REARRANGEMENTS OF CONDITIONALLY INTEGRABLE FUNCTIONS

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ABSTRACT

From an s-variate continuous function whose positive and negative parts both have infinite Lebesgue integral, we construct a continuous rearrangement whose generalized Riemann integral equals any prescribed value. The original function is assumed to be finite a.e., with domain an interval in $\mathbb{R}^s$. A similar result is obtained for functions defined on a ball.

1. INTRODUCTION

Recall the following definition.

Definition 1.1: *The Lebesgue measurable functions f and g are called* **identically distributed***, denoted $f \sim g$, if they are finite almost everywhere, and if, for every real α, the sets $\{x : f(x) < \alpha\}$ and $\{x : g(x) < \alpha\}$ have the same measure. If f and g have the same domain, one is said to be a* **rearrangement** *of the other.*

An elementary fact from measure theory states that a Lebesgue integrable function and its rearrangements have the same Lebesgue integral.

In his survey lecture [4], Shisha compares the generalized Riemann integral (**GRI**) to the Lebesgue integral (**LI**) and makes the analogy that the GR-integrability of a function is to its L-integrability as the convergence of a series is to its absolute convergence. Hearing this, and noticing that L-integrable functions and absolutely convergent series behave alike under rearrangements, the listener might wonder if rearrangements effect a "conditionally integrable" function in the same way that they effect a conditionally convergent series. It turns out that they do. We'll show that if a (continuous) function is GR- but not L-integrable in $\mathbb{R}^s$, then it possesses a (continuous) rearrangement whose GRI takes any prescribed value.

(The analogous result for series is due to Riemann, whose name therefore arises twice here in connection with two different areas).

A brief description of the GRI appears in Section 3. Over a large class of functions, the GRI is identical to the LI. For instance, if $f \geq 0$, then the GRI and LI of f over the Lebesgue measurable set $E \subseteq \mathbb{R}^s$ are the same, in the sense that the existence of one implies the existence of the other and their equality. If $|f|$ is integrable in either sense, then the GRI and LI of f exist and are equal.

However, there are functions which are not absolutely integrable in either sense, but which are GR-integrable. Such a function has positive and negative parts with infinite GRI. Obviously, then, if f GR-integrable over the measurable set E, then we obviously cannot conclude that that it is GR-integrable on every subset of E. It is true, however, that if f is GR-integrable over the closed interval I in $\mathbb{R}^s$ (defined in Section 2), then it is GR-integrable over every closed subinterval of I. Fubini's theorem for the GRI states that if a multivariate function is GR-integrable over a product of intervals, then its GRI is computable via iterated integration; absolute integrability is not required.

An interesting property of the GRI is its ability to give meaning to integrals that are improper in the L-sense. Consider, for example, the following result ([2], §1.5).

Theorem 1.2: *Let $f : [a, b] \to \mathbb{R}$ have a GRI on $[a, s]$ for every $s \in (a, b)$. Then the GRI $\int_a^b f\,dx$ exists if and only if the limit of GRIs $\lim_{s \to b^-} \int_a^s f\,dx$ exists. Moreover, when they exist, the two are equal.*

Thus, some singular integrals, i.e., integrals that exist only improperly in the L-sense, exist as proper GRIs—some, that is, but not all. For example, for every positive ϵ, $\left(\int_{-1}^{-\epsilon} + \int_{\epsilon}^{1}\right) x^{-1}\,dx$ is zero, but x^{-1} is not GR-integrable on $(-1, 1)$. The following definition is made to include functions that are integrable in the GR- but not L-sense, as well as functions like $f(x) = x^{-1}$ on $(-1, 1)$.

Definition 1.3: *A Lebesgue measurable function f of s variables is* **conditionally integrable** *on the measurable set E if f is finite almost everywhere and if both Lebesgue integrals $\int_E f^+$ and $\int_E f^-$ are infinite.*

Here $f^+ := \max(f, 0)$ and $f^- := \max(-f, 0)$, the **positive part** and **negative part** of f. The integrability of f is said to be conditional since, as we shall show, the existence and value of the GRI varies among f's rearrangements.

2. NOTATION

We use $:=$ to denote definition, as in $f(x) := x^2$.

A sequence is said to converge if it has a limit in $[-\infty, \infty]$, and diverge otherwise.

We denote the domain of a function f by $\operatorname{dom} f$; if $M \subseteq \operatorname{dom} f$, then $f|_M$ is the restriction of f to M. Denote the greatest integer, or floor, function by $\lfloor\cdot\rfloor$, and the identity function by $(\cdot)$. By **monotone**, we mean either nondecreasing or nonincreasing.

For $x \in \mathbb{R}^s$, we refer to the ith coordinate of x by $x(i)$; subscripts are reserved to denote sequences. Define $r(x)$ to be the Euclidean norm of x. Let $B(R)$ denote $\{x \in \mathbb{R}^s : r(x) < R\}$, the open ball of radius R centered at the origin. $B^c(R)$ shall denote the closed ball. $A(R, S)$ shall denote the open annulus $\{x \in \mathbb{R}^s : R < r(x) < S\}$, and $A^c(R, S)$ the closed annulus. In this context, R and S will always stand for positive, real numbers. By **radial** function, we mean a function of the form $g \circ r$, where $\operatorname{dom} g \subseteq [0, \infty)$. We refer to $g \circ r$ as the **radial extension** of g, and say that $g \circ r$ is **nondecreasing** if g is.

Define π_s to be $m_s\big(B(1)\big)$, the volume of the unit ball.

In spherical coordinates, the integral of $g \circ r$ over $B(R)$ is $\int_0^R g(r)\pi_s\,dr^s$, where dr^s is $sr^{s-1}\,dr$, provided $g \circ r$ is absolutely integrable on $B(R)$.

By **measurable**, we will always mean Lebesgue measurable. We'll denote the LI of f over the E by $\int_E f\,dm$, and its GRI by $\int_E f\,dx$, with the usual convention when $E = (a, b)$. We may refer simply to the integral of f, denoted $\int_E f$, when it is unambiguous to do so. The s-dimensional measure of E is written $m(E)$, or $m_s(E)$, if there exists a chance for confusion over the dimension.

The set $I \subseteq \mathbb{R}^s$ is called an **interval** if it is the Cartesian product of s intervals in $\mathbb{R}$. Unless otherwise specified, all intervals are bounded and nondegenerate; i.e., $0 < m_s(I) < \infty$. The interior of an interval I is denoted I°.

The **essential supremum** of f, written $\operatorname{ess\,sup} f$, is the infimum of the collection of real β for which $m\big(f^{-1}(\beta,\infty]\big) = 0$. The **essential infimum** of f is defined as $\operatorname{ess\,inf} f = -\operatorname{ess\,sup} -f$.

3. THE GENERALIZED RIEMANN INTEGRAL

See [1] for an overview of the GRI and its current status; for a more detailed introduction to the GRI, see [2].

Developed by Henstock and Kurzweil in the mid 1950's, the generalized Riemann integral of a function over the closed and bounded interval $I \subset \mathbb{R}^s$, is defined as a generalized limit of Riemann sums, as follows.

A **gauge** γ is a function which associates to every z in I an s-dimensional open interval $\gamma(z)$ containing z. It is not required that $\gamma(z) \subseteq I$, that $\gamma(z)$ be bounded, or that γ be in any way continuous.

