Extra problems for *Introductory Analysis*, by John Fridy

1.1.51: Prove that if x and y are real numbers, then exactly one of the following is true:
1) $x > y$,
2) $x < y$, or
3) $x = y$.

1.1.52: If $x < y$, then $x < \frac{x+y}{2} < y$.

1.1.53: Given real numbers B and C, if $A \leq C$ for every real number such that $A < B$, then $B \leq C$.

1.1.54: If $0 < x < y$, then $x < \sqrt{xy} < y$. (Compare with 1.1.52. $(x + y)/2$ and $\sqrt{xy}$ are called the *arithmetic* and *geometric means* resp. of x and y.)

1.1.55: If x and y are nonnegative, then $\sqrt{xy} \leq \frac{x+y}{2}$ with equality iff $x = y$. (Hint: $(\sqrt{x} - \sqrt{y})^2 \geq 0$.)

1.1.56: Give an example of a set S other than $\mathbb{P}$ or $-\mathbb{P}$ for which, for each x in $\mathbb{R}$, exactly one of the following is true:
(i) $x = 0$;
(ii) $x \in S$;
(iii) $-x \in S$.

1.3.51: Give an example of a bounded set A that contains its lub but not its glb.

1.3.52: Prove that lub(A) and glb(A), if they exist, are unique.

1.3.53: Prove that if $B \subset A$ and A is bouded above, then so is B, and $\operatorname{lub} B \leq \operatorname{lub} A$.

1.3.54: Let A be a bounded set. Prove that if $\alpha = \operatorname{lub}(A)$, then $\forall \varepsilon > 0$, $\exists a \in A$ such that $\alpha \geq a > \alpha - \varepsilon$. State and prove a similar statement for glb(A).

1.3.55: Let A be a bounded set. Suppose that some $a \in A$ is an upper bounded for A. Prove that $a = \operatorname{lub}(A)$.

1.3.56: Let a and C be real numbers. Prove $|a| \leq C$ iff $a \leq C$ and $-a \leq C$.

2.1.51: Prove that $\lim_n s_n = 0$ iff $\lim_n |s_n| = 0$.

2.1.52: Prove that, if a_n converges to A, then $|a_n|$ converges to $|A|$. Is the converse true? Prove the converse or provide a conterexample.

2.1.53: Prove that if $|r| < 1$, then $\lim_n r^n = 0$. Hint: use (do not reprove) the result of Example 2.5.

2.1.54: Prove that, if $0 < r < 1$, then $\lim_n nr^n = 0$. Hint: the displayed equation of Example 2.5 could also have read

$$(1/r)^n = \ldots > \frac{n(n-1)}{2} h^2.$$

2.1.55: Prove that, if $0 < r < 1$, and p is any positive integer, then $\lim_n n^p r^n = 0$.

2.1.56: Extend the results in Problems 2.1.54 and 2.1.55 to the case $|r| < 1$.

2.1.57: Prove that if $A \subset \mathbb{R}$ is bounded above then there exists a sequence $\{a_n\}$ of points in A such that $\lim_{n\to\infty} a_n = \text{lub}\, A$.

2.1.58: Prove that if $\lim_n a_n = L$, then $\lim_n (a_1 + a_2 + \cdots + a_n)/n$ exists and also equals L.

2.1.59: Prove that if $\lim_n a_n = L$, then $\lim_n \sqrt[n]{a_1 a_2 \cdots a_n}$ exists and also equals L.

2.1.60: Give an example a divergent sequence a for which $\lim_n (a_1 + a_2 + \cdots + a_n)/n$ exists.

2.2.51: Prove that, if r, s, and t are sequences such that $r_n \le s_n \le t_n$ for all natural numbers n, and if $\lim_n r_n = \lim_n t_n = L$, then s_n also converges to L. (Hint: use—do not reprove—Proposition 2.1 and Lemma 2.1)

2.2.52: Let $\lim_n a_n = A$. Prove that the sequence $\{b_n\}$ given by $b_n = (a_n + a_{n+1})/2$ also converges to A.

