12.1: Three-dimensional space

Rectangular Coordinates
The \( xy \)-plane is sometimes called \( \mathbb{R}^2 \), **two-dimensional space**, or simply **two space**, because any point in the plane can be represented by its two rectangular coordinates \((x, y)\) (also called Cartesian coordinates).

To define rectangular coordinates \((x, y, z)\) for **three-dimensional space** (or \( \mathbb{R}^3 \), or **three space**), picture the \( z \)-axis perpendicular to the \( xy \)-plane and passing through the origin.

![Diagram of 3D space with axes](image)

**Tip:** draw the axes in three space from a point of view in the first octant (where \( x, y, \) and \( z \) are positive).

Some authors draw the same three axes rotated to look like the picture to the right of this paragraph. To see that this and figure 12.1.2 are the same after a rotation, check that both are a **right-handed system**: when pointing your right index finger in the direction of the positive \( x \)-axis and your right middle finger in the direction of the positive \( y \)-axis, your right thumb points in the direction of the positive \( z \)-axis.

Implicit equations of curves and surfaces
The graph in the \( xy \)-plane of an equation is often (but not always) a line or curve.
The graph in \( xyz \)-space of a single equation is often (but not always) a surface.
The graph of a system of two \( xyz \)-equations is often (b.n.a.) a curve.

12.1.1 Find the equation(s) of the given coordinate plane or axes.

a. the \( xy \)-plane  
b. the \( xz \)-plane  
c. the \( yz \)-plane  
d. the \( x \)-axis  
e. the \( y \)-axis  
f. the \( z \)-axis
Orthogonal Projection

The orthogonal projection of a point onto a line (or plane) is the point on that line (or plane) closest to the given point. The line segment to the point from its projection is orthogonal to the line (or plane). Projecting onto one of the coordinate axes or planes is just a matter of resetting some coordinates equal to zero.

12.1.re2. Find the projection of the given point onto the coordinate line or plane.
   a. $(2, -3, 4)$ onto the $xy$-plane  
   b. $(2, -3, 4)$ onto the $xz$-plane  
   c. $(2, -3, 4)$ onto the $yz$-plane  
   d. $(2, -3, 4)$ onto the $x$-axis  
   e. $(2, -3, 4)$ onto the $y$-axis  
   f. $(2, -3, 4)$ onto the $z$-axis

More on orthogonal projection onto lines in section 12.3.

Distance in three space

The distance from a point $(x, y, z)$ to the origin is

$$\sqrt{x^2 + y^2 + z^2}.$$  

The distance from one point $(x_0, y_0, z_0)$ to another $(x_1, y_1, z_1)$ is

$$\sqrt{(x_1 - x_0)^2 + (y_1 - y_0)^2 + (z_1 - z_0)^2}.$$  

12.1.re3. Find an equation of the sphere centered at $(2, -1, 3)$ having radius 5.

12.1.re4. Find the center and radius of the given sphere.
   a. $x^2 + (y - 3)^2 + (z + 2)^2 = 25$  
   b. $x^2 + 2x + y^2 - 4y + z^2 - 6z = 2$  
   c. $x^2 - x + y^2 + 3y + z^2 + 5z = \frac{1}{4}$
12.1.re5. Which of the axes below are the same as in figure 12.1.2?

\[ \begin{align*}
\text{a.} & \quad \text{y} & \quad \text{z} & \quad \text{x} \\
\text{b.} & \quad \text{x} & \quad \text{z} & \quad \text{y} \\
\text{c.} & \quad \text{z} & \quad \text{y} & \quad \text{x} \\
\text{d.} & \quad \text{z} & \quad \text{x} & \quad \text{y} \\
\text{e.} & \quad \text{y} & \quad \text{x} & \quad \text{z} \\
\text{f.} & \quad \text{y} & \quad \text{z} & \quad \text{x}
\end{align*} \]

Answers

12.1.re1a. \( z = 0 \)  12.1.re1b. \( y = 0 \)  12.1.re1c. \( x = 0 \)  12.1.re1d. \( y = 0 \) and \( z = 0 \)  12.1.re1e. \( x = 0 \) and \( z = 0 \)  12.1.re1f. \( x = 0 \) and \( y = 0 \)  12.1.re2a. \((2,-3,0)\)  12.1.re2b. \((2,0,4)\)  12.1.re2c. \((0,-3,4)\)  12.1.re2d. \((2,0,0)\)  12.1.re2e. \((0,-3,0)\)  12.1.re2f. \((0,0,4)\)  12.1.re3. \((x-2)^2 + (y+1)^2 + (z-3)^2 = 25\)  12.1.re4a. \( \text{ctr} = (0,3,-2), \ \text{rad} = 5 \)  12.1.re4b. \( \text{ctr} = (-1,2,3), \ \text{rad} = 4 \)  12.1.re4c. \( \text{ctr} = \left(\frac{1}{4},-\frac{3}{4},-\frac{5}{4}\right), \ \text{rad} = 3 \)  12.1.re5. a,e,f.
12.2: Vectors

In Calculus III, a vector is a directed line segment in \( \mathbb{R}^2 \) or \( \mathbb{R}^3 \). A same vector can have different initial and terminal points.

12.2.re1. The vector from the initial point \((0, 0, 0)\) to the terminal point \((2, 3, 4)\) is denoted \(\langle 2, 3, 4 \rangle\). The vector with initial point \((2, 4, -1)\) and terminal point \((4, 7, 3)\) is the same:

\[
\langle 4 - 2, 7 - 4, 3 - (-1) \rangle = \langle 2, 3, 4 \rangle
\]

A vector-valued variable is usually denoted either in bold or with an arrow: \(\mathbf{u}\) or \(\vec{u}\). To distinguish them from vectors, real numbers and real-valued variables are called scalars (which can be used either either a noun or an adjective).

12.2.re2. \(x, y, \) and \(z\) are scalar variables. \(\mathbf{r} = \langle x, y, z \rangle\) is a vector-valued variable.

**Vector Arithmetic**

**Vector Addition** is the addition of vectors component by component. **Scalar Multiplication** is the multiplication of every component of a vector by a scalar.

12.2.re3. \[
\langle 2, 1 \rangle + \langle 3, -2 \rangle = \langle 5, -1 \rangle \\
3\langle 2, 1 \rangle = \langle 6, 3 \rangle \\
-\frac{1}{2}\langle 2, 1 \rangle = \langle -1, -\frac{1}{2} \rangle
\]

If the initial point of \(\mathbf{v}\) is placed at the terminal point of \(\mathbf{u}\), then \(\mathbf{u} + \mathbf{v}\) reaches from the initial point of \(\mathbf{u}\) to the terminal point of \(\mathbf{v}\).

12.2.re4. Express the vector \(\mathbf{?}\) shown in the figure in terms of \(\mathbf{u}\) and \(\mathbf{v}\). (Hint: what is \(\mathbf{?} + \mathbf{v}\)?)

If \(\mathbf{u}, \mathbf{v}, \mathbf{w}\) are vectors and \(s\) and \(t\) are scalars, then

- a. \(\mathbf{u} + \mathbf{v} = \mathbf{v} + \mathbf{u}\)  
- b. \(\mathbf{u} + 0 = \mathbf{u}\)  
- c. \(s(\mathbf{v} + \mathbf{w}) = s\mathbf{v} + s\mathbf{w}\)  
- d. \(s(t\mathbf{u}) = (st)\mathbf{u}\)  
- e. \(\mathbf{u} + (\mathbf{v} + \mathbf{w}) = (\mathbf{u} + \mathbf{v}) + \mathbf{w}\)  
- f. \(\mathbf{u} + (-\mathbf{u}) = 0\)  
- g. \((s + t)\mathbf{u} = s\mathbf{u} + t\mathbf{u}\)  
- h. \(1\mathbf{u} = \mathbf{u}\)
Some Special Vectors

in $\mathbb{R}^3$:  
$\mathbf{0} = \langle 0, 0, 0 \rangle$  
$\mathbf{i} = \langle 1, 0, 0 \rangle$  
$\mathbf{j} = \langle 0, 1, 0 \rangle$  
$\mathbf{k} = \langle 0, 0, 1 \rangle$

in $\mathbb{R}^2$:  
$\mathbf{0} = \langle 0, 0 \rangle$  
$\mathbf{i} = \langle 1, 0 \rangle$  
$\mathbf{j} = \langle 0, 1 \rangle$

Always be careful to distinguish between the vector $\mathbf{0}$ and the scalar $0$.

Every vector can be written uniquely as $a\mathbf{i} + b\mathbf{j} + c\mathbf{k}$ for some scalars $a, b, c$.

12.2.re.5. Express the given vector in terms of $\mathbf{i}, \mathbf{j},$ and $\mathbf{k}$.

a. $\langle 2, 3, -1 \rangle$  
b. $\langle -1, 0, 1 \rangle$  
c. $\langle 3, \pi \rangle$

Magnitude

If $\mathbf{u} = \langle u_1, u_2, u_3 \rangle$, the magnitude or length of $\mathbf{u}$ is

$$|\mathbf{u}| = \sqrt{u_1^2 + u_2^2 + u_3^2}$$

If $\mathbf{u}$ is a vector and $s$ is a scalar, then

$$|s\mathbf{u}| = |s||\mathbf{u}|$$

A unit vector is a vector of length 1. If $\mathbf{u} \neq \mathbf{0}$, the unit vector

$$\frac{1}{|\mathbf{u}|}\mathbf{u}$$

is the normalization of $\mathbf{u}$.