A **division** $\mathcal{D}$ of I is a finite collection of closed intervals J and **tags** $z_J \in J$ such that $I = \bigcup J$ and $J \cap H$ has measure zero for every $J \neq H$ in $\mathcal{D}$. A division $\mathcal{D}$ is said to be γ-**fine** if $J \subseteq \gamma(z_J)$ for every $J \in \mathcal{D}$. It can be shown that for any γ there exists a γ-fine division.

The gauge γ' is said to be **finer** than γ if $\gamma'(z) \subseteq \gamma(z)$ for all $z \in I$. Clearly, any γ'-fine division is also γ-fine.

For f defined on I, and for $\mathcal{D}$ a division of I, the Riemann sum $fm(\mathcal{D})$ is defined as $\sum_{\mathcal{D}} f(z_J)m(J)$. The GRI $\int_I f\,dx$ is said to exist if, for every positive ϵ, there exists a gauge γ such that

$$\left|\int_I f\,dx - fm(\mathcal{D})\right| < \epsilon \tag{3.1}$$

for any γ-fine division $\mathcal{D}$ of I.

For example, take $s = 1$. If, for every $\epsilon > 0$, there is a $\delta > 0$ so that, when $\gamma(z) = (z - \delta, z + \delta)$, any γ-fine $\mathcal{D}$ satisfies (3.1), then f is Riemann integrable on I.

The concept of GRI exists for unbounded intervals but is not relevant to our discussion. The GRI of f over a set $E \subseteq I$ is defined to be $\int_I g\,dx$, where $g = f$ on E and $g = 0$ on $I \setminus E$.

Specializing the arguments in [3], we state and prove a multivariate generalization by McLeod of Theorem (1.2).

Let I' be an s-dimensional interval containing all but one of its $(s-1)$-dimensional faces. Let I be the closure of I'. Let $f : I \to \mathbb{R}$ be identically zero on $I \setminus I'$ and

GR-integrable on every closed subinterval J of I'. For $\mathcal{D}$ a division of I, define

$$\nu(\mathcal{D}) := \sum_{\substack{J\in\mathcal{D}\\ J\subseteq I'}} \int_J f\, dx.$$

If G is a real-valued function whose domain is the set of all divisions on I, we say that $A = \lim_{\mathcal{D}} G(\mathcal{D})$ if, for every positive ϵ, there exists a gauge γ on I such that $|A - \nu(\mathcal{D})| < \epsilon$ for every γ-fine division $\mathcal{D}$ of I. For example, the GRI of f on I is $\lim_{\mathcal{D}} fm(\mathcal{D})$.

In contrast to the standard limit theorems, McLeod's theorem below does not require that f be absolutely integrable on I.

Theorem 3.2: *Under the above assumptions, the GRI of f over I is the same as $\lim_{\mathcal{D}} \nu(\mathcal{D})$, in the sense if either exists, the two are equal.*

Proof: It will suffice to prove $\lim_{\mathcal{D}}\big(\nu(\mathcal{D}) - fm(\mathcal{D})\big) = 0$.

Let $A : \mathbb{R}^s \to \mathbb{R}^s$ be an invertible map of the form $A(x) = A(0) + PDx$, where P is a permutation matrix and D is a diagonal matrix. It is not difficult to show via Riemann sums that f is GR-integrable on I if and only if $f \circ A$ is GR-integrable on $A^{-1}I$. It is without loss of generality, then, to let $I = [0,1]^s$ and $I' = [0,1) \times [0,1]^{s-1}$.

For $n \geq 1$, set $I_n := [0, n/(n+1)] \times [0,1]^{s-1}$, and let γ_n be a gauge on I_n that satisfies

a: $\left|\int_{I_n} f\, dx - fm(\mathcal{D})\right| < \epsilon 2^{-n}$ for every γ_n-fine division $\mathcal{D}$ on I_n;
b: if $z(1) < n/(n+1)$, then $I^\circ \cap \gamma_n(z) \subseteq I_n^\circ$; and
c: if $z(1) = n/(n+1)$ for some $n \geq 1$, then $\gamma_{n+1}(z) \subseteq \gamma_n(z)$.

That there exists a γ_n to satisfy *a* follows from the integrability of f on I_n. Having such a γ_n, one can replace γ_n by a finer gauge that satisfies *a* and *b*, and then by an even finer gauge that satisfies *a*, *b*, and *c*.

Define γ on I as follows. If $z(1) = 1$, set $\gamma(z) = \mathbb{R}^s$. Otherwise, set $\gamma(z) = \gamma_n(z)$ for the unique n that satisfies $(n-1)/n \leq z(1) < n/(n+1)$. Then

b′: if $z(1) < n/(n+1)$, then $I^\circ \cap \gamma(z) \subseteq I_n^\circ$.

Because I_n° is an increasing sequence of sets, it is sufficient to prove b' for the minimal n such that $z(1) < n/(n+1)$. But then $\gamma(z) = \gamma_n(z)$ and b' reduces to b.

Let $\mathcal{D}$ be any γ-fine division of I. We claim that

$$\big|\nu(\mathcal{D}) - fm(\mathcal{D})\big| < \epsilon, \tag{3.3}$$

which will complete the proof.

Partition $\mathcal{D}$ as follows. Let $\mathcal{D}_0$ consist of all $J \in \mathcal{D}$ that intersect $I \setminus I'$. Let $\mathcal{D}_n$ consist of all $J \in \mathcal{D}$ for which n is the smallest integer satisfying $J \subseteq I_n$. By definition, $\nu(\mathcal{D}_0) = 0$, and since $J \in \mathcal{D}_0$ implies $z_J(1) = 0$, the Riemann sum $fm(\mathcal{D}_0) = 0$ also. Therefore

$$\nu(\mathcal{D}) - fm(\mathcal{D}) = \sum_{n=1}^{\infty} \nu(\mathcal{D}_n) - fm(\mathcal{D}_n). \tag{3.4}$$

The summand is zero when $\mathcal{D}_n$ is empty, which occurs for all but finitely many n.

Let $n \geq 1$ and consider $J \in \mathcal{D}_n$. The corresponding tag satisfies $z_J \in J \subseteq I_n$, so $z_J(1) \leq n/(n+1)$. Furthermore, $z_J(1) < (n-1)/n$ implies that $n > 1$, and, with b', that $I^\circ \cap \gamma(z_J) \subseteq I_{n-1}^\circ$, contradicting $J \not\subseteq I_{n-1}$. Therefore

$$\frac{n-1}{n} \leq z_J(1) \leq \frac{n}{n+1}.$$

If $z_J(1) < n/(n+1)$, then $\gamma(z_J) = \gamma_n(z_J)$, and if $z_J(1) = n/(n+1)$, then $\gamma(z_J) = \gamma_{n+1}(z_J) \subseteq \gamma_n(z_J)$ by c. Thus $J \in \mathcal{D}_n$ implies that $\gamma(z_J) \subseteq \gamma_n(z_J)$. Because $J \subseteq \gamma(z_J)$, there exists a γ_n-fine division of I_n, say $\mathcal{C}_n$, which contains $\mathcal{D}_n$. Henstock's Lemma ([2]) implies that $fm(\mathcal{D}_n)$ approximates $\nu(\mathcal{D}_n)$ almost as well as $fm(\mathcal{C}_n)$ approximates $\int_{I_n} f\,dx$. Specifically,

$$\left|\nu(\mathcal{D}_n) - fm(\mathcal{D}_n)\right| \leq \epsilon 2^{-n}$$

(compare with a). Since the left side is zero when $\mathcal{D}_n = \emptyset$,

$$\sum_{n=1}^{\infty} |\nu(\mathcal{D}_n) - fm(\mathcal{D}_n)| < \epsilon.$$

Combined with (3.4), this proves (3.3). ■

4. A UNIVARIATE REARRANGEMENT

We begin this section with two simple but useful lemmas.