2.2.53: See Problem 2.2.52 above. If b_n converges, must a_n converge?

2.2.54: Prove or disprove: if both s_n and $s_n + t_n$ converge, then t_n must also converge.

2.2.55: Show by example that if s_n and $s_n t_n$ both converge, it is not necessary that t_n converges. To conclude that t_n converges, what must be true about the $\lim_n s_n$?

2.5.51: Prove that if s is a monotonic sequence, and if some subsequence of s converges, then so must s.

3.4.51: Prove directly (do not use Theorem 3.2) that if $\{a_n\}$ and $\{b_n\}$ are Cauchy sequences, then so is $\{a_n + b_n\}$.

3.4.52: Prove directly (do not use Theorem 3.2) that if $\{a_n\}$ and $\{b_n\}$ are Cauchy sequences, then so is $\{a_n b_n\}$. You may want to use Lemma 3.2.

3.4.53: Let $s_n = \sin n$ (that is, the sine of n *radians*). Prove that the sequence s has a convergent subsequence.

4.4.51: Give an example of a function defined on $\mathbb{R}$ that is continuous at $x = 0$ and $x = 2$ but nowhere else.

4.7.51: Prove that, if $\lim_{x\to a} f(x) > c$ (or $< c$), then there exists a $\delta > 0$ such that $0 < |x - a| < \delta$ implies $f(x) > c$ ($< c$, resp.).

4.7.52: Prove that function limits are unique. That is, if both of the numbers L and M satisfy the definition of $\lim_{x\to a} f(x)$, then L must equal M.

5.1.51: Prove that if the functions f and g are bounded on the set D, then so are $f + g$ and fg.

5.1.52: Give an example to show that the sum of two unbounded functions can be bounded. Give another example to show that the product of two unbounded functions can be bounded. Make sure the domain D is clear in your examples.

5.1.53: If f and g are functions on D and f and $f+g$ are bounded, must g also be bounded? Prove or provide a counterexample.

5.1.54: If f and g are functions on D and f and fg are bounded, must g also be bounded? Prove or provide a counterexample.

5.1.55: If $f(x)$ is monotonic on some interval $[c,\infty)$, then $\lim_{x\to\infty} f(x)$ exists iff f is bounded on $[c,\infty)$.

5.4.51: The function f is said to be **Lipschitz continuous** on D if there exists a constant M such that for all x_1 and x_2 in D,

$$|f(x_1) - f(x_2)| \leq M|x_1 - x_2|$$

(Note that M is necessarily nonnegative). Prove that if f is Lipschitz continuous on D, then it is uniformly continuous there.

5.4.52: Give an alternate proof of Theorem 5.5 by using the following outline. Assume $\exists\varepsilon^*\ 0$ such that, $\forall n$ a positive integer, $\exists x_n$ and y_n in $[a,b]$ such that $|x_n - y_n| < \frac{1}{n}$ but $|f(x_n) - f(y_n)| \geq \varepsilon^*$. Apply Bolzano-Weierstrauss to $\{x_n\}$. Prove that the corresponding subsequence of $\{y_n\}$ must also converge. Apply the SCFL and arrive at a contradiction.

5.4.53: Give an example of a function f and sets $A \subset \mathbb{R}$ and $B \subset \mathbb{R}$ such that f is uniformly continuous on A and uniformly continuous on B, but f is *not* uniformly continuous on $A \cup B$.

6.2.51: Prove of give a counterexample to disprove: if f is a differentiable one-to-one function, then $f'(x)$ is never zero.

6.3.51: If the function f has a bounded derivative on the interval (a,b), then f is Lipschitz continuous there.

6.3.52: Prove that if f and g are continuous on $[a,b]$ and differentiable on (a,b), and if $f(a) = g(a)$ and $f(b) = g(b)$, then there exists a point c in (a,b) such that $f'(c) = g'(c)$.