12.2.re.6. Let

$\mathbf{u} = \langle 2, 1, -2 \rangle$  
$\mathbf{v} = \langle 5, 2, -1 \rangle$  
$\mathbf{w} = \langle 3, -4 \rangle$  
$\mathbf{p} = 5\mathbf{i} - 4\mathbf{j} - 2\mathbf{k}$  
$\mathbf{q} = -2\mathbf{i} + 4\mathbf{j} - \mathbf{k}$

and find the following, if they exist.

a. $\mathbf{u} - 2\mathbf{v}$  
b. $3\mathbf{u} - \mathbf{v} + 2\mathbf{p}$  
c. $3\mathbf{u} - 2\mathbf{w}$  
d. $\mathbf{u} - 2$

e. $\mathbf{v} - 2\mathbf{i}$  
f. $|\mathbf{v}|$  
g. $|2\mathbf{u}|$  
h. $| - 2\mathbf{w} | + |\mathbf{w}|$

i. the normalization of $\mathbf{u}$

j. a vector of length 3 in the opposite direction of $\mathbf{w}$

Vectors in Physics

Vectors are used to model things that have magnitude and direction.

12.2.re.7. A ship sails due west with speed 4 knots. Taking the magnitude of velocity to be speed, and assuming $\mathbf{j}$ and $\mathbf{i}$ represent the directions north and east, resp., find a vector representing the velocity of the ship.

12.2.re.8. Find the vector representing a force with magnitude 3 Newtons in the direction of $\langle 7, -4, -4 \rangle$.

Answers

12.2.re.4. $\mathbf{u} - \mathbf{v}$  
12.2.re.5a. $2\mathbf{i} + 3\mathbf{j} - \mathbf{k}$  
12.2.re.5b. $-\mathbf{i} + \mathbf{k}$  
12.2.re.5c. $3\mathbf{i} + \pi\mathbf{j}$  
12.2.re.6a. $\langle -8, -3, 0 \rangle$  
12.2.re.6b. $\langle 11, -7, -9 \rangle$  
12.2.re.6c. dne.  
12.2.re.6d. dne.  
12.2.re.6e. $3\mathbf{i} + 2\mathbf{j} - \mathbf{k}$  
12.2.re.6f. $\sqrt{30}$  
12.2.re.6g. 6  
12.2.re.6h. 15  
12.2.re.6i. $\langle \frac{2}{3}, \frac{1}{3}, -\frac{4}{3} \rangle$  
12.2.re.6j. $\langle -\frac{2}{5}, \frac{42}{5} \rangle$  
12.2.re.7. $-4\mathbf{i}$  
12.2.re.8. $\langle \frac{7}{5}, -\frac{4}{5}, \frac{4}{5} \rangle$. 

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12.3: The Dot Product

**Definition.** The **dot product** of the vectors \( \mathbf{u} = \langle u_1, u_2, u_3 \rangle \) and \( \mathbf{w} = \langle w_1, w_2, w_3 \rangle \) is

\[
\mathbf{u} \cdot \mathbf{w} = u_1 w_1 + u_2 w_2 + u_3 w_3.
\]

The dot product is sometimes called the **inner product** or **scalar product**.

**12.3.1.**

a. \( \langle 4, 3 \rangle \cdot \langle 1, 5 \rangle = 4 \cdot 1 + 3 \cdot 5 = 19 \).

b. \( \langle 2, -4, 3 \rangle \cdot \langle 3, 3, 2 \rangle = 2 \cdot 3 - 4 \cdot 3 + 3 \cdot 2 = 0 \).

If \( \mathbf{u}, \mathbf{v}, \mathbf{w} \) are vectors and \( s \) is a scalar and \( \theta \) is the angle between \( \mathbf{u} \) and \( \mathbf{v} \), then

\[
\begin{align*}
\text{a. } &0 \cdot \mathbf{u} = 0 & \text{e. } &\mathbf{u} \cdot \mathbf{u} = |\mathbf{u}|^2 \\
\text{b. } &\mathbf{u} \cdot \mathbf{v} = \mathbf{v} \cdot \mathbf{u} & \text{f. } &\mathbf{u} \cdot \mathbf{v} = ||\mathbf{u}|| ||\mathbf{v}|| \cos \theta \\
\text{c. } &\mathbf{u} \cdot (\mathbf{v} + \mathbf{w}) = \mathbf{u} \cdot \mathbf{v} + \mathbf{u} \cdot \mathbf{w} & \text{g. } &\mathbf{u} \perp \mathbf{v} \text{ iff } \mathbf{u} \cdot \mathbf{v} = 0 \\
\text{d. } &(s\mathbf{u}) \cdot \mathbf{v} = s(\mathbf{u} \cdot \mathbf{v}) = \mathbf{u} \cdot (s\mathbf{v}) & \text{h. } &\mathbf{u} \cdot \mathbf{v} > 0 \text{ iff } 0 \leq \theta < \pi/2 \\
\text{ } & & &\mathbf{u} \cdot \mathbf{v} < 0 \text{ iff } \pi/2 < \theta \leq \pi
\end{align*}
\]

**Orthogonal Projection**

If \( \mathbf{u} \) and \( \mathbf{v} \) are vectors, and \( \mathbf{v} \neq \mathbf{0} \), then the vector

\[
\text{proj}_\mathbf{v} \mathbf{u} = \frac{\mathbf{u} \cdot \mathbf{v}}{\mathbf{v} \cdot \mathbf{v}} \mathbf{v}
\]

is called the **orthogonal projection** or **vector projection** of \( \mathbf{u} \) onto \( \mathbf{v} \), and the scalar

\[
\text{comp}_\mathbf{v} \mathbf{u} = \frac{\mathbf{u} \cdot \mathbf{v}}{||\mathbf{v}||}
\]

is the called the **scalar projection** of \( \mathbf{u} \) onto \( \mathbf{v} \), or the **component** of \( \mathbf{u} \) in the direction of \( \mathbf{v} \).

The vector projection \( \text{proj}_\mathbf{v} \mathbf{u} \) is the “shadow” of \( \mathbf{u} \) cast on the line spanned by \( \mathbf{v} \) by a ray of light orthogonal to \( \mathbf{v} \). The scalar projection \( \text{comp}_\mathbf{v} \mathbf{u} \) is the signed length of \( \text{proj}_\mathbf{v} \mathbf{u} \).

\[
\begin{align*}
\text{comp}_\mathbf{v} \mathbf{u} > 0 & \text{ iff } \mathbf{u} \cdot \mathbf{v} > 0 & \text{iff } \text{proj}_\mathbf{v} \mathbf{u} \text{ is in same direction as } \mathbf{v} \\
\text{comp}_\mathbf{v} \mathbf{u} < 0 & \text{ iff } \mathbf{u} \cdot \mathbf{v} < 0 & \text{iff } \text{proj}_\mathbf{v} \mathbf{u} \text{ is in opposite direction as } \mathbf{v}
\end{align*}
\]
12.3.re2. Let 
\[ u = \langle 1, 2, 2 \rangle \quad v = \langle 3, 0, -2 \rangle \quad w = \langle 2, 1, -1 \rangle \quad p = \langle -2, 1, 0 \rangle \]
and find the following, if they exist.

- \( \text{proj}_u v \)
- \( \text{comp}_u v \)
- \( \text{proj}_v u \)
- \( \text{comp}_v u \)
- \( \text{proj}_u w \)
- \( \text{proj}_u p \)
- \( u \cdot (w - \text{proj}_u w) \)
- \( w \cdot (w - \text{comp}_u w) \)

12.3.re3. Use orthogonal projection to find the point on the line closest to the given point.

- \((2, 3), y = x\)
- \((1, -1), 2y + 3x = 0\)

Work
The work done by a constant force \( F \) moving an object along a straight line is
\[ W = F \cdot D \]
where \( D \) is the change in position, or displacement, of the object.

12.3.re4. Find the work done by a force of magnitude 3 N in the direction of \( \langle 8, -4, 1 \rangle \) in moving an object in a straight line from the point \((1, 0, 1)\) to the point \((3, 1, -1)\). (Assume coordinates are measured in meters).

Answers
12.3.re2a. \((-\frac{1}{9}, -\frac{2}{9}, -\frac{2}{9})\) 12.3.re2b. \(-\frac{1}{9}\) 12.3.re2c. \((-\frac{10}{13}, 0, \frac{5}{13})\) 12.3.re2d. \(-\frac{1}{\sqrt{13}}\)
12.3.re2e. \((\frac{2}{9}, \frac{4}{9}, \frac{1}{9})\) 12.3.re2f. \(0\) 12.3.re2g. \(0\) 12.3.re2h. dne. 12.3.re3a. \((\frac{5}{2}, \frac{5}{2})\)
12.3.re3b. \((\frac{10}{13}, -\frac{15}{13})\) 12.3.re4. \( \mathbf{F} = \frac{3}{2} \langle 8, -4, 1 \rangle \) \( \mathbf{D} = \langle 2, 1, -2 \rangle \) Work = \( \frac{10}{3} \) ft-lbs.
12.4: The Cross Product

**Definition.** The **cross product** of the vectors \( \mathbf{u} = \langle u_1, u_2, u_3 \rangle \) and \( \mathbf{w} = \langle w_1, w_2, w_3 \rangle \) is the determinant

\[
\mathbf{u} \times \mathbf{w} = \begin{vmatrix}
  \mathbf{i} & \mathbf{j} & \mathbf{k} \\
  u_1 & u_2 & u_3 \\
  w_1 & w_2 & w_3
\end{vmatrix},
\]

which can be calculated by expansion along the top row:

\[
= \mathbf{i} \begin{vmatrix}
  u_2 & u_3 \\
  w_2 & w_3
\end{vmatrix} - \mathbf{j} \begin{vmatrix}
  u_1 & u_3 \\
  w_1 & w_3
\end{vmatrix} + \mathbf{k} \begin{vmatrix}
  u_1 & u_2 \\
  w_1 & w_2
\end{vmatrix},
\]