Lemma 4.1: *If g is nondecreasing on (R,S), and if the radial function $\rho \circ r$ is identically distributed to $r|_{A(R,S)}$ then $g \circ \rho \circ r \sim g \circ r$.*

Proof: To see this, let α be a real number, and define S_α to be $\sup\{x > R : g(x) < \alpha\}$. Then

$$(g \circ \rho \circ r)^{-1}(-\infty, \alpha) = (\rho \circ r)^{-1}(R, S_\alpha).$$

By hypothesis, this set has the same measure as $r^{-1}(R, S_\alpha) = (g \circ r)^{-1}(-\infty, \alpha)$, completing the proof. ■

Note that when $s = 1$, the hypothesis on ρ is equivalent to $\rho \sim (\cdot)|_{(R,S)}$, and the conclusion is equivalent to $g \circ \rho \sim g$.

Often, as in the proof of the next lemma, it is useful to note that identically distributed functions have domains of equal measure.

Lemma 4.2: *Let $\{f_n\}$ and $\{g_n\}$ be sequences of functions satisfying $f_n \sim g_n$ and $\operatorname{ess\,sup} f_n \leq \operatorname{ess\,inf} f_{n+1}$. Assume that, for $n \neq m$, both $\operatorname{dom} f_n \cap \operatorname{dom} f_m$ and $\operatorname{dom} g_n \cap \operatorname{dom} g_m$ are of measure zero. Define f a.e. on $\bigcup \operatorname{dom} f_n$ by the rule $f|_{\operatorname{dom} f_n} := f_n$; define g similarly. Then $f \sim g$.*

Proof: Since $f_n \sim g_n$, these two functions have the same essential supremum and infimum. Consider the increasing sequence

$$\cdots \leq \operatorname{ess\,inf} f_{n-1} \leq \operatorname{ess\,sup} f_{n-1} \leq \operatorname{ess\,inf} f_n \leq \operatorname{ess\,sup} f_n \leq \cdots. \tag{4.3}$$

If (4.3) has a largest member $\leq \alpha$, then $\operatorname{ess\,sup} f_{n-1} \leq \alpha < \operatorname{ess\,sup} f_n$ for some n, and

$$m\big(f^{-1}(-\infty, \alpha)\big) = m\big(f_n^{-1}(-\infty, \alpha)\big) + \sum_{k<n} m(\operatorname{dom} f_k).$$

The same is true when f is replaced by g, and the right-hand sides of each are equal, as desired.

On the other hand, if there is no largest member of (4.3) $\leq \alpha$, then either $\alpha >$ ess sup f_n for every n, or $\alpha <$ ess inf f_n for every n. In either case, $m\big(f^{-1}(-\infty,\alpha)\big) = m\big(g^{-1}(-\infty,\alpha)\big)$. ■

Consider now a conditionally integrable function f on $(0,1)$, and a number $\xi \in [-\infty,\infty]$. It follows from Riemann's result on rearrangements of conditionally convergent series that f has a rearrangement whose GRI equals ξ, as we now show.

Replace f by its nondecreasing rearrangement ([5], p. 29). Since $\int_0^1 f^+\,dx$ and $\int_0^1 f^-\,dx$ are infinite,

i) there exists $x_0 \in (0,1)$ such that $f \geq 0$ on $(x_0,1)$ $f \leq 0$ on $(0,x_0)$;

ii) one can define $x_m \in (0,1)$ for $m = \pm1, \pm2, \ldots$ by the rule

$$I_m := \int_{x_{m-1}}^{x_m} f\,dx = \begin{cases} 2m^{-1} & \text{if } m > 0, \text{ and} \\ 2m+1^{-1} & \text{if } m < 0; \end{cases}$$

and *iii)* $\lim_{m\to\infty} x_m = 1$ and $\lim_{m\to-\infty} x_m = 0$.

Take a one-to-one function σ from $\mathbb{Z}$ onto $\mathbb{N}$ for which $\sum_{m=1}^{\infty} I_{\sigma(m)} = \xi$. We'll take a corresponding rearrangement g for which

$$\lim_{\alpha\to1^-} \int_0^\alpha g\,dx = \lim_{N\to\infty} \sum_{m=1}^{N} I_{\sigma(m)}. \tag{4.4}$$

Then, by Theorem (1.2), the GRI of g exists and also equals ξ.

To construct g, begin with the infinite partition of $(0,1)$ consisting of the intervals (x_{m-1},x_m). Set $\Delta_m := x_m - x_{m-1}$ for $m \in \mathbb{Z}$ and $s_m := \sum_{i=1}^{m} \Delta_{\sigma(i)}$ for $m \geq 0$. On (s_{m-1},s_m), define $\rho_m(x) := x - s_m + x_{\sigma(m)}$. Clearly, $\rho_m \sim (\cdot)|_{(x_{\sigma(m)-1},x_{\sigma(m)})}$. Define ρ on $\bigcup \operatorname{dom}\rho_m$ as in Lemma (4.2). Then $\rho \sim (\cdot)|_{(0,1)}$; consequently, $m(0,1) = m(\operatorname{dom}\rho)$, so $\operatorname{dom}\rho = [0,1)$. Therefore ρ is a rearrangement of $(\cdot)|_{(0,1)}$. By Lemma (4.1), $g := f \circ \rho$ is a rearrangement of f.

One can check directly that

$$\int_{s_{m-1}}^{s_m} g\,dx = \int_{x_{\sigma(m)-1}}^{x_{\sigma(m)}} f\,dx.$$

Viewed as a function of its upper limit, the integral $\int_0^\alpha g\,dx$ is monotone on (s_{m-1},s_m), since g has constant sign on each such interval. Equation (4.4) follows, as desired.

Finally, we note that there exist σ such that the partial sums of $\sum I_{\sigma(m)}$ diverge. By Theorem (1.2), the corresponding rearrangement g will fail to be GR-integrable.

We summarize the results of this section in the following theorem, making the obvious generalization from $(0,1)$ to (a,b).

Theorem 4.5: *Let f be a conditionally integrable function on the finite interval (a,b), and let $\xi \in [-\infty,\infty]$. Then f has a rearrangement whose GRI equals ξ. There also exists a rearrangement of f that is not GR-integrable.*

5. CONTINUOUS REARRANGEMENTS ON DISKS

In all likelihood, the rearrangement constructed in Theorem (4.5) will be discontinuous. In this section and the next, we deal with the more challenging problem of finding continuous rearrangements of continuous functions.

We begin with the following lemma.

Lemma 5.1: *Let f be a s-variate measurable function whose domain* $\operatorname{dom} f$ *has finite measure. Let f be finite almost everywhere. Then there exists a nondecreasing function g, with* $\operatorname{dom} g = (0, A)$ *(for some $A < \infty$), whose radial extension is identically distributed to f. The function $g \circ r$ is finite on a punctured open ball with volume equal to* $m_s(\operatorname{dom} f)$.

Proof: To construct g, we modify a construction appearing in [5].

