m_t|$  and  $|m_{i+1} - m_t|$ . By (2.2) and (2.3), this is bounded by  $2n|I_i|$ . Therefore, with constants  $c_2$  and  $c_3$  depending only on  $n$  and  $\theta$ ,

$$\|D^{n-l} \psi_{s+1}\| \leq c_2 |I_i|^{l-1}$$

and

$$\|D^n(\theta_h \psi_{s+1})\| \leq c_3 |I_i|^{-1}. \quad (2.5)$$

Differentiating (2.4), applying (2.3) to the term  $(m_{s+n} - m_s)$ , and combining the result with (2.5) gives (1.3), completing the proof of the theorem. ■

In his proof, Favard uses interpolating polynomials of degree  $n$  when, as we've seen,  $n - 1$  suffices. However, the greatest difference the proof of this theorem and Favard's proof is the use of an operator of the form (1.1). Favard writes his extension on each interval  $I_i$  as the sum of polynomials and piecewise polynomial "smoothing functions," both of degree  $n$ .

## REFERENCES

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