12.4.re1. \((2\mathbf{i} + \mathbf{j} - 3\mathbf{k}) \times (-\mathbf{i} - \mathbf{j} + \mathbf{k}) =
\[
\begin{vmatrix}
  \mathbf{i} & \mathbf{j} & \mathbf{k} \\
  2 & 1 & -3 \\
  -1 & -1 & 1
\end{vmatrix} = \mathbf{i} \begin{vmatrix}
  1 & -3 \\
  -1 & 1
\end{vmatrix} - \mathbf{j} \begin{vmatrix}
  2 & -3 \\
  -1 & 1
\end{vmatrix} + \mathbf{k} \begin{vmatrix}
  2 & 1 \\
  -1 & -1
\end{vmatrix}
\]

\[
= \mathbf{i}(1 \cdot 1 - (-1)(-3)) - \mathbf{j}(2 \cdot 1 - (-1)(-3)) + \mathbf{k}(2(-1) - (-1)1)
\]

\[
= \langle -2, 1, -1 \rangle
\]

12.4.re2. Find the following, if they exist.

a. \( \langle 2, -4, 1 \rangle \times \langle 1, 0, 1 \rangle \)  
   b. \( \langle 1, 0, 1 \rangle \times \langle 2, -4, 1 \rangle \)  
   c. \( \langle 2, -4, 1 \rangle \times \langle -4, 8, -2 \rangle \)  
   d. \( \langle 2, -4, 1 \rangle \times \mathbf{k} \)  
   e. \( \langle 2, -4, 1 \rangle \times \mathbf{i} \)  
   f. \( \mathbf{i} \times \mathbf{j} \)

If \( \mathbf{u}, \mathbf{v}, \mathbf{w} \) are vectors and \( s \) is a scalar and \( \theta \) is the angle between \( \mathbf{u} \) and \( \mathbf{v} \), then

a. \( \mathbf{u} \cdot (\mathbf{u} \times \mathbf{v}) = \mathbf{v} \cdot (\mathbf{u} \times \mathbf{v}) = 0 \)
   
   d. \( (s\mathbf{u}) \times \mathbf{v} = s(\mathbf{u} \times \mathbf{v}) = \mathbf{u} \times (s\mathbf{v}) \)

b. \( |\mathbf{u} \times \mathbf{v}| = |\mathbf{u}| |\mathbf{v}| \sin \theta \)
   
   e. \( \mathbf{u} \times (\mathbf{v} + \mathbf{w}) = \mathbf{u} \times \mathbf{v} + \mathbf{u} \times \mathbf{w} \)

f. \( \mathbf{u}, \mathbf{v}, \mathbf{w} \) is a right-handed system,

a. means that \( \mathbf{u} \times \mathbf{v} \) is orthogonal to both \( \mathbf{u} \) and \( \mathbf{v} \). b. implies that \( |\mathbf{u} \times \mathbf{v}| \) is the area of the parallelogram with sides \( \mathbf{u} \) and \( \mathbf{v} \), and that \( \mathbf{u} \times \mathbf{v} = \mathbf{0} \) iff \( \mathbf{u} = \mathbf{0}, \mathbf{v} = \mathbf{0}, \) or \( \mathbf{u} \parallel \mathbf{v} \).

f. means that when you point to \( \mathbf{u} \) with your open right hand, and then curl your fingers closed in the direction of \( \mathbf{v} \), your thumb points in the direction of \( \mathbf{u} \times \mathbf{v} \). For instance, \( \mathbf{i}, \mathbf{j}, \mathbf{k} \) is a right-handed system.

The **triple product** of the vectors \( \mathbf{u} = \langle u_1, u_2, u_3 \rangle, \mathbf{v} = \langle v_1, v_2, v_3 \rangle, \) and \( \mathbf{w} = \langle w_1, w_2, w_3 \rangle \) is

\[
\mathbf{u} \cdot (\mathbf{v} \times \mathbf{w}) = \begin{vmatrix}
  u_1 & u_2 & u_3 \\
  v_1 & v_2 & v_3 \\
  w_1 & w_2 & w_3
\end{vmatrix}
\]

\( |\mathbf{u} \cdot (\mathbf{v} \times \mathbf{w})| \) is the volume of the parallelepiped with sides \( \mathbf{u}, \mathbf{v}, \mathbf{w} \).
12.4.re3. Find the area of the parallelogram with the given vertices.
   a. \((0, 0, 0), (3, 1, 2), (2, -1, 4), (5, 0, 6)\)
   b. \((7, -1), (12, 2), (3, 1), (8, 4)\)

12.4.re4. Find the volume of the parallelepiped with edges \(u, v, w\).
   \[
   u = i + j \quad v = j - k \quad w = i - 2j + k
   \]

12.4.re5. Let \(u = \langle 1, 2, 2 \rangle, v = \langle 3, 0, -2 \rangle, w = \langle 2, 1, -1 \rangle\) and find the following, if they exist.
   a. \(u \times v\)
   b. \(v \times w\)
   c. \(v \times (w \times w)\)
   d. \(u \cdot (v \times w)\)
   e. \(u \cdot (v \times u)\)
   f. \((u \cdot v) \cdot (v \cdot w)\)
   g. \((u \times v) \times (v \times w)\)
   h. \(\text{comp}_{v \times w} u\)

**Torque**

The torque of a force vector \(\mathbf{F}\) and position vector \(\mathbf{r}\) is defined to be \(\mathbf{\tau} = \mathbf{r} \times \mathbf{F}\). Torque can be thought of as the magnitude and direction of a turning force acting on a (right-handed) bolt at the origin when the force \(\mathbf{F}\) is applied to a wrench \(\mathbf{r}\).

12.4.re6. A wrench 0.5 m long lies along the line \(y = x\) in quadrant I in the \(xy\)-plane and grips a bolt at the origin. A force of magnitude 2 N in the direction \(3\mathbf{i} + 4\mathbf{j}\) applied to the end of the wrench. Find the magnitude of the torque applied to the bolt.

Answers

12.4.re2a. \((-4, -1, 4)\) 12.4.re2b. \((4, 1, -4)\) 12.4.re2c. \(0\) 12.4.re2d. \((-4, -2, 0)\) 12.4.re2e. \((0, 1, 4)\)
12.4.re2f. \(\mathbf{k}\) 12.4.re3a. \(5\sqrt{5}\) 12.4.re3b. 22 12.4.re4. 2 12.4.re5a. \((-4, 8, -6)\) 12.4.re5b. \((2, -1, 3)\)
12.4.re5c. \(0\) 12.4.re5d. 6 12.4.re5e. 0 12.4.re5f. dne 12.4.re5g. \((18, 0, -12)\) 12.4.re5h. \(6/\sqrt{14}\)
12.4.re6. \(\frac{1}{2}\sqrt{2}\) Nm.
12.5: Equations of lines and planes

Lines

A line is determined by a point on the line a vector parallel the line.

12.5.re1. Find the equation(s) of the line passing through \((2, 1, -1)\) parallel to \(\langle 10, 9, 8 \rangle\).

Solution one: a point \((x, y, z)\) lies on the line iff the vector from \((2, 1, -1)\) to \((x, y, z)\) is parallel to \(\langle 10, 9, 8 \rangle\):

\[
\begin{aligned}
\langle x, y, z \rangle - \langle 2, 1, -1 \rangle &= t\langle 10, 9, 8 \rangle \\
\langle x, y, z \rangle &= \langle 2, 1, -1 \rangle + t\langle 10, 9, 8 \rangle
\end{aligned}
\]

for some value of \(t\), so the vector-valued function

\[
r(t) = \langle 2, 1, -1 \rangle + t\langle 10, 9, 8 \rangle,
\]

or \(\langle 2 + 10t, 1 + 9t, -1 + 8t \rangle\), will trace out the line as \(t\) goes from \(-\infty\) to \(\infty\). This is called a (parametric) vector form of the line.

Solution two: setting \(x, y, \) and \(z\) equal the components of \(r\) gives

\[
\begin{aligned}
x &= 2 + 10t \\
y &= 1 + 9t \\
z &= -1 + 8t
\end{aligned}
\]

Together, these three equations are called a parametric form of the line.

Solution three: solving for \(t\) in terms of \(x, y, \) and \(z\) and setting these expressions equal gives

\[
\frac{x - 2}{10} = \frac{y - 1}{9} = \frac{z + 1}{8}
\]

These two equations are called a symmetric form of the line.

12.5.re2. Find an equation of the given line.

a. Through \((2, 0, 3)\), parallel \(\langle 1, 2, 3 \rangle\).  
b. Through the points \((2, 0, 3)\) and \((9, 1, 5)\).  
c. Through \((2, 8, 3)\), parallel \(\langle 0, 4, 5 \rangle\).

12.5.re3. Find a point on and a vector parallel to the given line. (There are many correct answers.)

a. \(x = 2 - t, \ y = 10 - t, \ z = 2 + 5t\).  
b. \(\frac{x - 3}{4} = y - 2 = 2z + 5\)  
c. \(r = \langle 2 + 3t, 3 - t, 8 \rangle\).
Planes

A plane is determined by a point on the plane a vector orthogonal to the plane (called a **normal vector**).

12.5.re4. Find an equation of the plane passing through \((7, 8, -9)\) normal to \((2, 3, 4)\).

Solution: a point \((x, y, z)\) lies on the plane iff the vector from \((7, 8, -9)\) to \((x, y, z)\) is orthogonal to \((2, 3, 4)\):

\[
\langle 2, 3, 4 \rangle \perp \langle x - 7, y - 8, z + 9 \rangle,
\]

which is true iff their dot product is zero. Therefore, the plane is the solution set to the equation

\[
2(x - 7) + 3(y - 8) + 4(z + 9) = 0.
\]

There are other equations for the same plane, obtainable from this one by some algebra, e.g.

\[
2x + 3y + 4z = 2.
\]

12.5.re5. Find an equation of the given plane.

a. Through the point \((2, 1, 0)\) and normal to \((1, 2, 3)\).

b. Through the points \((2, 1, 0), (3, 2, 1)\), and \((9, 1, 5)\).

c. Through the points \((2, 1, 0)\) and parallel the plane \(x - 5z = 10\).

d. The plane containing \((0, 1, -1)\) and the line \(x = 1 + t, y = 2t - 1, z = 3t\).