For real x, set $p(x) := m_s\big(f^{-1}(-\infty, x)\big)$. Then p is left-continuous and nondecreasing. If $x < \inf f$ or $x > \sup f$, then $p(x) = 0$ or $p(x) = m_s(\operatorname{dom} f)$, respectively. Define f_* to be the inverse function p^{-1}, with the understanding that the jump discontinuities of p will correspond to intervals of constancy of f_* and at the jump discontinuities of f_* (corresponding to intervals of constancy of p), f_* will be defined to left continuous. Then f_* has domain $[0, m_s(\operatorname{dom} f)]$ and range $[\inf f, \sup f]$. Furthermore, f_* is nondecreasing, and, for $\alpha \in \mathbb{R}$,

$$m_1\big(f_*^{-1}(\infty, \alpha)\big) = m_s\big(f^{-1}(-\infty, \alpha)\big), \tag{5.2}$$

since both sides equal $p(\alpha)$.

If $s = 1$ and $\operatorname{dom} f = [0, c]$ for some real c, then f_* is the nondecreasing rearrangement used in the proof of Theorem (4.5).

Set $g(r) := f_*(\pi_d r^d)$ for all allowable $r \geq 0$. Then g is nondecreasing and its domain is of the required form. For any real α, the set $m_s\big((g \circ r)^{-1}(-\infty, \alpha)\big)$ is empty if $\alpha < \inf f$, and otherwise it is a ball with radius

$$\sup\{r : f_*(\pi_s r^s) < \alpha\} = \sup\{r : \pi_s r^s < m_s\big(f^{-1}(-\infty, \alpha)\big)\},$$

so that its volume is $m_s\big(f^{-1}(-\infty, \alpha)\big)$. Hence $g \circ r$ is identically distributed to f. Since $\operatorname{dom} f$ and $\operatorname{dom} g \circ r$ have the same measure, the proof is complete. ■

From the proof of Lemma (5.1), one observes the following corollary.

Corollary 5.3: *If f is bounded below, then $g \circ r$ is defined at the origin and equals* $\inf f$ *there. If, in addition, f_* is continuous on $\big[0, m_s(\operatorname{dom} f)\big)$ then $g \circ r$ is continuous.*

We next construct a radial function $\rho \circ r$ that is identically distributed to $r|_{A(R,S)}$.

Let $X := \{x_0, x_1, \ldots\}$ be an increasing sequence of nonnegative reals with the property that

$$x_n^s + (S^s - R^s)2^{-\lfloor n/2 \rfloor - 2} \leq x_{n+1}^s \quad (n \geq 0). \tag{5.4}$$

This condition ensures that the intersection of any two of the intervals

$$\left[x_n, \sqrt[s]{x_n^s + (S^s - R^s)2^{-\lfloor n/2 \rfloor - 2}}\right] \tag{5.5}$$

has measure zero. Define $\rho := \rho(\cdot \mid X, R, S)$ on the union of all such intervals as follows. For n even, set ρ on (5.5) to be

$$\rho(x) := \sqrt[s]{2^{\lfloor n/2 \rfloor + 2}(x^s - x_n^s) + R^s}$$

and for n odd,

$$\rho(x) := \sqrt[s]{2^{\lfloor n/2 \rfloor + 2}(x_n^s - x^s) + S^s}$$

Note that on the even intervals, ρ increases from R to S, and on the odd intervals, it decreases from S to R.

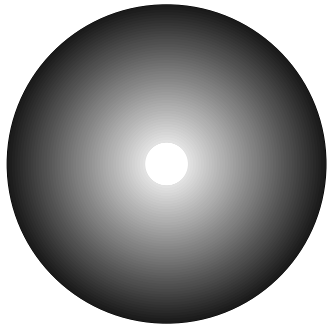

Figure 1. $r|_{A(R,S)}$

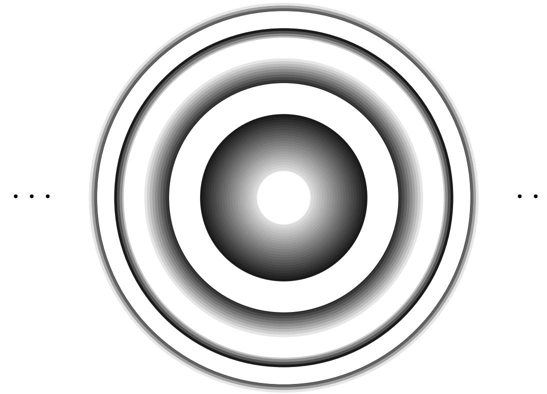

Figure 2. $\rho(r(\cdot) \mid X, R, S)$

Lemma 5.6: *For X satisfying (5.4) and for $0 \le R < S < \infty$, the function $r|_{A(R,S)}$ is identically distributed to $\rho\big(r(\cdot) \mid X, R, S\big)$.*

Proof: The domain of $\rho \circ r$ is the union of closed annuli of the form

$$A^c\Big(x_n, \sqrt[s]{x_n^s + (S^s - R^s)2^{-\lfloor n/2 \rfloor - 2}}\Big).$$

To see that $\rho \circ r$ is identically distributed to r's restriction to $A(R, S)$, note first that if $\alpha \le R$, then both $(\rho \circ r)^{-1}(-\infty, \alpha)$ and $r^{-1}(-\infty, \alpha)$ are empty, and if $\alpha \ge S$,

$$\begin{aligned} m_s\big((\rho \circ r)^{-1}(-\infty, \alpha)\big) &= m_s\big(\operatorname{dom} \rho \circ r\big) \\ &= \sum_{n \ge 0} \pi_s\big((x_n^s + (S^s - R^s)2^{-\lfloor n/2 \rfloor - 2}) - x_n^s\big) \\ &= \pi_s(S^s - R^s) = m_s\big(r^{-1}(-\infty, \alpha)\big). \end{aligned} \tag{5.7}$$

Finally, if $R < \alpha < S$, then it is not hard to see that $(\rho \circ r)^{-1}(-\infty, \alpha)$ is a union of annuli whose volumes sum geometrically to $\pi_s(\alpha^s - R^s) = m_s\big(r^{-1}(-\infty, \alpha)\big)$. ∎

Figures 1 and 2 show the density plots of $r|_{A(R,S)}$ and $\rho\big(r(\cdot) \mid X, R, S\big)$ (for a particular X, R, and S). The domain of $\rho \circ r$ is an infinite sequence of annuli, and $\rho \circ r$ alternates between increasing and decreasing on these. If one interprets r and $\rho \circ r$ as point densities, then the objects in figures 1 and 2 have the same total mass. In figure 2, this mass is divided into parts $\frac{1}{4}, \frac{1}{4}, \frac{1}{8}, \frac{1}{8}, \ldots$, which are summed in (5.7).

We now state and prove main result of this section.

Theorem 5.8: *Let f be a conditionally integrable continuous function on $B(R) \subset \mathbb{R}^s$, and let $\xi \in [-\infty, \infty]$. Then f has a continuous radial rearrangement $g \circ r$ such that the singular integral*

$$\lim_{\alpha \to R^-} \int_{B(\alpha)} g \circ r = \xi.$$

Furthermore, f has a continuous rearrangement $g \circ r$ whose singular integral over $B(R)$ does not exist.

Since g is continuous, $\int_{B(\alpha)} g \circ r = \pi_s \int_0^\alpha g(r)\, dr^s$ for $\alpha < R$. Hence Theorem (1.2) implies that the GRI $\pi_s \int_0^R g(r)\, dr^s$ equals ξ in the first case, and in the second does not exist. When $s = 1$, the radial functions are simply the even functions, and the existence of $\int_0^R g(r)\, dr$ is equivalent to the existence of $\int_{-R}^R g(|x|)\, dx$. Hence Theorem (5.8) has the following corollary.