6.3.53: Prove that if f and g are n times differentiable on an open interval (a,b) containing the $n+1$ points $x_0, x_1, \ldots, x_n$, and if $f = g$ at $x_0, x_1, \ldots, x_n$, then there exists a point c in (a,b) such that $f^{(n)}(c) = g^{(n)}(c)$.

6.4.51: Prove that, if f and g are continuous on $[a,b]$ and differentiable on (a,b), then there is a point c between a and b such that

$$\big(g(b) - g(a)\big)f'(c) - \big(f(b) - f(a)\big)g'(c) = 0.$$

(This implies that the columns of the 2×2 matrix

$$\begin{pmatrix} f(b) - f(a) & f'(c) \\ g(b) - g(a) & g'(c) \end{pmatrix}$$

are linearly dependent, so, if neither column is zero, one is a nonzero multiple of the other. That is, if the point $\big(f(a), g(a)\big)$ is different from the point $\big(f(b), g(b)\big)$, and if at least one of $f'(c)$ or $g'(c)$ is nonzero, then the tangent vector to the parametric curve

$$\Big\{\big(f(t), g(t)\big) : a \le t \le b\Big\}$$

at $t = c$ is parallel to the line segment joining the beginning and end of the curve.)

6.4.52: Find the point on the curve

$$\Big\{\big(t^3 - t, t^2\big) : -2 \le t \le 2\Big\}$$

at which the tangent vector is parallel to the line segment joining the two endpoints of the curve.

6.5.51: Prove that, if $|x| < \pi/2$, then

$$1 - \frac{x^2}{2} \le \cos x \le 1 - \frac{x^2}{2} + \frac{x^4}{24}$$

6.5.52: Let $q(x)$ be the second degree Taylor polynomial for $\sqrt{1+x}$ at $x = 0$. Prove that $\sqrt{1+x} \le q(x)$ for every $x \ge 0$.

7.1.51: Prove that the function $f(x) = x$ is integrable on any interval and that $\int_a^b x\,dx = \frac{1}{2}(b^2 - a^2)$ by completing the following steps.

a. If $\mathcal{P} = \{x_k\}_{k=0}^n$ is a partition on $[a, b]$, define $L_\mathcal{P} = \sum_{k=1}^n x_{k-1}(x_k - x_{k-1})$ and $U_\mathcal{P} = \sum_{k=1}^n x_k(x_k - x_{k-1})$. Show that $U_\mathcal{P} + L_\mathcal{P} = (b^2 - a^2)$ and $|U_\mathcal{P} - L_\mathcal{P}| \le \|\mathcal{P}\|(b-a)$ for any $\mathcal{P}$.

b. Show that for every positive ε there exists a positive δ such that $\|\mathcal{P}\| < \delta$ implies that $|U_\mathcal{P} - \frac{1}{2}(b^2 - a^2)| = |U_\mathcal{P} - \frac{1}{2}(U_\mathcal{P} + L_\mathcal{P})|$ and $|L_\mathcal{P} - \frac{1}{2}(b^2 - a^2)|$ are both less than ε.

c. Show that $L_\mathcal{P} \le \mathcal{P}(f, \mu) \le U_\mathcal{P}$ for any $\mathcal{P}$ and μ and that, if $\|\mathcal{P}\|$ is less than the δ chosen in step b., then $|\mathcal{P}(f, \mu) - \frac{1}{2}(b^2 - a^2)| < \varepsilon$.

7.3.51: Prove, for any bounded function f on $[a, b]$, and for any partition $\mathcal{P}$ on $[a, b]$, that $\mathcal{P}(f, m) = -\mathcal{P}(-f, M)$ and $\mathcal{P}(f, M) = -\mathcal{P}(-f, m)$.

7.3.52: Prove, for any bounded function f on $[a, b]$, that $\Lambda(f) = -\lambda(-f)$.