12.5.re6. Find a point on and a vector normal to the given plane. (There are many correct answers.)

a. \(3x + 10y + 5z = 6\).

b. \(2x + 1 = 4y - z\)

12.5.re7. Does the given equation(s) describe a line or a plane?

a. \(x = 4t, y = 2 - t, z = 5 - 6t\).

b. \(\frac{x - 2}{3} = 2y - 2 = \frac{z + 1}{5}\)

c. \(4x - y - 6z = 0\)

12.5.re8. Find the point of intersection, if there is one.

a. The line \(x = 1 - t, y = 2t - 1, z = 3 + 2t\) and the plane \(2x + 3y - z = -6\)

b. The lines \(x = 1 + 3t, y = 2 - 4t, z = 4 + t\) and \(x = 3 + 2s\) \(y = -s + 1, z = -2s + 1\).

2 = 4x - y - 6z = 0

Answers

12.5.re2a. vector form: \(\mathbf{r} = (2 + t, 2t, 3 + 3t)\). symmetric form: \(x - 2 = \frac{t}{3} = \frac{z - 3}{3}\).

12.5.re2b. vector form: \(\mathbf{r} = (2 + 7t, t, 3 + 2t)\). symmetric form: \(\frac{x - 2}{7} = \frac{y - t}{1} = \frac{z - 3}{2}\).

12.5.re2c. vector form: \(\mathbf{r} = (2, 8 + 4t, 3 + 5t)\).

symmetric form: \(x = 2; \ \frac{y - 8}{4} = \frac{z - 3}{5}\).

12.5.re3a. \((2, 10, 2); \ (1, -1, -1)\).

12.5.re3b. \((3, 2, -\frac{2}{3}); \ (4, 1, \frac{2}{3})\).

12.5.re3c. \((5, 2, 8)\) (when \(t = 1)\), \((3, -1, 0)\).

12.5.re5a. \(x - 2 + 2(y - 1) + 3z = 0, \ or \ x + 2y + 3z = 4\).

12.5.re5b. \(5x + 2y - 7z = 12\).

12.5.re5c. \(x - 5z = 2\).

12.5.re5d. \(4x + y - 2z = 3\).

12.5.re6a. \((2, 0, 0)\) \((3, 10, 5)\).

12.5.re6b. \((1, 0, -3); \ (2, -4, 1)\). 12.5.re7a. line (in parametric form).

12.5.re7b. line (in symmetric form).

12.5.re7c. plane.

12.5.re8a. \((x, y, z) = (2, -3, 1)\) \((at \ t = -1)\). 12.5.re8b. none.

12.5.re8c. both \(= (1, 2, 4)\) at \(t = -2, s = 1\).
12.6: Cylinders and Quadratic Surfaces

See [http://kunklet.people.cofc.edu/MATH221/transformations221.pdf](http://kunklet.people.cofc.edu/MATH221/transformations221.pdf) for a review of how changes to an equation change the corresponding graph.

**Elementary conic sections in the xy-plane**

1. **Parabolas**
   \[ y = kx^2 \]
   \[ x = ky^2 \]

   ![Parabola graphs](image)

2. **Ellipses**
   \[ \frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \]
   
   semimajor axis = \( \max\{a, b\} \)
   semiminor axis = \( \min\{a, b\} \)

   ![Ellipse graph](image)

3. **Hyperbolas**
   \[ \frac{x^2}{a^2} - \frac{y^2}{b^2} = 1 \]
   \[ \frac{y^2}{b^2} - \frac{x^2}{a^2} = 1 \]

   ![Hyperbola graphs](image)

For practice with conics, see [http://kunklet.people.cofc.edu/MATH221/stew1005prob.pdf](http://kunklet.people.cofc.edu/MATH221/stew1005prob.pdf).
Cylinders

A cylinder is a surface obtained by dragging a planar curve in the direction perpendicular to its plane. Any equation in \( x, y \), or \( x, z \), or \( y, z \) generates a cylinder in \( xyz \) space.

12.6.re1. The graph of \( x^2 + y^2 = 1 \) is a circle in the \( xy \) plane, where \( z = 0 \). Since the equation is independent of \( z \), its graph is the (right circular) cylinder made up of copies of the same circle at all other \( z \)-values. (below left)

12.6.re2. The graph of \( z = 1 - y^2 \) is a parabola in the \( yz \)-plane \( x = 0 \). Dragging this curve in the \( x \)-direction generates the graph of the equation in \( xyz \)-space. (above right)

12.6.re3. Sketch the graph of the given equation.

a. \( z = \sin y \)    b. \( xy = -1 \)    c. \( y^2 - z^2 = 4 \)    d. \( x = 2z - z^2 \)
Quadratic surfaces

12.6.re4. \( z = x^2 + y^2 \)

<table>
<thead>
<tr>
<th>When</th>
<th>equation is</th>
<th>cross-section is</th>
</tr>
</thead>
<tbody>
<tr>
<td>( x = 0 )</td>
<td>( z = y^2 )</td>
<td>a parabola;</td>
</tr>
<tr>
<td>( y = 0 )</td>
<td>( z = x^2 )</td>
<td>a parabola;</td>
</tr>
<tr>
<td>( z = \text{const.} )</td>
<td>( x^2 + y^2 = \text{const.} )</td>
<td>a circle.</td>
</tr>
</tbody>
</table>

**elliptic paraboloid**

12.6.re5. \( x^2 + y^2 = z^2 \)

<table>
<thead>
<tr>
<th>When</th>
<th>equation is</th>
<th>cross-section is</th>
</tr>
</thead>
<tbody>
<tr>
<td>( x = 0 )</td>
<td>( y^2 = z^2 )</td>
<td>a pair of lines ( z = \pm y );</td>
</tr>
<tr>
<td>( y = 0 )</td>
<td>( x^2 = z^2 )</td>
<td>a pair of lines ( z = \pm x );</td>
</tr>
<tr>
<td>( z = \text{const.} )</td>
<td>( x^2 + y^2 = \text{const.} )</td>
<td>a circle.</td>
</tr>
</tbody>
</table>

**elliptical cone**

12.6.re6. \( x^2 + y^2 + \frac{4}{9}z^2 = 1 \)

<table>
<thead>
<tr>
<th>When</th>
<th>equation is</th>
<th>cross-section is</th>
</tr>
</thead>
<tbody>
<tr>
<td>( x = 0 )</td>
<td>( y^2 + \frac{4}{9}z^2 = 1 )</td>
<td>an ellipse;</td>
</tr>
<tr>
<td>( y = 0 )</td>
<td>( x^2 + \frac{4}{9}z^2 = 1 )</td>
<td>an ellipse;</td>
</tr>
<tr>
<td>( z = \text{const.} )</td>
<td>( x^2 + y^2 = \text{const.} )</td>
<td>a circle.</td>
</tr>
</tbody>
</table>

Tip: graph of \( x^2 + y^2 + \left(\frac{z}{3/2}\right)^2 = 1 \) is obtained from unit sphere \( x^2 = y^2 = z^2 = 1 \) by scaling in \( z \)-direction by a factor of \( 3/2 \).
### 12.6.re7. \( x^2 + y^2 - z^2 = 1 \)

<table>
<thead>
<tr>
<th>When</th>
<th>Equation is</th>
<th>Cross-section is</th>
</tr>
</thead>
<tbody>
<tr>
<td>( x = 0 )</td>
<td>( y^2 - z^2 = 1 )</td>
<td>a hyperbola (( y \neq 0 ));</td>
</tr>
<tr>
<td>( y = 0 )</td>
<td>( x^2 - z^2 = 1 )</td>
<td>a hyperbola (( x \neq 0 ));</td>
</tr>
<tr>
<td>( z = \text{const.} )</td>
<td>( x^2 + y^2 = \text{const.} )</td>
<td>a circle.</td>
</tr>
</tbody>
</table>
See the Table 1 in 12.6 for a summary of quadratic surfaces in general. These six quadratic surfaces can always be draw in the positions shown above after shifting (possibly by completing the square) and rotating axes (as in example 12.1.re5)

12.6.re10. Describe and sketch the graph of the given equation.

a. $x^2 + 4y^2 + 8y + z^2 - 2z = 4$

b. $4x^2 - z^2 = y^2$

c. $-x^2 + y^2 - z^2 = 9$

d. $-x^2 + y^2 + 2y - z^2 = 0$

e. $x^2 + 4y^2 = 4$

f. $-x^2 + y^2 + z^2 = 9$

g. $-x^2 + 4x + y^2 - z^2 = 3$

h. $x^2 = z^2 - y$

i. $2 = -x + y^2 + 4z^2$

d. $0 = x^2 - 4y^2 + 4z^2$

j. $0 = x^2 - 4y^2 + 4z^2$

Answers

12.6.re3. a.,b.,c.,d.:

12.6.re10a. ellipsoid centered at $(0, -1, 1)$. semi-axes are 3 in $x$ and $z$ directions and $\frac{3}{2}$ in $y$ direction.

12.6.re10b. circular cone; axis of symmetry is $x$-axis. 12.6.re10c. hyperboloid; $y \neq 0$, two sheets; axis is $y$-axis. 12.6.re10d. hyperboloid; $y \neq -1$ two sheets; centered at $(0, -1, 0)$. 12.6.re10e. equation without $z$, hence a cylinder. semi-axis in $x$ direction is 2; in $y$ direction is 1. axis of symmetry is $z$-axis.

12.6.re10f. hyperboloid; $y$ and $z$ not both $= 0$, so of one sheet; axis is $x$-axis. 12.6.re10g. hyperboloid of one sheet ($x, z$ not both $0$) centered at $(2, 0, 0)$. 12.6.re10h. hyperbolic paraboloid, includes the cross-sections $x = 0; y = z^2$ and $z = 0; y = -x^2$. 12.6.re10i. elliptic paraboloid with axis of symmetry $x$-axis and vertex $(-2, 0, 0)$. $x + 2 \geq 0$. 12.6.re10j. elliptic cone with axis of symmetry $y$-axis. Elliptical cross-sections at $y = \text{const.}$ are twice as long in $x$ direction as in $z$ direction. 12.6.re10k. equation without $y$, hence a cylinder; in fact, it consists of the two planes $x = 2z$ and $x = -2z$. 