Corollary 5.9: *Let f be a univariate function that is continuous and conditionally integrable on $(-R,R)$. Let $\xi \in [-\infty,\infty]$. Then f has a continuous even rearrangement $g(|\cdot|)$ whose GRI on $(-R,R)$ equals ξ. There also exists a continuous rearrangement of f which is not GR-integrable on $(-R,R)$.*

For $s>1$, it would be interesting to know whether or not the GRIs $\int_{B(R)} g\circ r\,dx$ and $\pi_s\int_0^R g(r)\,dr^s$ are the same for g continuous on $[0,R)$. If so, one could restate the conclusion of Theorem (5.8) in terms of the GRI of the rearrangement, as we have in Corollary (5.9) and will do in the next section. However, the multivariate change-of-variables rule requires absolute integrability, as do most limit theorems that could be used to prove $\lim_{\alpha\to R^-}\int_{B(\alpha)} = \int_{B(R)}$.

The proof is similar to that of Theorem (4.5). We'll assume, without loss of generality, that $R=1$, then choose a series which converges conditionally to ξ/π_s (or fails to converge), and then construct a (radial) rearrangement $g\circ r$ such that, as α increases to 1, $\int_0^\alpha g(r)\,dr^s$ behaves like the partial sums of the series.

Proof of (5.8): Without loss of generality, $R=1$. Furthermore, it will suffice to prove Theorem (5.8) in case $\xi\geq 0$. Then, if $\xi<0$, we can obtain a rearrangement of $-f$ whose GRI equals $-\xi$, and therefore a rearrangement of f whose GRI is ξ.

Apply Lemma (5.1) to the restriction $f|_{f^{-1}[0,\infty)}$, obtaining a nondecreasing function g_+ whose radial extension is identically distributed to $f|_{f^{-1}[0,\infty)}$ and has domain $B(R_+)$, where $\pi_s R_+^s = m_s(f^{-1}[0,\infty))$. Likewise, let $g_-\circ r$ be a nonincreasing radial rearrangement of $f|_{f^{-1}(-\infty,0)}$ with domain $B(R_-)$. We claim that g_+ and g_- are continuous.

Consider g_+ first.

Corollary (5.3) implies that, because $f|_{f^{-1}[0,\infty)}$ is bounded below, we need only show that $\big(f|_{f^{-1}[0,\infty)}\big)^*$, or f^* for short, is continuous on $[0,R_+)$. Since f^* is nondecreasing, it will suffice to show that it has no jump discontinuities. These exist if and only if there are intervals on which $m_s\big(f^{-1}(0,x)\big)$ is constant, or, equivalently, if $m_s\big(f^{-1}(a,b)\big)=0$ for some nontrivial (a,b). But because f is continuous and is unbounded above and below, $f^{-1}(a,b)$ is nonempty and open, and hence has positive measure. Therefore g_+ is continuous. The proof of g_-'s continuity is similar.

Choose $\sum_{m\geq 2} a_m$ to be a positive series summing to ∞ as follows.

If $\xi=0$, take $a_{2m}=a_{2m+1}=(m+3)^{-1}$; if $\xi=\infty$ take $a_{2m}\equiv 2$ and $a_{2m+1}\equiv 1$; and if ξ is finite and positive, take $a_m = c/(m+1)$, where $c>0$ is chosen so that, as in the other cases,

$$\sum_{m=2}^{\infty}(-1)^m a_m = \frac{\xi}{\pi_s}. \tag{5.10}$$

To construct a rearrangement whose GRI does not exist, take $a_m\equiv 1$, so that

$$\sum_{m=2}^{\infty}(-1)^m a_m \text{ diverges.} \tag{5.11}$$

With $\sum a_m$ so chosen, both $\sum_{m=1}^\infty 2a_{2m}-a_{2m-2}$ and $\sum_{m=1}^\infty 2a_{2m+1}-a_{2m-1}$ are positive series summing to infinity. (We adopt the convention that $a_m=0$ for

$m < 2$.) Thus, it is possible to define the sequences (r_m) and (s_m) by

$$\begin{aligned} r_0 = s_0 &= 0; \\ \int_{r_{m-1}}^{r_m} g_+(r)\, dr^s &= 2a_{2m} - a_{2m-2} \quad m \geq 1; \quad \text{and} \\ \int_{s_{m-1}}^{s_m} g_-(r)\, dr^s &= \; - 2a_{2m+1} + a_{2m-1} \quad m \geq 1. \end{aligned} \tag{5.12}$$

Furthermore, r_m (or s_m) increases to R_+ (or R_-) as $m \to \infty$.

Now define the functions g_m for $m \in \mathbb{Z} \setminus 0$ by

$$g_m := \begin{cases} g_+|_{[r_{m-1}, r_m]} & \text{if } m > 0; \text{ and} \\ g_-|_{[s_{-m-1}, s_{-m}]} & \text{if } m < 0. \end{cases} \tag{5.13}$$

Next we construct, for every $m \geq 1$, a sequence $X_m = \{x_{m,0}, x_{m,1}, \ldots\}$ of non-negative reals satisfying (5.4). The associated

$$\rho_m := \rho(\cdot \mid X_m, r_{m-1}, r_m)$$

is defined on closed intervals with left endpoint $x_{m,n}$ and right endpoint

$$\mathrm{rep}^x_{m,n} := \sqrt[s]{x^s_{m,n} + (r^s_m - r^s_{m-1})2^{-\lfloor n/2 \rfloor - 2}}.$$

We also construct $Y_m = \{y_{m,0}, y_{m,1}, \ldots\}$ also to satisfy (5.4). For $m > 0$, define

$$\rho_{-m} := \rho(\cdot \mid Y_m, s_{m-1}, s_m),$$

whose domain consists of closed intervals from $y_{m,n}$ to

$$\mathrm{rep}^y_{m,n} := \sqrt[s]{y^s_{m,n} + (s^s_m - s^s_{m-1})2^{-\lfloor n/2 \rfloor - 2}}.$$

We define the $x_{m,n}$s and $y_{m,n}$s by passing through the following list and assigning each entry to be the rep of its predecessor.

$$\begin{array}{rllllll} 0 =: x_{1,0} & x_{1,1} \\ y_{1,0} & y_{1,1} \\ x_{1,2} & x_{2,0} & x_{2,1} & x_{1,3} \\ y_{1,2} & y_{2,0} & y_{2,1} & y_{1,3} \\ x_{1,4} & x_{2,2} & x_{3,0} & x_{3,1} & x_{2,3} & x_{1,5} \\ y_{1,4} & y_{2,2} & y_{3,0} & y_{3,1} & y_{2,3} & y_{1,5} & \cdots \end{array} \tag{5.14}$$

That is, we start by setting $x_{1,0}$ equal to 0, and then set $x_{1,1} = \mathrm{rep}^x_{1,0}$, then $y_{1,0} = \mathrm{rep}^x_{1,1}$, then $y_{1,1} = \mathrm{rep}^y_{1,0}$, and $x_{1,2} = \mathrm{rep}^y_{1,1}$, etc.. The typical mth row of xs in this list is

$$\begin{array}{ccccccccc} x_{1,2m-2} & x_{2,2m-4} & \cdots & x_{m-1,2} & x_{m,0} \\ & x_{m,1} & x_{m-1,3} & \cdots & x_{2,2m-3} & x_{1,2m-1} \end{array} \tag{5.14.m}$$

(to be followed by $y_{1,2m-2}, y_{2,2m-4}$, etc.), in which the first two elements of X_m are defined. In the $(m+1)$th row, the next two members of X_m are computed, and so on, so that every $x_{m,n}$ (and $y_{m,n}$) is eventually given a value. The condition (5.4) is satisfied by each X_m (and Y_m), since X_m (and Y_m) appears, in order, as a subsequence of (5.14). Thus the function ρ_m is well-defined as in Lemma (5.6) for every $m \neq 0$.