7.3.53: Suppose f is bounded on $[a, b]$. Prove that f is integrable on $[a, b]$ iff $\forall \varepsilon < 0$, there exist partitions $\mathcal{P}_1$ and $\mathcal{P}_2$ such that $\mathcal{P}_1(f, M) - \mathcal{P}_2(f, m) < \varepsilon$.

7.5.51: Prove that if α and β are positive, then $\int_0^1 x^\alpha (1-x)^\beta\,dx \le \big((2\alpha+1)(2\beta+1)\big)^{-1/2}$.

7.5.52: Prove that $\displaystyle\int_0^1 x^{100} \sin x\,dx \le \sqrt{\frac{2 + \sin 2}{804}}$.

7.5.53: Prove that $\displaystyle\int_0^1 x^{100} \sin x\,dx \le \frac{\sin 1}{101}$. Hint: use Theorem 7.4. These two problems demonstrate that even though $f(x) = x^{100}$ ranges between 0 and 1 on $[0, 1]$, the integral above is small because the integral of f or f^2 is small there. (That is, f is *on average* small on $[0,1]$.)

7.5.54: Prove that the function f is integrable on $[0,1]$, where

$$f(x) = \begin{cases} 0 & \text{if } 0 \le x < 1/2, \\ 1/2 & \text{if } 1/2 \le x < 3/4, \\ 3/4 & \text{if } 3/4 \le x < 7/8, \\ 7/8 & \text{if } 7/8 \le x < 15/16, \\ & \vdots \\ 1 & \text{if } x = 1. \end{cases}$$

7.5.55: Prove that the function g is *not* integrable on $[0,1]$, where

$$g(x) = \begin{cases} 1 & \text{if } 0 \le x < 1/2, \\ 2 & \text{if } 1/2 \le x < 3/4, \\ 4 & \text{if } 3/4 \le x < 7/8, \\ 8 & \text{if } 7/8 \le x < 15/16, \\ & \vdots \\ 0 & \text{if } x = 1. \end{cases}$$

7.5.56: If f is continuous on $[a,b]$ and if g is integrable on $[a,b]$, and if g is either never negative or never positive on $[a,b]$, then there exists a point c in $[a,b]$ for which $\int_a^b fg = f(c)\int_a^b g$. (Hint: mimic the proof of MVTI, exercise 7.2.7, page 97.)

7.6.51: Find $\displaystyle\lim_{n\to\infty} \frac{\pi}{n} \sum_{k=1}^{n} \sin\left(\frac{k\pi}{n}\right)$.

9.1.51: Show that to every number sequence $\{s_n\}_{n=1}^{\infty}$ there corresponds a unique series $\sum a$ such that s_n is the nth partial sum of $\sum a$.

9.1.52: In contrast to Proposition 9.2, show by example that if $\int_0^\infty f$ exists, $f(x)$ needn't converge to zero as $x \to \infty$. Prove that if $\int_0^\infty f$ exists and if h is any positive number, then $\lim_{x\to\infty} \int_x^{x+h} f = 0$.

9.1.53: Show that if $\sum a_k$ and $\sum b_k$ both converge, and if c is a real number, then $\sum(a_k+b_k)$ and $\sum c a_k$ converge, and $\sum_{k=1}^\infty (a_k+b_k) = \sum_{k=1}^\infty a_k + \sum_{k=1}^\infty b_k$ and $\sum_{k=1}^\infty c a_k = c\sum_{k=1}^\infty a_k$.

9.5.51: Give an example to show that $\sum a_k^2$ can converge while $\sum a_k$ does not.

9.5.52: Show that if $\{a_k\}$ is a sequence of positive numbers and if $\lim_{k\to\infty} a_{k+1}/a_k = L$, then $\lim_{k\to\infty} \sqrt[k]{a_k}$ also equals L. Hint: use Problem 2.1.59. This fact means that, while it sometimes may be harder to use, the root test is stronger that the ratio test, since whatever conclusion the ratio test provides would also be provided by the root test.