13.1: Vector-valued functions and the representation of curves by equations

Vector-valued functions
When \( x = x(t), \ y = y(t), \) and \( z = z(t) \) are scalar-valued functions of the scalar variable \( t, \) then \( r(t) = \langle x(t), y(t), z(t) \rangle \) is a vector-valued function of \( t. \) Its domain is the set of \( t \)-values at which \( x(t), y(t), \) and \( z(t) \) are all well-defined.

13.1.re1. Find the domain of vector-valued function.
   a. \( \langle \sqrt{t-2}, \ln(5-t) \rangle \)
   b. \( \langle e^t, \frac{t+1}{t-1}, \sqrt{t} \rangle \)
   c. \( \langle \frac{1}{\sqrt{2t-3}}, \sin(t^2), \frac{2t-1}{t^2-5t+6} \rangle \)

Limits of vector-valued functions are computed component-wise.

13.1.re2. \( \lim_{t \to 1} \langle e^{t-1}, \frac{t^2-1}{t-1}, \frac{\ln t}{t-1} \rangle = \langle \lim_{t \to 1} e^{t-1}, \lim_{t \to 1} \frac{t^2-1}{t-1}, \lim_{t \to 1} \frac{\ln t}{t-1} \rangle. \)

The first of these three limits equals 1 by continuity. The second is the same as \( \lim_{t \to 1} \frac{(t+1)(t-1)}{t-1} = 2. \) The third, by l'Hospital’s Rule, is \( \lim_{t \to 1} \frac{t-1}{2t} = \frac{1}{2}. \) Therefore, the limit of \( r(t) \) is \( (1, 2, \frac{1}{2}) \).

13.1.re3. Calculate the limit
   a. \( \lim_{t \to 0} \langle \frac{\sin t}{t}, \frac{1-\cos t}{t^2}, \frac{1-\cos t}{t^2} \rangle \)
   b. \( \lim_{t \to 4} \langle \sin(t\pi), \cos((t+1)\frac{\pi}{2}), \frac{2-\sqrt{t}}{t-4} \rangle \)
   c. \( \lim_{h \to 0} \langle \frac{(t+h)^3-t^3}{h}, \frac{\ln(t+h)-\ln t}{h}, \frac{e^{-t-h}-e^{-t}}{h} \rangle \)

(Hint for part c: what is \( \lim_{h \to 0} \frac{f(t+h)-f(t)}{h} \)?)

Representations of curves by equations

<table>
<thead>
<tr>
<th>( \mathbb{R}^2 )</th>
<th>PARAMETRIC EQUATIONS</th>
<th>IMPLICIT EQUATIONS</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>( x = x(t) )</td>
<td>( f(x, y) = 0 )</td>
</tr>
<tr>
<td></td>
<td>( y = y(t) )</td>
<td></td>
</tr>
<tr>
<td>( \mathbb{R}^3 )</td>
<td>( x = x(t) )</td>
<td>( f(x, y, z) = 0 )</td>
</tr>
<tr>
<td></td>
<td>( y = y(t) )</td>
<td>( g(x, y, z) = 0 )</td>
</tr>
<tr>
<td></td>
<td>( z = z(t) )</td>
<td></td>
</tr>
</tbody>
</table>

13.1.re4. The unit circle in \( \mathbb{R}^2 \) can be expressed implicitly by \( x^2+y^2 = 1 \) and parametrically by \( x = \cos t, \ y = \sin t. \)
We sometimes express the parametric functions for $x$, $y$, and $z$ along a curve in a single vector-valued function $\mathbf{r}(t)$, as in the next example.

**13.1.re5.** The line in $\mathbb{R}^3$ passing through the point $(0, 1, -2)$ parallel to $\langle 3, -4, 5 \rangle$ can be expressed parametrically by

$$x = 3t, \quad y = 1 - 4t, \quad z = -2 + 5t,$$

or

$$\mathbf{r}(t) = \langle 3t, 1 - 4t, -2 + 5t \rangle.$$

The same line can be expressed implicitly by the two equations of its symmetric form

$$\frac{x}{3} = \frac{y - 1}{-4} = \frac{z + 2}{5},$$

or, if you prefer, $\frac{x}{3} + \frac{y - 1}{4} = 0$, $\frac{y - 1}{4} + \frac{z + 2}{5} = 0$.

---

**Sketching curves in space**

Sketching curves in space by hand is a worthwhile exercise (though, in practice, best left to machines). It often helps to identify the equation of a surface to which the curve belongs, that is, one of the equations of its implicit representation.

**13.1.re6.** Sketch the curve given parametrically by $\langle t, t^2, t^3 \rangle$.

It’s difficult to capture the shape of this curve in a single drawing. It might help to eliminate the parameter $t$ to obtain an $xy$ equation, an $xz$ equation, and a $yz$ equation. Then draw these curves in the three coordinate planes. These are views of the curve from the positive $z$-, negative $y$- and positive $x$-axes.

![Graphs of $y = x^2$, $z = x^3$, and $y = \sqrt[3]{z^2}$](image)
Based on these, we produce a sketch like the graph below left. To make the drawing clearer, include the cylinder \( y = x^2 \) on which the curve lies, below right.

13.1.re7. Describe and sketch the given parametrically by \( \mathbf{r}(t) \).

a. \( \langle t, \sin t, -t \rangle \)  
   b. \( \langle \sin t, \sin t, -\cos t \rangle \)  
   c. \( \langle t, 1 - t^2, 1 \rangle \)  
   d. \( \langle 2 \cos t, t, \sin t \rangle \)  
   e. \( \langle t \sin t, t \cos t, t \rangle \)  
   f. \( \langle t, 1 - t^2, t^2 \rangle \)

13.1.re8. Find a parametric representation of the curve given implicitly by the system of equations.

a. \( x + y = 1 \), \( x^2 - y^2 = z \)  
   b. \( z = (x - 1)^2 + y^2 \), \( x^2 + y^2 = 1 \)  
   c. \( xy = 1 \), \( z = e^{(x+y)^2} \)  
   d. \( (x - 3)^2 + z^2 = 1 \), \( x^2 - y^2 + z^2 = 2 \), \( y > 0 \)

Answers
13.1.rel1a. \([2, 5]\)  
13.1.rel1b. \([0, 1) \cup (1, \infty)\)  
13.1.rel1c. \(\left( \frac{3}{2}, 2 \right) \cup (2, 3) \cup (3, \infty)\)  
13.1.rel3a. \(\langle 1, 0, \frac{1}{2} \rangle\)  
13.1.rel3b. \(\langle 0, 0, -\frac{1}{4} \rangle\)  
13.1.rel3c. \(\langle 3t^2, \frac{1}{t}, -e^{-t} \rangle\)  
13.1.re7a. A sinusoidal curve long the line \( x + z = 0; y = 0 \)  
13.1.re7b. An ellipse in the plane \( x = y \) whose shadow in the \( xz \)-plane is the unit circle.  
13.1.re7c. The parabola \( y = 1 - x^2 \) in the plane \( z = 1 \)  
13.1.re7d. A helix on the elliptical cylinder \( \frac{1}{4}x^2 + z^2 = 1 \)  
13.1.re7e. A helix on the cone \( x^2 + y^2 = z^2 \)  
13.1.re7f. The parabola in the plane \( y + z = 1 \)  

Graphs a-f below.

13.1.re8a. \(\langle t, 1 - t, 2t - 1 \rangle\)  
13.1.re8b. \(\langle \cos t, \sin t, 2 - 2 \cos t \rangle\)  
13.1.re8c. \(\langle t, t^{-1}, e^{t^2 + 2 + t^2} \rangle \) (\( t \neq 0 \))  
13.1.re8d. \(\langle 3 + \cos t, \sqrt{8 + 6 \cos t}, \sin t \rangle\)
13.2: Calculus on vector-valued functions

**Differentiation**

If \( \mathbf{r}(t) = \langle x(t), y(t), z(t) \rangle \), then \( \frac{d}{dt} \mathbf{r} = \langle \frac{dx}{dt}, \frac{dy}{dt}, \frac{dz}{dt} \rangle \).

13.2.re1. \( \frac{d}{dt} \langle e^t, \sec t, \tan^{-1} t \rangle = \langle e^t, \sec t \tan t, \frac{1}{\sec^2 t+1} \rangle \)

See page 858 of text for important rules of vector differentiation.

13.2.re2. Find the derivative of vector-valued function.
   a. \( \langle \sqrt{t-2}, \ln|t-5| \rangle \)  
   b. \( \langle e^t, \frac{t+1}{t-1}, \sqrt{t^2+1} \rangle \)  
   c. \( \langle -\frac{1}{\sqrt{3t-1}}, \sinh(t^2), \cosh^2 t \rangle \)

**Integration**

Like differentiation, integration of vector-valued functions is performed component-wise. The Fundamental Theorem of Calculus for vector-valued functions says that

\[ \int_a^b \frac{d}{dt} \mathbf{r} \, dt = \mathbf{r}(b) - \mathbf{r}(a), \]

provided \( \frac{d}{dt} \mathbf{r} \) is continuous.