For $m \neq n$, the sets $\operatorname{dom}\rho_n$ and $\operatorname{dom}\rho_m$ intersect at most on a set of measure zero, since any two intervals of the form

$$[x_{m,n}, \operatorname{rep}^x_{m,n}] \quad \text{or} \quad [y_{m,n}, \operatorname{rep}^y_{m,n}]$$

intersect at most at an endpoint. Thus the sets $\{\operatorname{dom}\rho_m \circ r\}$ are mutually almost disjoint, intersecting on a the surface of a ball if at all. As in Lemma (4.2), we define the radial function $g \circ r$ on their union. Since $g_m \circ \rho_m \circ r \sim g_m \circ r$ for every m, the functions f and $g \circ r$ are identically distributed.

We now show $\operatorname{dom} g \circ r = \operatorname{dom} f$, so that $g \circ r$ is a rearrangement of f. Since $\operatorname{dom} g \circ r = \bigcup \operatorname{dom} g_m \circ \rho_m \circ r = \bigcup \operatorname{dom}\rho_m \circ r$, it will suffice to show that $\bigcup \operatorname{dom}\rho_m$ is $[0, 1)$.

Since every $x_{m,n}$ and $y_{m,n}$ is nonnegative, each $\operatorname{dom}\rho_m$ lies entirely in $[0, \infty)$. By construction, $\bigcup \operatorname{dom}\rho_m$ is connected, its smallest member is 0, and it contains no largest member; therefore it is of the form $[0, c)$ for some $c \leq \infty$. From $g \circ r \sim f$, it follows that $m_s(\operatorname{dom} f) = m_s(\operatorname{dom} g \circ r) = \pi_s c^s$, so $c = 1$, as desired.

To see that $g \circ r$ is continuous on $B(1)$, or, equivalently, that g is continuous on $[0, 1)$, first note that g_m is continuous, and that ρ_m is continuous on each interval of its domain. Hence, one need only verify that g is continuous on any nonempty $\operatorname{dom}\rho_m \cap \operatorname{dom}\rho_n$. To this end, we list all possible intersections between intervals of the forms

$$[x_{m,n}, \operatorname{rep}^x_{m,n}] \quad \text{or} \quad [y_{m,n}, \operatorname{rep}^y_{m,n}]$$

below. The intersection point is the left endpoint of the second interval and right endpoint of the first as listed; the right endpoint of each interval is simply rep of its left. We therefore suppress the argument of rep.

$$\begin{array}{rc} a: & [x_{l,2(m-l)}, \operatorname{rep}] \cap [x_{l+1,2(m-l)-2}, \operatorname{rep}] \\ b: & [x_{m,0}, \operatorname{rep}] \cap [x_{m,1}, \operatorname{rep}] \\ c: & [x_{l+1,2(m-l)-1}, \operatorname{rep}] \cap [x_{l,2(m-l)+1}, \operatorname{rep}] \\ d: & [x_{1,2m-1}, \operatorname{rep}] \cap [y_{1,2m-2}, \operatorname{rep}] \\ a', b', c': & \text{(Replace } x \text{ by } y \text{ throughout } b,\ c,\ d.) \\ d': & [y_{1,2m-1}, \operatorname{rep}] \cap [x_{1,2m}, \operatorname{rep}] \end{array}$$

Recall that, when $m \geq 1$, the function ρ_m increases on $[x_{m,2k}, \operatorname{rep}]$ and decreases on $[x_{m,2k+1}, \operatorname{rep}]$. Specifically,

$$\begin{aligned} \rho_m(x_{m,2k}) = r_{m-1} = \rho_m(\operatorname{rep}^x_{m,2k+1}), \quad &\text{and} \\ \rho_m(x_{m,2k+1}) = r_m = \rho_m(\operatorname{rep}^x_{m,2k}). & \end{aligned} \tag{5.15}$$

Likewise, ρ_{-m} takes the extreme values

$$\begin{aligned} \rho_{-m}(y_{m,2k}) = s_{m-1} = \rho_{-m}(\operatorname{rep}^y_{m,2k+1}) \quad &\text{and} \\ \rho_{-m}(y_{m,2k+1}) = s_m = \rho_{-m}(\operatorname{rep}^y_{m,2k}). & \end{aligned} \tag{5.16}$$

Using (5.15), (5.16), and $g_+(0) = g_-(0) = 0$, it is not hard to check continuity at each intersection point. For example, g is continuous at point a, since from the left g approaches

$$g_l(\rho_l(\operatorname{rep}^x_{l,2k})) = g_l(r_l)$$

and from the right

$$g_{l+1}(\rho_{l+1}(x_{l+1,2\tilde{k}})) = g_{l+1}(r_l)$$

and these are equal by (5.13). The proof of continuity at c, a', and c' are similar. At b, the left-hand limit is $g_m(\rho_m(\mathrm{rep}^x_{m,0}))$ and the right hand limit is $g_m(\rho_m(x_{m,1}))$. Both of these equal $g_m(r_m)$ by (5.15); continuity at b' follows similarly. The left-hand limit at d is

$$g_1(\rho_1(\mathrm{rep}^x_{1,2m-1})) = g_1(r_0) = g_+(0) = 0,$$

while the right-hand limit is

$$g_{-1}(\rho_{-1}(y_{1,2m-2})) = g_{-1}(s_0) = g_-(0) = 0.$$

Hence g is continuous at d; continuity at d' is proved similarly.

Thus g is continuous on $[0, 1)$, and $g \circ r$ is continuous on $B(1)$.

Finally, we consider the integral of $g(r)\, dr^s$ on $(0, 1)$.

We claim

$$\int_{x_{1,2k-2}}^{\mathrm{rep}^x_{1,2k-1}} g(r)\, dr^s = a_{2k} \quad (k \geq 1) \tag{5.17}$$

and

$$\int_{y_{1,2k-2}}^{\mathrm{rep}^y_{1,2k-1}} g(r)\, dr^s = -a_{2k+1} \quad (k \geq 1). \tag{5.18}$$

By a simple change of variables, one can check that

$$\int_{x_{m,n}}^{\mathrm{rep}^x_{m,n}} g(r)\, dr^s = 2^{-\lfloor n/2 \rfloor - 2} \int_{r_{m-1}}^{r_m} g_+(u)\, du^s \qquad (m \geq 1, \quad n \geq 0) \tag{5.19}$$

and

$$\int_{y_{m,n}}^{\mathrm{rep}^y_{m,n}} g(r)\, dr^s = 2^{-\lfloor n/2 \rfloor - 2} \int_{s_{m-1}}^{s_m} g_-(u)\, du^s \qquad (m \geq 1, \quad n \geq 0) \tag{5.20}$$

As a consequence,

$$\int_{x_{m,n}}^{\mathrm{rep}^x_{m,n}} g(r)\, dr^s = \frac{1}{2} \int_{x_{m,n-2}}^{\mathrm{rep}^x_{m,n-2}} g(r)\, dr^s \tag{5.21}$$

and

$$\int_{y_{m,n}}^{\mathrm{rep}^y_{m,n}} g(r)\, dr^s = \frac{1}{2} \int_{y_{m,n-2}}^{\mathrm{rep}^y_{m,n-2}} g(r)\, dr^s. \tag{5.22}$$