9.5.53: Find positive series $\sum a_k$ and $\sum b_k$ for which $\lim_{k\to\infty} \sqrt[k]{a_k}$ exists and is less than 1, and $\lim_{k\to\infty} \sqrt[k]{b_k}$ exists and is greater than 1, but neither $\lim_{k\to\infty} a_{k+1}/a_k$ nor

$\lim_{k\to\infty} b_{k+1}/b_k$ exist. That is, find convergent and divergent series to which the root test applies but the ratio test does not.

9.6.51: Give three examples of nonconvergent series with bounded partial sums.

9.7.51: Determine the absolute/conditional convergence/divergence of

$$\frac{1}{\sqrt{2}+1} - \frac{1}{\sqrt{2}-1} + \frac{1}{\sqrt{3}+1} - \frac{1}{\sqrt{3}-1} + \frac{1}{\sqrt{4}+1} - \frac{1}{\sqrt{4}-1} + \cdots$$

10.1.51: If f and g both have bounded variation on $[a, b]$, and if c is a scalar, then $f + g$ and cf have bounded variation on $[a, b]$, and $V_a^b cf = |c| V_a^b f$ and $V_a^b(f + g) \leq V_a^b f + V_a^b g$.

10.1.52: Find two functions f and g for which $V_0^1(f - g) = V_0^1 f + V_0^1 g$.

10.1.53: If $\mathcal{P}$ is a partition on $[a, b]$ and f is a function on $[a, b]$, and if $\mathcal{P}'$ is a refinement of $\mathcal{P}$, then $\mathcal{P}(f) \leq \mathcal{P}'(f)$.

10.1.54: (a) Show that

$$f(x) = \begin{cases} x^{3/2} \cos \dfrac{1}{x} & \text{if } x \neq 0, \text{ and} \\ 0 & \text{if } x = 0 \end{cases}$$

is differentiable at $x = 0$. but that f' is not bounded on $[0, 1]$.

10.1.54: (b) Show that f has bounded variation on $[0, 1]$. Hint: Suppose $\mathcal{P}$ is a partition on $[0, 1]$. For each of the finitely many intervals of the form $[\frac{1}{(k+1)\pi}, \frac{1}{k\pi}]$ (on which f is monotone) that contain a point of $\mathcal{P}$, add the points $\frac{1}{(k+1)\pi}$ and $\frac{1}{k\pi}$ to $\mathcal{P}$, obtaining a new partition $\mathcal{P}'$. By 10.1.53, $\mathcal{P}(f) \leq \mathcal{P}'(f)$. Show $\mathcal{P}'(f) \leq \sum_{k=1}^{\infty} \left| f\left(\frac{1}{k\pi}\right) - f\left(\frac{1}{(k+1)\pi}\right) \right|$, and that this sum in finite.

10.1.55: Find a function that is continuous on $[0, 1]$, has bounded variation on $[0, 1]$, is differentiable on $(0, 1)$, but whose derivative is unbounded. Don't use the function in Problem 10.1.54.

10.1.56: Prove that if f' (and therefore $|f'|$) is Riemann integrable on $[a, b]$, then $V_a^b f = \int_a^b |f'|$.

10.3.51: Prove that $\int_a^b f\,dg$, if it exists, is unique. That is, given f and g, two functions defined on $[a, b]$, there cannot be two different numbers J that both satisfy Definition 10.2.

11.2.51: Suppose f be a real-valued function defined on $[0, \infty)$ satisfying $f(0) = 0$ and $\lim_{n\to\infty} f(x) = 0$. Define $g_n(x) = f(nx)$. Prove that g_n converges pointwise to the zero function on $[0, \infty)$. Furthermore, if f is not identically zero, then g_n does **not** converge uniformly to zero.

11.2.52: Suppose f is a decreasing function on $[0, \infty)$ satisfying $\lim_{x\to\infty} f(x) = 0$. Define $g_n(x) = f(x + n)$. Prove that g_n converges uniformly to the zero function on $[0, \infty)$.