13.2.re3. 
   a. \( \int (t^2 + 1, \sec t) \, dt = \left\langle \frac{1}{3}t^3 + t + C_1, \ln|\sec t + \tan t| + C_2 \right\rangle \), or \( \left\langle \frac{1}{3}t^3 + t, \ln|\sec t + \tan t| \right\rangle + C \), where \( C \) is a constant vector in \( \mathbb{R}^2 \).
   b. \( \int_{-1}^{1} (t^2 + t, 2te^{-t^2}) \, dt = \left. \left\langle \frac{1}{3}t^3 + \frac{1}{2}t^2, -e^{-t^2} \right\rangle \right|_{-1}^{1} = \left\langle \frac{1}{3} + \frac{1}{2}, -e^{-1} \right\rangle - \left\langle -\frac{1}{3} + \frac{1}{2}, -e^{-1} \right\rangle = \left\langle \frac{2}{3}, 0 \right\rangle \)

13.2.re4. Integrate.
   a. \( \int \langle \frac{t}{2-t}, \sin t \cos^3 t, te^t \rangle \, dt \)  
   b. \( \int \langle \frac{1}{\sqrt{t-1}}, e^x \sec^2(e^x), \tan t \rangle \, dt \)  
   c. \( \int_{0}^{1} \langle \frac{1}{\sqrt{t+1}}, \sin(\pi t), (t+1)^4 \rangle \, dt \)
**Tangent & unit tangent vectors**

If \( \frac{dr}{dt} \neq 0 \), then \( \frac{dr}{dt} \) is a tangent vector to the curve parametrized by \( r \), and

\[
\mathbf{T} = \frac{\mathbf{r}'(t)}{|\mathbf{r}'(t)|}
\]

is the **unit tangent** vector to the curve.

The vector-valued function \( \mathbf{r} \) is said to be **smooth** if \( \frac{dr}{dt} \) is continuous and never equal \( 0 \).

13.2.re5. For the given \( \mathbf{r} \), find \( \frac{dr}{dt} \), \( \mathbf{T} \), and the line tangent to the curve parametrized by \( \mathbf{r} \) at the point corresponding to the given time.

a. \( \mathbf{r} = \langle t^2 + 1, t^3 - t, t \rangle \), \( t = 1 \)
b. \( \mathbf{r} = \langle \frac{1}{2} t^2, \ln |t| \rangle \), \( t = -1 \)
c. \( \mathbf{r} = e^t \mathbf{i} + e^t \sin t \mathbf{j} + e^t \cos t \mathbf{k} \), \( t = 0 \)

**Answers**

13.2.re2a. \( \langle \frac{1}{2}(t - 2)^{-1/2}, (t - 5)^{-1} \rangle \).
13.2.re2b. \( \langle e^t, -2(t - 1)^{-2}, t(t^2 + 1)^{-1/2} \rangle \).
13.2.re2c. \( -(3t - 1)^{-4/3}, 2t \cosh(t^2), 2 \cosh t \sinh t \).
13.2.re4a. \( \langle \frac{1}{2}(\ln |t^2 - 4|, -\frac{1}{4} \cos^4 t, te^t - e^t) + \mathbf{C} \rangle \).
13.2.re4b. \( \langle \frac{1}{4}(\ln |t - 2| - \ln |t + 2|), \tan(e^t), \ln |\sec t| \rangle + \mathbf{C} \).
13.2.re4c. \( \langle \frac{3}{4}, \frac{3}{4}, 1 \rangle \).
13.2.re5a. \( \frac{dr}{dt} = \langle 2t, 3t^2 - 1, 1 \rangle \). \( \mathbf{T} = \frac{1}{\sqrt{1 + 4t^2 + (3t^2 - 1)^2}} \langle 2t, 3t^2 - 1, 1 \rangle \). Line is \( \langle 2, 0, 1 \rangle + t \langle 2, 2, 1 \rangle \).
13.2.re5b. \( \frac{dr}{dt} = \langle t, t^{-1} \rangle \). \( \mathbf{T} = \frac{1}{\sqrt{t^2 + t^{-2}}} \langle t, t^{-1} \rangle \). Line is \( \langle \frac{3}{4}, 0 \rangle - t \langle 1, 1 \rangle \).
13.2.re5c. \( \frac{dr}{dt} = e^t \langle 1, \sin t + \cos t, \cos t - \sin t \rangle \). \( \mathbf{T} = \frac{1}{\sqrt{3}} \langle 1, \sin t + \cos t, \cos t - \sin t \rangle \). Line is \( \langle 1, 0, 1 \rangle + t \langle 1, 1, 1 \rangle \).
13.3: Arc length, curvature, and the TNB frame

Arc length

The total length of the curve parametrized by \( \mathbf{r}(t) = \langle x(t), y(t), z(t) \rangle \) from \( t = a \) to \( t = b \) is

\[
\int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2} \, dt = \int_a^b d\mathbf{r} \, dt
\]

The length of a curve is also called its **arc length**, but arc length can also refer to a variable \( s \) that increases from 0 at the beginning of the curve to its total length at the end of the curve. If you drove a car along the curve, and if you set the trip odometer to zero at start of the curve, then the odometer will display the value of \( s \) as you travel. At time \( t \), the current value of \( s \) is

\[
s = \int_a^t \left| \frac{d\mathbf{r}}{dt^*} \right| \, dt^*
\]

(where \( t^* \) is dummy variable of integration). As a consequence, the particle’s speed

\[
\frac{ds}{dt} = \left| \frac{d\mathbf{r}}{dt} \right|
\]

The direction of \( \frac{d\mathbf{r}}{dt} \) is the direction of the particle’s motion, and the magnitude of \( \frac{d\mathbf{r}}{dt} \) is the speed of the particle.

13.3.re1. Find the length of the helix parametrized by \( \mathbf{r} = \langle \sin t, \cos t, t \rangle \) for \( 0 \leq t \leq \pi \).

Solution: \( \frac{ds}{dt} = |\langle \cos t, -\sin t, 1 \rangle| = \sqrt{\cos^2 t + \sin^2 t + 1} = \sqrt{2} \). Therefore, the arc length equals \( \int_0^\pi \sqrt{2} \, dt = \sqrt{2}\pi \).

13.3.re2. Find the length of the given curve.

a. \( \mathbf{r} = \langle \frac{1}{2} t^2, \frac{4}{3} t^{3/2}, 2t \rangle \) \( 0 < t < 1 \)

b. \( \mathbf{r} = \langle \ln t, \frac{1}{2} t^2, \sqrt{2}t \rangle \) \( 1 < t < e \)

c. \( y = x^2, z = \frac{2}{3} x^3 \), from \( (-1, 1, \frac{2}{3}) \) to \( (1, 1, \frac{2}{3}) \)
The **Unit Tangent**, **Unit Normal**, and **Binormal** are three mutually orthogonal unit vectors given by

\[ T = \frac{dr}{dt} / \left\| \frac{dr}{dt} \right\| \quad N = \frac{dT}{dt} / \left\| \frac{dT}{dt} \right\| \quad B = T \times N \]

(assuming \( \frac{dr}{dt} \neq 0 \)). \( T, N, B \) is a right-handed system. You can think of them as a set of coordinate axes that travels along the curve, twisting so that \( T \) is always tangent to the curve and \( N \) always points in the direction the curve is turning. There’s nice animation of the TNB “frame” moving along a curve in space at [https://youtu.be/JZGFcwipHYY](https://youtu.be/JZGFcwipHYY).

**Curvature** is the scalar given by

\[
\kappa = \frac{\left\| \frac{dT}{ds} \right\|}{\left\| \frac{dT}{dt} \right\|} = \frac{\frac{dT}{dt}}{ds} = \frac{|r'(t) \times r''(t)|}{|r'(t)|^3}
\]

\( \kappa \) is the speed at which \( T \) turns when we travel along the curve with the constant speed 1.

The curvature of a straight line is zero, and the curvature of a circle is the reciprocal of its radius. Along most curves, curvature is not constant.

While \( r \) and its derivatives depend on the motion of the particle tracing out the curve, \( T \), \( N \), \( B \), and \( \kappa \) are geometric properties of the curve itself.
13.3.3. Find \( \mathbf{T} \), \( \mathbf{N} \), \( \mathbf{B} \), and \( \kappa \) along the curve given by \( \mathbf{r} = \langle \frac{1}{2} t^2, \frac{4}{7} t^{3/2}, 2t \rangle \) \((t > 0)\).

Two tips in calculations such as these:

- If \( c \) is a scalar and \( \mathbf{u} \) a vector, then \( \frac{d}{dt}(cu) = \frac{dc}{dt} \mathbf{u} + c \frac{d\mathbf{u}}{dt} \).
- If \( c \) is positive, then \( \mathbf{u} \) and \( cu \) have the same normalization.

Solution. \( \frac{d\mathbf{r}}{dt} = \langle t, 2t^{1/2}, 2 \rangle \), and \( \frac{d\mathbf{r}}{dt} \), the length of \( \frac{d\mathbf{r}}{dt} \), is \( \sqrt{t^2 + 4t + 4} = \sqrt{(t + 2)^2} = t + 2 \) (since \( t + 2 > 0 \)). Normalize \( \frac{d\mathbf{r}}{dt} \) to obtain

\[
\mathbf{T} = (t + 2)^{-1} \langle t, 2t^{1/2}, 2 \rangle
\]

Differentiate using the product rule:

\[
\frac{d\mathbf{T}}{dt} = (t + 2)^{-1} \langle 1, t^{-1/2}, 0 \rangle - (t + 2)^{-2} \langle t, 2t^{1/2}, 2 \rangle
\]

We can obtain \( \mathbf{N} \) by normalizing

\[
(t + 2)^2 \frac{d\mathbf{T}}{dt} = (t + 2)\langle 1, t^{-1/2}, 0 \rangle - \langle t, 2t^{1/2}, 2 \rangle
\]

\[
= (2, 2t^{-1/2} - t^{1/2}, -2)
\]

the magnitude of which is

\[
\sqrt{4 + (2t^{-1/2} - t^{1/2})^2 + 4} = \sqrt{4 + (4t^{-1} - 4 + t) + 4} = \sqrt{4t^{-1} + 4 + t} = 2t^{-1/2} + t^{1/2}
\]

Therefore,

\[
\mathbf{N} = (2t^{-1/2} + t^{1/2})^{-1} \langle 2, 2t^{-1/2} - t^{1/2}, -2 \rangle
\]