By (5.19) and (5.12), the integral of g from $x_{m,0}$ to $\mathrm{rep}^x_{m,0} = x_{m,1}$ and from there to $\mathrm{rep}^x_{m,1}$ is

$$\int_{x_{m,0}}^{\mathrm{rep}^x_{m,1}} g(r)\, dr^s = \frac{1}{2} \int_{r_{m-1}}^{r_m} g_+(u)\, du^s = a_{2m} - \frac{1}{2} a_{2m-2}. \tag{5.23}$$

Similarly, (5.20) and (5.12) imply

$$\int_{y_{m,0}}^{\mathrm{rep}^y_{m,1}} g(r)\, dr^s = \frac{1}{2} \int_{s_{m-1}}^{s_m} g_-(u)\, du^s = -a_{2m+1} + \frac{1}{2} a_{2m-1}. \tag{5.24}$$

Letting $m = 1$ in (5.23) and (5.24) verifies the $k = 1$ cases of (5.17) and (5.18). We now induct on k. By (5.14.m), (suppressing $g(r)\,dr^s$)

$$\int_{x_{1,2k-2}}^{\mathrm{rep}^x_{1,2k-1}} = \int_{x_{1,2k-2}}^{\mathrm{rep}^x_{k-1,2}} + \int_{x_{k-1,3}}^{\mathrm{rep}^x_{1,2k-1}} + \int_{x_{k,0}}^{\mathrm{rep}^x_{k,1}} .$$

By (5.23), the last of these is $a_{2k} - \frac{1}{2}a_{2k-2}$, and, by (5.21) and (5.14.m), the first two equal

$$\frac{1}{2}\left(\int_{x_{1,2k-4}}^{\mathrm{rep}^x_{k-1,0}} + \int_{x_{k-1,1}}^{\mathrm{rep}^x_{1,2k-3}}\right),$$

which by the induction hypothesis is $\frac{1}{2}a_{2k-2}$, proving (5.17). The proof of (5.18) is similar.

Keeping the order of (5.14) in mind, we can rewrite

$$\int_0^{\mathrm{rep}^y_{1,2k-1}} g(r)\,dr^s = \sum_{n=1}^{k} \int_{x_{1,2n-2}}^{\mathrm{rep}^x_{1,2n-1}} g(r)\,dr^s + \sum_{n=1}^{k} \int_{y_{1,2n-2}}^{\mathrm{rep}^y_{1,2n-1}} g(r)\,dr^s.$$

By virtue of (5.17) and (5.18),

$$\int_0^{\mathrm{rep}^y_{1,2k-1}} g(r)\,dr^s = \sum_{m=2}^{2k+1} (-1)^m a_m,$$

and similarly,

$$\int_0^{\mathrm{rep}^x_{1,2k-1}} g(r)\,dr^s = \sum_{m=2}^{2k} (-1)^m a_m.$$

Viewed as a function of its upper limit, $\int_0^\alpha g(r)\,dr^s$ is monotone for α between $\mathrm{rep}^x_{1,2k-1} = y_{1,2k-2}$ and $\mathrm{rep}^y_{1,2k-1}$, and for α between $\mathrm{rep}^y_{1,2k-1} = x_{1,2k}$ and $\mathrm{rep}^x_{1,2k+1}$, because g is of constant sign on those intervals. Since

$$\lim_{k\to\infty} x_{1,k} = \lim_{k\to\infty} y_{1,k} = R,$$

this implies

$$\lim_{\alpha\to R^-} \int_0^\alpha g(r)\,dr^s = \sum_{m=2}^{\infty} (-1)^m a_m.$$

Whether the series converges to ξ/π_s (as in (5.10)), or fails to converge (as in (5.11)), this finishes the proof of Theorem (5.8). ■

The appendix shows an example of this construction in one variable.

We note that if $\operatorname{dom} f = \mathbb{R}^s$, then the same construction yields a rearrangement whose GRI equals ξ or fails to exist, as desired, provided that both f^+ and f^- have nondecreasing radial rearrangements. For this, it is necessary that $m\big(f^{-1}(a,b)\big) < \infty$ whenever $0 < b - a < \infty$.

6. CONTINUOUS REARRANGEMENTS ON INTERVALS

In this section, we construct a rearrangement on an interval in $\mathbb{R}^s$ and investigate the resulting GRI.

Theorem 6.1: *Let f be a conditionally integrable continuous function on H, an open and bounded interval in $\mathbb{R}^s$. Then f has a continuous rearrangement which is not GR-integrable on H. Furthermore, for every $\xi \in [-\infty, \infty]$, there exists a continuous rearrangement of f whose GRI over H equals ξ.*

Proof: Without loss of generality, we take $\xi \geq 0$ and $H = [-1, 1] \times [0, 1]^{s-1}$.

Define f_* as in Lemma (5.1); replace f_* by $f_*(\cdot + 1)$, so that f_* is nondecreasing with domain $(-1, 1)$. For every real α, (5.2) is satisfied, and, because $m_s\big(f^{-1}(a, b)\big) > 0$ for (a, b) any nontrivial interval, f_* is continuous.

Define the linear projector P on H by $P(x) = x(1)$. Then $f_* \circ P$ is a continuous rearrangement of f, since

$$m_s\big(P^{-1} f_*^{-1}(-\infty, \alpha)\big) = m_s\big(f_*^{-1}(-\infty, \alpha) \times [0, 1]^{s-1}\big)$$

and this is $m_s\big(f^{-1}(-\infty, \alpha)\big)$ by (5.2). Choose $x_0 \in (-1, 1)$ so that $f_* \leq 0$ on $[-1, x_0]$ and $f_* \geq 0$ on $[x_0, 1]$. The integral of $f_* \circ P$ over $[-1, x_0] \times [0, 1]^{s-1}$ in infinite, because $(f_* \circ P)^- \sim f^-$. Since $f_* \circ P$ is not GR-integrable on a subinterval of H, it cannot be integrable on H, proving the first conclusion of Theorem (6.1).

To construct a rearrangement whose GRI is ξ, apply Corollary (5.9) to obtain an even rearrangement $g(|\cdot|)$ of f_* such that $\int_{-1}^{1} g(|x|)\, dx = \xi$. Then

$$G := g\left(|P(\cdot)|\right)$$

is defined on H and, for every real α,

$$\begin{aligned} m_s\big(G^{-1}(-\infty, \alpha)\big) &= m_s\big(g(|\cdot|)^{-1}(-\infty, \alpha) \times [0, 1]^{s-1}\big) \\ &= m_1\big(f_*^{-1}(-\infty, \alpha)\big) \end{aligned}$$

By (5.2), $G \sim f$.

Let $I := [0, 1]^s$, and $I' = [0, 1) \times [0, 1]^{s-1}$. By Theorem (3.2), if

$$\lim_{\mathcal{D}} \nu(\mathcal{D}) = \frac{\xi}{2}, \tag{6.2}$$

where

$$\nu(\mathcal{D}) := \sum_{\substack{J \in \mathcal{D} \\ J \subseteq I'}} \int_J G(x)\, dx$$

for $\mathcal{D}$ a division of I, then $\int_I G\, dx = \xi/2$. It will then follow that $\int_H G\, dx = \xi$, as desired.