11.2.53: Suppose that f and g are bounded functions on D. Prove that $\operatorname{lub}_{x\in D}\big(f(x) + g(x)\big) \leq \operatorname{lub}_{x\in D} f(x) + \operatorname{lub}_{x\in D} g(x)$.

11.3.51: Prove that if f_n converges uniformly to F on D, and if each f_n is *uniformly* continuous on D, then so is F.

11.3.52: Give an example of a sequence of discontinuous functions f_n which converges uniformly on $[-1, 1]$ to a discontinuous function F. Hint: Try piecewise constant functions that are all discontinuous at $x = 0$.

11.3.53: Give an example of a sequence of discontinuous functions f_n which converge uniformly on $[-1, 1]$ to a continuous function F.

11.6.51: Let B_n be the nth Bernstein polynomial for the function f. Prove that $B_n > 0$ on $[0, 1]$ if $f > 0$ on $[0, 1]$.

11.6.52: Prove that $B_n'(x) = n \sum_{k=0}^{n-1} \left(f\left(\frac{k+1}{n}\right) - f\left(\frac{k}{n}\right) \right) \binom{n-1}{k} x^k (1-x)^{n-1-k}$. Furthermore, if f is an increasing function on $[0, 1]$, then so is B_n.

11.6.53: Give an example to show that the degree of $B_n(x)$ can be strictly less than n.

11.6.54: B_n is not guaranteed to be the *best* polynomial of degree n or less that approximates f. That is, if f is continuous on $[0, 1]$, and n is a natural number, there could exist a polynomial $p(x)$ of degree n or less for which $\text{lub}_{0\le x\le 1} |f(x) - p(x)| < \text{lub}_{0\le x\le 1} |f(x) - B_n(x)|$. For example, let $f(x) = x^2$ and find $B_2(x)$, $B_3(x)$, and $B_4(x)$. What polynomial p would satisfy the inequality above for all of these three values of n?

11.6.55: Give a proof of Corollary 11.5 that does not refer to the proof of Theorem 11.5. That is, prove Corollary 11.5 directly from the statement of Theorem 11.5.

11.7.51: Define $f_k(x) = \dfrac{\sin kx + \cos kx}{k^3}$. Prove that $\sum_{k=1}^{\infty} f_k(x)$ is a differentiable function of x, and express its derivative as a series.

12.1.51: Find the interval of convergence of the power series $\sum_{k=2}^{\infty} x^k/(\ln k)^k$

12.1.52: Prove that the Cauchy product of two power series centered at a is also a power series centered at a.

12.3.51: Show that any function $f(x)$ can be written as the sum of an even function and an odd function. (f needn't be analytic. Hint: see the definitions of cosh and sinh and how they imply that cosh is even and sinh is odd and $e^x = \cosh(x) = \sinh(x)$.)

12.3.52: Suppose $f(x) = \sum_{k=0}^{\infty} a_k x^k$ has a nonzero radius of convergence. Prove that, if f is even, then $a_k = 0$ for all odd k, and, if f is odd, then $a_k = 0$ for all even k.

12.3.53: Prove that is f and g are analytic on $(a - R, a + R)$, then $f + g$ and $f \cdot g$ are also. (Hint for $f \cdot g$: Theorem 9.14.)

12.4.51: As the text mentions, the Cauchy form (theorem 12.9) of R_n can be derived from the integral form (theorem 12.8) via the Mean Value Theorem for Integrals. Derive the Lagrange form (theorem 12.7) from the integral form. Specifically, prove theorem 12.7 using only theorem 12.8 and exercise 7.5.56 above.

13.1.51: Is $d(x, y) = |[x - y]|$ a metric on $\mathbb{R}$?

13.1.52: Let $C[a, b]$, or C for short, denote the set of all continuous functions on $[a, b]$. For f and g in C, define $d(f, g) = \max_{x \in [a,b]} |f(x) - g(x)|$. Show that $d(f, g)$ is well defined and that d is a metric on C.