Now \( \mathbf{B} = \mathbf{T} \times \mathbf{N} = \)

\[
(t + 2)^{-1} (2t^{-1/2} + t^{1/2})^{-1} \left(\langle t, 2t^{1/2}, 2 \rangle \times \langle 2, 2t^{-1/2} - t^{1/2}, -2 \rangle\right)
\]

\[
= (t^{-1/2}(t + 2)^2)^{-1} \left| \begin{array}{ccc}
\mathbf{i} & \mathbf{j} & \mathbf{k} \\
t & 2t^{1/2} & 2 \\
2 & 2t^{-1/2} - t^{1/2} & -2
\end{array} \right|
\]

\[
= \frac{t^{1/2}}{(t + 2)^2} \left( \mathbf{i} \left| \begin{array}{c}
2t^{1/2} \\
2t^{-1/2} - t^{1/2} \\
2
\end{array} \right| - \mathbf{j} \left| \begin{array}{c}
t \\
2 \\
-2
\end{array} \right| + \mathbf{k} \left| \begin{array}{c}
t \\
2 \\
2t^{1/2} - t^{1/2}
\end{array} \right| \right)
\]

\[
= \frac{t^{1/2}}{(t + 2)^2} \left( -(2t^{1/2} + 4t^{-1/2}) \mathbf{i} + (2t + 4) \mathbf{j} - (2t^{1/2} + t^{3/2}) \mathbf{k} \right)
\]
or, remarkably,
\[ \left\langle \frac{-2}{t + 2}, \frac{2t^{1/2}}{t + 2}, -t \right\rangle. \]

Finally, we can calculate \( \kappa \) either from
\[
\left| \frac{d\mathbf{T}}{dt} \right| \div \frac{ds}{dt} = \left( \frac{2t^{-1/2} + t^{1/2}}{(t + 2)^2} \right) \div (t + 2) = \frac{t^{-1/2}}{(t + 2)^2}
\]
or by calculating
\[
\mathbf{r}' \times \mathbf{r}'' = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ t & 2t^{1/2} & 2 \\ 1 & t^{-1/2} & 0 \end{vmatrix} = \left\langle -2t^{-1/2}, 2, -t^{1/2} \right\rangle = t^{-1/2}\left\langle -2, 2t^{1/2} - t \right\rangle,
\]
the length of which is \( t^{-1/2}(t + 2) \), which we divide by \( \left( \frac{ds}{dt} \right)^3 \) to obtain again
\[ \kappa = t^{-1/2}(t + 2) \div (t + 2)^3 = \frac{t^{-1/2}}{(t + 2)^2}. \]

Note that
\[ \mathbf{B} = (t + 2)^{-1}\left\langle -2, 2t^{1/2}, -t \right\rangle \]
is a positive-scalar multiple of \( \mathbf{r}' \times \mathbf{r}'' \) and so could be obtained by normalizing this cross product. We’ll see in section 13.4 that this is always the case.

13.3.re4. Find \( \mathbf{T} \), \( \mathbf{N} \), \( \mathbf{B} \), and \( \kappa \) along the given curve.

a. \( \mathbf{r} = \langle 4 \sin t, 3t, -4 \cos t \rangle \)
b. \( \mathbf{r} = \langle \ln t, \frac{1}{2}t^2, \sqrt{2}t \rangle \)
c. \( \mathbf{r} = \langle \cos^3 t, \sin^3 t, 0 \rangle \) \( 0 < t < \pi/2 \)
d. \( \mathbf{r} = \langle 3 \sin t, 4 \sin t, 5 \cos t \rangle \)
The normal and osculating planes

As \( \mathbf{TNB} \) travel along the curve, two planes travel along with them. At any point on the curve, the normal plane passes through that point and is orthogonal to \( \mathbf{T} \), and the osculating plane passes through that point and is orthogonal to \( \mathbf{B} \).

![The normal plane](image1)

![the osculating plane](image2)

Figure 13.3.1

It is not necessary to compute \( \mathbf{T} \) to find the normal plane, since \( \mathbf{r}' \) is also orthogonal to this plane. It is also unnecessary to compute \( \mathbf{B} \) to find the osculating plane, since, as we’ll see in 13.4, \( \mathbf{r}'(t) \times \mathbf{r}''(t) \) is orthogonal to this plane.

13.3.re5. Find the normal and osculating planes to the curve at the given point.

a. \( \mathbf{r} = \langle t, t \sin t, t \cos t \rangle \), at \((2\pi, 0, 2\pi)\)
b. \( \mathbf{r} = \langle t^2, t^2 \sin t^2, t^2 \cos t^2 \rangle \), at \((2\pi, 0, 2\pi)\)
c. \( y = z^2 \) and \( xy = 1 \), at \((\frac{1}{4}, 4, -2)\)

Answers

13.3.re2a. \( 5/2 \) 13.3.re2b. \( \frac{2 \pi}{3} \) 13.3.re2c. \( 10/3 \) 13.3.re4a. \( \mathbf{T} = \frac{1}{5} \langle 4 \cos t, 3, 4 \sin t \rangle \), \( \mathbf{N} = \langle -\sin t, 0, \cos t \rangle \), \( \mathbf{B} = \frac{1}{5} \langle 3 \cos t, 4, 3 \sin t \rangle \), \( \kappa = 4/25 \) 13.3.re4b. \( \mathbf{T} = (t^2 + 1)^{-1} \langle 1, t^2, \sqrt{2}t \rangle \), \( \mathbf{N} = (t^2 + 1)^{-1} \langle -\sqrt{2}t, \sqrt{2}t, 1 - t^2 \rangle \), \( \mathbf{B} = (t^2 + 1)^{-2} \langle -t^2, -1, \sqrt{2}t \rangle \), \( \kappa = 2^{1/2} t^{-2} (t^2 + 1)^{-2} \) 13.3.re4c. \( \mathbf{T} = \langle -\cos t, \sin t, 0 \rangle \), \( \mathbf{N} = \langle \sin t, \cos t, 0 \rangle \), \( \mathbf{B} = -\mathbf{k} \), \( \kappa = \frac{1}{\csc t} \) 13.3.re4d. \( \mathbf{T} = \frac{1}{5} \langle 3 \cos t, 4 \cos t, -5 \sin t \rangle \), \( \mathbf{N} = -\frac{1}{5} \langle 3 \sin t, 4 \sin t, \cos t \rangle \), \( \mathbf{B} = \langle -4/5, 3/5, 0 \rangle \), \( \kappa = \frac{1}{5} \) 13.3.re5a. n.p.: \( x + 2\pi y + z = 4\pi \); o.p.: \( -2\pi^2 - 1)(x - 2\pi) + \pi y + z - 2\pi = 0 \) 13.3.re5b. same as in a. 13.3.re5c. n.p.: \( \frac{1}{5}(x - \frac{1}{4}) + 4(y - 4) + (z + 2) = 0 \) o.p.: \( -2(x - \frac{1}{4}) + \frac{8}{5}(y - 4) - (z + 2) = 0 \).
13.4: Velocity and acceleration

It \( r(t) \) represents the position of an object at time \( t \), then its first two derivatives are named velocity and acceleration. The magnitude of velocity is speed. We sometimes “suppress the \( t \)” when writing these functions, e.g., when we write \( r \) instead of \( r(t) \).

\[
\begin{align*}
\mathbf{r} & \quad = \text{position (vector)} \\
\mathbf{v} & = \frac{d\mathbf{r}}{dt} = \text{velocity (vector)} \\
\mathbf{a} & = \frac{d\mathbf{v}}{dt} = \text{acceleration (vector)} \\
ds & = |\mathbf{v}| = \text{speed (scalar)}
\end{align*}
\]

13.4.re1. Find velocity, acceleration, and speed for the given position.

a. \( \mathbf{r} = \langle 4 \sin t, 3t, -4 \cos t \rangle \)  
b. \( \mathbf{r} = \langle \ln t, {1 \over 2} t^2, \sqrt{2}t \rangle \)  
c. \( \mathbf{r} = \langle \cos^3 t, \sin^3 t, t \rangle \)

Initial value problems

13.4.re2. Find position \( \mathbf{r} \) if \( \mathbf{a}(t) = \langle te^t, 2t, 1 \rangle \), \( \mathbf{v}(0) = \langle -1, 1, 0 \rangle \), and \( \mathbf{r}(0) = \langle -1, 0, -1 \rangle \).

Solution: Integrate once to find \( \mathbf{v} \) and again to find \( \mathbf{r} \). Use the given values of \( \mathbf{v} \) and \( \mathbf{r} \) to solve for constants of integration. (Integrate \( te^t \) by parts.)

\[
\begin{align*}
\mathbf{v} & = \langle te^t - e^t, t^2, t \rangle + \mathbf{C} \\
\langle -1, 1, 0 \rangle & = \langle -1, 0, 0 \rangle + \mathbf{C} \quad \Rightarrow \quad \mathbf{C} = \langle 0, 1, 0 \rangle \\
\mathbf{v} & = \langle te^t - e^t, t^2, t \rangle + \langle 0, 1, 0 \rangle \\
& = \langle te^t - e^t, t^2 + 1, t \rangle \\
\mathbf{r} & = \langle te^t - 2e^t, {1 \over 3} t^3 + t, {1 \over 2} t^2 \rangle + \mathbf{D} \\
\langle -1, 0, -1 \rangle & = \langle -2, 0, 0 \rangle + \mathbf{D} \quad \Rightarrow \quad \mathbf{D} = \langle 1, 0, -1 \rangle \\
\mathbf{r} & = \langle te^t - 2e^t, {1 \over 3} t^3 + t, {1 \over 2} t^2 \rangle + \langle 1, 0, -1 \rangle \\
& = \langle te^t - 2e^t + 1, {1 \over 3} t^3 + t, {1 \over 2} t^2 - 1 \rangle
\end{align*}
\]

13.4.re3. A constant force of magnitude 15 in the direction of \(-3\mathbf{i} + 4\mathbf{k}\) acts on an object of mass \(1/2\). If, at time 0, the object’s position and velocity are \(2\mathbf{i} - \mathbf{k}\) and \(\mathbf{j} - \mathbf{i}\) respectively, find the object’s position at time \(t\). Hint: Newton’s second law of motion states that \(\text{force} = \text{mass} \times \text{acceleration}\).