From the construction of g, there exists $a_n > 0$ such that $\sum_{n=2}^{\infty} (-1)^n a_n = \xi/2$, and there exists a sequence $0 = b_1 < b_2 < \cdots$ converging to 1 such that

$$\int_{b_{n-1}}^{b_n} g = (-1)^n a_n.$$

Furthermore, $(-1)^n g \geq 0$ on (b_{n-1}, b_n). Define $S_m := \sum_{n=2}^{m} (-1)^n a_n$.

For every $\epsilon > 0$, there exists N such that $|S_m - \xi/2| < \epsilon$ for all $m \geq N$. Define $\gamma = \gamma_\epsilon$ on I by

$$\gamma(z) = \begin{cases} (b_N, \infty) \times \mathbb{R}^{s-1} & \text{if } z(1) = 1; \text{ and} \\ \big(-\infty, \frac{1}{2}(z(1) + 1)\big) \times \mathbb{R}^{s-1} & \text{if } z(1) < 1. \end{cases}$$

Let $\mathcal{D} = \{(z_J, J)\}$ be any γ-fine division on I. We'll show that

$$\left|\nu(\mathcal{D}) - \frac{\xi}{2}\right| < \epsilon, \tag{6.3}$$

proving (6.2).

For $J \in \mathcal{D}$, we let $[j_1, j_2]$ denote the image of J under the projection P, and define $A_J := m_s(J)/(j_2 - j_1)$, the $(s-1)$-dimensional area of J's cross-section. Then, provided $j_2 < 1$, Fubini's theorem allows $\int_J G = A_J \int_{j_1}^{j_2} g$.

Let $\bigcup' J$ denote the union of all J in $\mathcal{D}$ with $J \not\subseteq I'$. Let $\mathcal{D}^n$ denote the collection of those $J \not\subseteq I'$ for which $j_1 \leq b_n$, and let $\bigcup^n$ and $\sum^n$ stand for a union and a sum over $\mathcal{D}^n$.

Define $I_n := [b_{n-1}, b_n] \times [0,1]^{s-1}$. Then $I_n \setminus \bigcup' J = I_n \setminus \bigcup^n J$. Since $(-1)^n G \geq 0$ on I_n,

$$\int_{I_n \setminus \bigcup^n J} G \leq \int_{I_n} G = a_n.$$

Define

$$w_n := a_n^{-1}(-1)^n \int_{I_n \setminus \bigcup^n J} G.$$

Then $w_n \in [0,1]$.

Fix n. Every $J \in \mathcal{D}^n$ has the form $[j_1, 1] \times K$ for $j_1 \leq b_n$ and K a closed interval in $\mathbb{R}^{s-1}$. For $J \in \mathcal{D}^n$ define $J^* := [b_{n-1}, 1] \times K$. Then $A_J = A_{J^*}$ and $I_n \setminus \bigcup^n J \supset I_n \setminus \bigcup^n J^*$ together imply

$$(-1)^n \int_{I_n \setminus \bigcup^n J} G \geq (-1)^n \int_{I_n \setminus \bigcup^n J^*} G = a_n(1 - \sum\nolimits^n A_J).$$

Hence $w_n \geq 1 - \sum^n A_J$. Likewise,

$$(-1)^{n+1} \int_{I_{n+1} \setminus \bigcup^{n+1} J} G \leq (-1)^{n+1} \int_{I_{n+1} \setminus \bigcup^n J} G = a_{n+1}(1 - \sum\nolimits^n A_J),$$

so that $w_{n+1} \leq 1 - \sum^n A_J$. Therefore, for every n,

$$w_n \geq w_{n+1}.$$

Since $\mathcal{D}$ is γ-fine, if $J \not\subseteq I'$, then $j_1 > b_N$, proving $\mathcal{D}^n = \emptyset$ for $n \leq N$. Also, for each of the (finitely many) $J \subseteq I'$, we have $j_2 < 1$, so there exists $M > N$ for which $j_2 < b_M$ for all $J \subseteq I'$. In terms of w_n, these two facts translate to

$$w_n = \begin{cases} 1 & \text{if } n \leq N, \text{ and} \\ 0 & \text{if } n > M. \end{cases}$$

We now consider $\nu(\mathcal{D})$, by definition $\sum_{J \subseteq I'} \int_J G$. We choose to write this as

$$\sum_{n=2}^{\infty} \int_{I_n \setminus \bigcup' J} G = \sum_{n=2}^{\infty} \int_{I_n \setminus \bigcup^n J} G.$$

The summand above is zero for all but finitely many terms. In fact,

$$\nu(\mathcal{D}) = \sum_{n=2}^{N} (-1)^n a_n + \sum_{n=N+1}^{M} (-1)^n a_n w_n.$$

Summation by parts gives

$$S_N + \sum_{n=N+1}^{M} (-1)^n a_n w_n = \sum_{n=N}^{M} S_n(w_n - w_{n+1}),$$

which, on the one hand, is

$$\begin{aligned} &\leq \left(\frac{\xi}{2} + \epsilon\right) \sum_{n=N}^{M} (w_n - w_{n+1}) \\ &= \frac{\xi}{2} + \epsilon, \end{aligned}$$

and, on the other, is $\geq \xi/2 - \epsilon$. Thus (6.3) is true, completing the proof. ■

APPENDIX

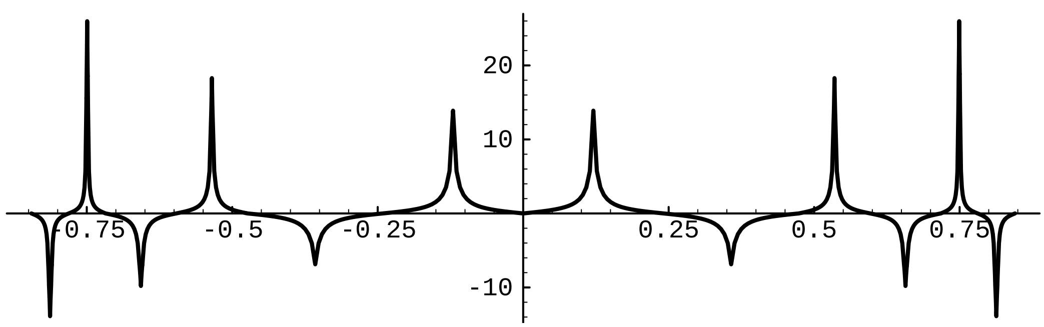

Figure 3: A rearrangement $g(|x|)$ of $f(x) = x/(1 - x^2)$.

Figure 3 shows a rearrangement of $f(x) = x/(1 - x^2)$ on the interval $(-1, 1)$, constructed as in the proof of Theorem (5.8). Since f is monotone, the radial (i.e., even) rearrangements of f^+ and $-f^-$ satisfy $g_+(|x|) = -g_-(|x|) = f(2|x|)$ on $(-\frac{1}{2}, \frac{1}{2})$. Calculations were made using $a_m = (m+1)^{-1}$, so that the GRI equals $\ln 4 - 1$.

Each ρ_m is piecewise linear, with its derivative taking the values $\pm 2^k$ for $k \geq 2$. Thus the graph of $g(|x|)$ is composed of horizontally scaled segments of the original graph, each appearing four times with a scaling factor $\pm 2^k$ for every $k \geq 3$. By (5.17) and (5.18), the integral of the rearrangement between any two successive x-intercepts equals one term of the series $\sum_{m \geq 2} (-1)^m a_m$.

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