13.1.53: See Problem 13.1.52. For f and g in C, define $e(f, g) = \min_{x \in [a,b]} |f(x) - g(x)|$. Show that $e(f, g)$ is well defined. Is e a metric on C?

13.1.54: See Problem 13.1.52. For f and g in C, define $\ell(f, g) = \int_a^b |f(x) - g(x)|\, dx$. Show that $\ell(f, g)$ is well defined. Is ℓ a metric on C?

13.1.55: Let Π_2 denote the set of all quadratic polynomials:

$$\Pi_2 = \{p_0 + p_1 x + p_2 x^2 : \text{ each } p_i \in \mathbb{R}\}.$$

For $p(x) = p_0 + p_1 x + p_2 x^2$ and $q(x) = q_0 + q_1 x + q_2 x^2$, define

$$d_1(p, q) = |p_0 - q_0| + |p_1 - q_1| + |p_2 - q_2|$$

and

$$d_2(p, q) = |p(0) - q(0)| + |p(1) - q(1)| + |p(2) - q(2)|.$$

Show that both d_1 and d_2 are metrics on Π_2. (Hint for d_2: If a quadratic polynomial has more than two zeros, it is identically zero.)

13.1.56: Let Π denote the set of all polynomials. For

$$p = p_0 + p_1 x + p_2 x^2 + \cdots + p_n x^n$$

and

$$q = q_0 + q_1 x + q_2 x^2 + \cdots + q_m x^m$$

define $d(p, q)$ to be $\max_{i \geq 0} |p_i - q_i|$. (If $i > n$, interpret p_i to be zero, and similarly for q_i.) Show that d is a metric on Π, so that Π is a metric space.

13.1.57: For all x, y, and z in a metric space, $|d(x, y) - d(x, z)| \leq d(y, z)$.

13.2.51: Let d and d^* denote the Euclidean and Taxicab metrics in E^n. Prove that $\forall x, y \in E^n$, $d^*(x, y) \leq d(x, y)\sqrt{n}$. Hint: CBS.

13.3.51: Let A be an subset of an arbitrary metric space X. Prove that A° is open.

13.3.52: Let A be an subset of an arbitrary metric space X. Prove that $A^{\circ\circ} = A^\circ$ and $\overline{\overline{A}} = \overline{A}$.

13.3.53: Let X be any metric space, let $x \in X$ and $r > 0$. Prove that if $y \in N_r(x)$ and $s = r - d(x, y)$, then $N_s(y) \subseteq N_r(x)$. Like many proofs in this chapter, a picture will help.

13.3.54: Let X be any metric space, let $x \in X$ and $r > 0$. Prove that $\overline{N_r}(x)$ is closed.

13.3.55: In contrast to problem 13.3.11, give an example of a metric space X, a point x in X, and a positive number r for which the closure of $N_r(x)$ is *not* $\overline{N}_r(x)$. Hint: try choosing X to be a subset of a known metric space. X will have to have some "holes."

13.3.56: If X is any metric space and $A \subseteq X$, then $\setminus\overline{A} = (\setminus A)^{\circ}$.

13.3.57: As proven in Proposition 13.1, every subset Y of a metric space X is a metric space in its own right. The purpose of this exercise is to demonstrate that whether a set $A \subset X$ can be open (or closed) in Y but not in X. Prove that the set $[0, 1)$ is not open or closed in the metric space $\mathbb{R}$, but that it is an open subset of the metric space $[0, \infty)$, and a closed subset of the metric space $(-\infty, 1)$.

13.3.58: See problem 13.3.57. Suppose that $A \subseteq Y \subseteq X$. Prove that if A is an open set in the metric space X, then A is an open set in the metric space Y, and if A is a closed set in X, then A is a closed set in Y. CHECK THIS.