13.4.re4. An acrobat is at to be shot from a cannon with speed \(32\sqrt{2}\) ft/sec at an upward angle \(\pi/4\) radians. So that we may correctly position the net to catch her, find the (horizontal) distance from the cannon at which the acrobat will descend to altitude of 12 ft. Assume the acrobat is launched from altitude zero and that, due to gravity, her acceleration is \(-32\) ft/sec\(^2\) downward.

Tip: place the cannon at the origin in the \(xy\)-plane, firing into the first quadrant. At what \(x\) will \(y = 12\)?
Tangential and normal components of acceleration

The tangential and normal components of \( \mathbf{a} \) are scalars \( a_T \) and \( a_N \) for which

\[
(13.4.1) \quad \mathbf{a} = a_T \mathbf{T} + a_N \mathbf{N}.
\]

Consequently, \( \mathbf{v} \) and \( \mathbf{a} \) lie in the same plane as \( \mathbf{T} \) and \( \mathbf{N} \), the osculating plane (as seen in figure 13.3.1). \( a_T \) and \( a_N \) are the components of \( \mathbf{a} \) in the directions \( \mathbf{T} \) and \( \mathbf{N} \) seen in section 12.3 and can be calculated any of these formulas

\[
\begin{align*}
  a_T &= \frac{d^2 s}{dt^2} = |\mathbf{a}| \cos \theta = \frac{\mathbf{v} \cdot \mathbf{a}}{|\mathbf{v}|} = |\mathbf{a}| \sin \theta = \frac{|\mathbf{v} \times \mathbf{a}|}{|\mathbf{v}|} = \sqrt{|\mathbf{a}|^2 - a_T^2} \\
  a_N &= \kappa \left( \frac{ds}{dt} \right)^2
\end{align*}
\]

Note that \( a_T \) is positive [negative] if the object is speeding up [slowing down]. Unless speed or curvature is zero, \( a_N \) is positive: \( \mathbf{a} \) and \( \mathbf{N} \) lie on the same side of the line containing \( \mathbf{v} \) in the osculating plane.

13.4.re5. Find \( a_T \) and \( a_N \) for the given \( \mathbf{r} \).

a. \( \mathbf{r} = (4 \sin t, 3t, -4 \cos t) \)  
  b. \( \mathbf{r} = (\ln t, \frac{1}{2}t^2, \sqrt{2}t) \)  
  c. \( \mathbf{r} = (\frac{1}{2}t^2, \frac{4}{3}t^{3/2}, 2t) \) (see 13.3.re3)
Calculating the TNB frame from \( r' \) and \( r'' \)

It's possible to find \( N \) and \( B \) without differentiating \( T \) (which is easily found by normalizing \( r' \)). Since \( r'' \) lies in the osculating plane on the same side of \( T \) as \( N \), one can find \( B \) by normalizing \( r' \times r'' \). And since \( TNB \) form a right-handed system, \( N \) must equal \( B \times T \):

\[
T = \frac{r'}{|r'|} \quad N = B \times T \quad B = \frac{r' \times r''}{|r' \times r''|}
\]

13.4.re6. Suppose that, at a particular time, \( v = (-2, 1, 2) \) and \( a = (1, 1, -1) \). Find the following at that time.

a. \( a_T \)  
b. \( a_N \)  
c. \( T \)  
d. \( N \)  
e. \( B \)
14.1: Real-valued functions of several real variables

Recall that a **function** is a rule that assigns to each elements of a set, called its **domain**, a unique element of another set, called its **range**. In Calculus III, we consider functions with domain $\subset \mathbb{R}^m$ and range $\subset \mathbb{R}^n$ for some $m$ and $n$.

Generally, a function’s domain is easier to determine than its range.

### 14.1.re1.

<table>
<thead>
<tr>
<th>function $f(x) = \frac{1}{x - 2}$</th>
<th>domain $(-\infty, 2) \cup (2, \infty) \subset \mathbb{R}^1$</th>
<th>range $(-\infty, 0) \cup (0, \infty) \subset \mathbb{R}^1$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$g(x) = \ln x$</td>
<td>$(0, \infty) \subset \mathbb{R}^1$</td>
<td>$\mathbb{R}^1$</td>
</tr>
<tr>
<td>$u(t) = ((t - 1)^2, \sin t, t)$</td>
<td>$\mathbb{R}^1$</td>
<td>a curve $\subset \mathbb{R}^3$</td>
</tr>
<tr>
<td>$v(x, y) = \frac{x^2 - y^2}{x + 2y - 1}$</td>
<td>${(x, y) \mid x + 2y \neq 1} \subset \mathbb{R}^2$</td>
<td>$\mathbb{R}^1$</td>
</tr>
<tr>
<td>$w(x, y, z) = \sqrt{1 - x^2 - 4z^2}$</td>
<td>${(x, y, z) \mid x^2 + 4z^2 \leq 1} \subset \mathbb{R}^3$</td>
<td>$[0, 1] \subset \mathbb{R}^1$</td>
</tr>
</tbody>
</table>

### 14.1.re2.

Sketch and describe the domain of the given function.

a. $u(x, y) = \frac{x^2 - y^2}{x + 2y - 1}$  
   b. $v(x, y, z) = \sqrt{1 - x^2 - 4z^2}$

**Solutions.**

a. The plane minus the line $x + 2y = 1$.  
   b. The cylinder $x^2 - 4z^2 = 1$ and its interior.

### 14.1.re3.

Describe and sketch the domain of the given function.

a. $\nu(x, y) = \sqrt{y - x + \sin^{-1}(x + y)}$  
   b. $\omega(x, y) = \ln((x - y)(y - 2))$

c. $\alpha(x, y, z) = \frac{x^2 + xz + z^2}{(x - y)(z - x^2 - y^2)}$  
   d. $\beta(x, y, z) = \sqrt{1 - y^2 - z^2} + \frac{1}{\sqrt{x^2 + y^2 + z^2 - 4}}$
Graphs of functions on $\mathbb{R}$ and $\mathbb{R}^2$

The graph of an equation $g(x, y) = 0$ [or $g(x, y, z) = 0$] is its solution set: the set of all points in the $xy$-plane [or in $xyz$-space] whose coordinates satisfy the equation. The graph of a function $f(x)$ [or $f(x, y)$] is the graph of the equation $y = f(x)$ [or $z = f(x, y)$], that is, the set of all the points $(x, f(x))$ in the plane [or the set of all points $(x, y, f(x, y))$ in space].

14.1.re4. The graph of $f(x) = x^2$ is the parabola $y = x^2$, below left. The graph of $g(x, y) = x^2 - y^2$ is the hyperbolic paraboloid $z = x^2 - y^2$, below right.

14.1.re5. Sketch the graph of the given function.

a. $f(x) = \frac{x^2 - 1}{x}$

b. $g(x, y) = -4x^2 - 8x - y^2$

c. $h(x, y) = 1 - \sqrt{x^2 + y^2}$

d. $k(x, y) = 4 - 2x - 3y$

Level curves and surfaces; contour maps

The level curves of a function $f(x, y)$ are the graphs of equations of the form $f(x, y) = k$, where $k$ is a constant. That is, level curves are the curves along which $f(x, y)$ is constant. The level surfaces of a function $f(x, y, z)$ are the graphs of equations $f(x, y, z) = k$, that is, the surfaces along which $f(x, y, z)$ is constant.

Each point in the domain of $f(x, y)$ lies on exactly one level curve of $f$. Consequently, the level curves of $f(x, y)$ are non-overlapping and completely fill the domain of $f$. Likewise, the level surfaces of $f(x, y, z)$ are nonoverlapping and completely fill the domain of $f(x, y, z)$.

A contour map for a function is a graph of a representative sample of is level curves. Typically, a contour map displays the curves

$$f(x, y) = k_0, \quad f(x, y) = k_1, \quad f(x, y) = k_2, \quad \ldots \quad f(x, y) = k_n$$

for some equally spaced numbers $k_0, k_1, \ldots k_n$. 
14.1.re6. The domain of \( f(x, y) = \sqrt{x^2 + y^2} \) is the entire \( xy \)-plane, and its level curves are circles centered at the origin. Each point in the plane lies on exactly one of the level curves of \( f \). The graph of \( f(x, y) \) appears below left. A contour map for \( f(x, y) \) for \( f(x, y) = 0, 1, 2, \ldots, 8 \) appears below right.

If the contour map displays the level curves for equally spaced values of \( f \), then we judge where and in which directions the function is increasing rapidly per unit

14.1.re7. More graphs and contour maps:

graph of \( x^2 + y^2 \)  
contour map of \( x^2 + y^2 \)

graph of \( x - \frac{1}{12}x^3 - \frac{1}{4}y^2 + \frac{1}{2} \)  
contour map of \( x - \frac{1}{12}x^3 - \frac{1}{4}y^2 + \frac{1}{2} \)
14.1.r8. We can’t display the graph of $x^2 + y^2 - z^2$ in three dimensions, but we can still see a contour map:
Answers
14.1.re3a. $y \geq x$ and $-1 \leq x + y \leq 1$. The region in the plane between $x + y = -1$ and $x + y = 1$ and above $y = x$. 14.1.re3b. $x(y - 2) > 0$. The region above $y = 2$ and below $y = x$ plus the region below $y = 2$ and above $y = x$ (excluding those lines). 14.1.re3c. $z \neq x^2 + y^2$. All of $\mathbb{R}^3$, minus the circular paraboloid $z = x^2 + y^2$ and the plane $x = y$. 14.1.re3d. $y^2 + z^1 \leq 1$ and $x^2 + y^2 + z^2 > 4$. The part of the cylinder $y^2 + z^1 = 1$ (and its interior) that lies outside the sphere $x^2 + y^2 + z^2 = 4$. 14.1.re5. Top row = \{a, b\}. Bottom row = \{c, d\}. 

\[ 
\begin{align*}
\text{Top row:} & \quad \{a, b\} \\
\text{Bottom row:} & \quad \{c, d\} 
\end{align*}
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