13.5.51: Show that $f_n \to f$ in $C[a, b]$ iff f_n converges uniformly to f on $[a, b]$. (See Problem 13.1.52 for the definition of the metric on $C[a, b]$.)

13.5.52: Let A be a closed set in the metric space X, and let $\{x^{(k)}\}_{k=1}^{\infty}$ be a sequence of points in A. Show that if $x^{(k)}$ converges in X to the point p, then $p \in A$.

13.7.51: Prove that D is dense in X iff $\overline{D} = X$.

13.7.52: Prove that D is dense in X iff for every point x in X there exists a sequence of points in D converging to x.

13.7.53: Prove that is D is dense in E^n, then some countable subset of D is dense in E^n.

13.7.54: Prove that the set of all polynomials is dense in $C[a, b]$. (See Problem 13.1.52 for the definition of the metric on $C[a, b]$.)

14.1.51: Let X be E^n in the Euclidean metric d, let Y be E^n under the taxicab metric d^*, and let Z be E^n under the max-metric $d_m(x, y) = max_i|x_i - y_i|$. Let T be the map that sends x to itself between any two of these spaces. Show that T is continuous. (Two parts of this problem, $X \to Y$ and $Y \to X$, were already done in other examples and exercises in the text.)

14.1.52: Let p be a point of a metric space X, and define the function $f : X \to \mathbb{R}$ by the rule $f(x) = d(x, p)$. Prove that f is continuous on X. (This fact can be useful in problem 14.3.8 in our text.)

14.2.51: Let I be the map that sends f to $\int_a^b f$. Prove that I is a continuous transformation from $C[a, b]$ into $\mathbb{R}$. (See Problem 13.1.52 for the definition of the metric on $C[a, b]$.)

14.2.52: Let T be a transformation from the metric space X to the metric space Y. Prove that T is continuous on X iff $T^{-1}(K)$ is closed for every closed set K in Y.

14.5.51: For $s \in \mathbb{R}$ and $x \in E^n$, prove that $\|sx\| = |s|\|x\|$.

14.5.52: Prove the triangle inequality in E^n: that $\|x + y\| \leq \|x\| + \|y\|$ for any x and y in E^n.

B.51: The *Cartesean Product* $S \times T$ of two sets S and T is the collection of all ordered pairs (s,t) such that $s \in S$ and $t \in T$. That is,

$$S \times T = \{(s,t) : s \in S, t \in T\}.$$

The Cartesean product of n ($\in \mathbb{N}$) copies of a set S is written S^n, i.e.,

$$\underbrace{S \times S \times \cdots \times S}_{n} = S^n.$$

For instance, $\mathbb{R}^3 = \{(x,y,z) : x,\ y, \text{ and } z \in \mathbb{R}\}$.

Prove that if S and T are countable and n is any natural number, then both $S \times T$ and S^n are countable.

B.52: Fix n a natural number, and let $\binom{\mathbb{N}}{n}$ denote the collection of all subsets of $\mathbb{N}$ of size n. Prove that $\binom{\mathbb{N}}{n}$ is countable.

B.53: Prove that the collection of all finite subsets of $\mathbb{N}$ is countable.

B.54: Prove that the collection Z of all infinite sequences consisting entirely of zeros and ones is uncountable. Hint: mimick the diagonalization argument used in Theorem B2.

B.55: Let $2^{\mathbb{N}}$ denote the set of all subsets of $\mathbb{N}$. Prove that $2^{\mathbb{N}}$ is uncountable. Hint: see Problem B.54 above. Show that $2^{\mathbb{N}}$ can be put in one-to-one correspondance with Z by associating to each $S \in 2^{\mathbb{N}}$ the sequence $x \in Z$ defined by

$$x_j = \begin{cases} 1 & \text{if } j \in S \\ 0 & \text{if } j \notin S. \end{cases}$$

B.56: Let Z_0 denote the collection of sequences in Z (defined in Problem B.54) that contain finitely many ones. Prove that Z_0 is countable.