12.1: Three-dimensional space

Rectangular Coordinates
The $xy$-plane is sometimes called $\mathbb{R}^2$, two-dimensional space, or simply two space, because any point in the plane can be represented by its two rectangular coordinates $(x, y)$ (also called Cartesian coordinates).

To define rectangular coordinates $(x, y, z)$ for three-dimensional space (or $\mathbb{R}^3$, or three space), picture the $z$-axis perpendicular to the $xy$-plane and passing through the origin.

![Figure 12.1.1](image1.png)

**Tip**: draw the axes in three space from a point of view in the first octant (where $x$, $y$, and $z$ are positive).

Some authors draw the same three axes rotated to look like the picture to the right of this paragraph. To see that this and figure 12.1.2 are the same after a rotation, check that both are a right-handed system: when pointing your right index finger in the direction of the positive $x$-axis and your right middle finger in the direction of the positive $y$-axis, your right thumb points in the direction of the positive $z$-axis.

![Figure 12.1.2](image2.png)

Implicit equations of curves and surfaces
The graph in the $xy$-plane of an equation is often (but not always) a line or curve.

The graph in $xyz$-space of a single equation is often (but not always) a surface.

The graph of a system of two $xyz$-equations is often (b.n.a.) a curve.

12.1.rev1. Find the equation(s) of the given coordinate plane or axes.

- a. the $xy$-plane
- b. the $xz$-plane
- c. the $yz$-plane
- d. the $x$-axis
- e. the $y$-axis
- f. the $z$-axis
Orthogonal Projection

The **orthogonal projection** of a point onto a line (or plane) is the point on that line (or plane) closest to the given point. The line segment to the point from its projection is orthogonal to the line (or plane). Projecting onto one of the coordinate axes or planes is just a matter of resetting some coordinates equal to zero.

12.1.re2. Find the projection of the given point onto the coordinate line or plane.

- a. $(2, -3, 4)$ onto the $xy$-plane
- b. $(2, -3, 4)$ onto the $xz$-plane
- c. $(2, -3, 4)$ onto the $yz$-plane
- d. $(2, -3, 4)$ onto the $x$-axis
- e. $(2, -3, 4)$ onto the $y$-axis
- f. $(2, -3, 4)$ onto the $z$-axis

More on orthogonal projection onto lines in section 12.3.

Distance in three space

The distance from a point $(x, y, z)$ to the origin is

$$\sqrt{x^2 + y^2 + z^2}.$$  

The distance from one point $(x_0, y_0, z_0)$ to another $(x_1, y_1, z_1)$ is

$$\sqrt{(x_1 - x_0)^2 + (y_1 - y_0)^2 + (z_1 - z_0)^2}.$$

12.1.re3. Find an equation of the sphere centered at $(2, -1, 3)$ having radius 5.

12.1.re4. Find the center and radius of the given sphere.

- a. $x^2 + (y - 3)^2 + (z + 2)^2 = 25$
- b. $x^2 + 2x + y^2 - 4y + z^2 - 6z = 2$
- c. $x^2 - x + y^2 + 3y + z^2 + 5z = \frac{1}{4}$
12.1.re5. Which of the axes below are the same as in figure 12.1.2?

![Diagrams of axes]

**Answers**

12.1.re1a. $z = 0$. 12.1.re1b. $y = 0$. 12.1.re1c. $x = 0$. 12.1.re1d. $y = 0$ and $z = 0$. 12.1.re1e. $x = 0$ and $z = 0$. 12.1.re1f. $x = 0$ and $y = 0$. 12.1.re2a. $(2, -3, 0)$ 12.1.re2b. $(2, 0, 4)$ 12.1.re2c. $(0, -3, 4)$ 12.1.re2d. $(2, 0, 0)$ 12.1.re2e. $(0, -3, 0)$ 12.1.re2f. $(0, 0, 4)$ 12.1.re3. $(x - 2)^2 + (y + 1)^2 + (z - 3)^2 = 25$. 12.1.re4a. ctr = $(0, 3, -2)$, rad = 5. 12.1.re4b. ctr = $(-1, 2, 3)$, rad = 4. 12.1.re4c. ctr = $(\frac{1}{2}, -\frac{1}{2}, -\frac{3}{2})$, rad = 3. 12.1.re5. a, e, f.
12.2: Vectors

In Calculus III, a vector is a directed line segment in \( \mathbb{R}^2 \) or \( \mathbb{R}^3 \). A same vector can have different initial and terminal points.

12.2.re1. The vector from the initial point (0, 0, 0) to the terminal point (2, 3, 4) is denoted \( \langle 2, 3, 4 \rangle \). The vector with initial point (2, 4, −1) and terminal point (4, 7, 3) is the same:

\[
(4 - 2, 7 - 4, 3 - (-1)) = (2, 3, 4)
\]

A vector-valued variable is usually denoted either in bold or with an arrow: \( \mathbf{u} \) or \( \vec{u} \). To distinguish them from vectors, real numbers and real-valued variables are called scalars (which can be used either a noun or an adjective).

12.2.re2. \( x, y, \) and \( z \) are scalar variables. \( \mathbf{r} = \langle x, y, z \rangle \) is a vector-valued variable.

Vector Arithmetic

**Vector Addition** is the addition of vectors component by component.

**Scalar Multiplication** is the multiplication of every component of a vector by a scalar.

12.2.re3.

\[
\langle 2, 1 \rangle + \langle 3, -2 \rangle = \langle 5, -1 \rangle \\
3 \langle 2, 1 \rangle = \langle 6, 3 \rangle \\
-\frac{1}{2} \langle 2, 1 \rangle = \langle -1, -\frac{1}{2} \rangle
\]

If the initial point of \( \mathbf{v} \) is placed at the terminal point of \( \mathbf{u} \), then \( \mathbf{u} + \mathbf{v} \) reaches from the initial point of \( \mathbf{u} \) to the terminal point of \( \mathbf{v} \).

12.2.re4. Express the vector ? shown in the figure in terms of \( \mathbf{u} \) and \( \mathbf{v} \). (Hint: what is \( ? + \mathbf{v} \)?)

If \( \mathbf{u}, \mathbf{v}, \mathbf{w} \) are vectors and \( s \) and \( t \) are scalars, then

a. \( \mathbf{u} + \mathbf{v} = \mathbf{v} + \mathbf{u} \)  
   e. \( \mathbf{u} + (\mathbf{v} + \mathbf{w}) = (\mathbf{u} + \mathbf{v}) + \mathbf{w} \)

b. \( \mathbf{u} + \mathbf{0} = \mathbf{u} \)  
   f. \( \mathbf{u} + (-\mathbf{u}) = \mathbf{0} \)

c. \( s(\mathbf{v} + \mathbf{w}) = sv + sw \)  
   g. \( (s + t)\mathbf{u} = s\mathbf{u} + t\mathbf{u} \)

d. \( s(t\mathbf{u}) = (st)\mathbf{u} \)  
   h. \( 1\mathbf{u} = \mathbf{u} \)
Some Special Vectors

in \( \mathbb{R}^3 \):

\[
\begin{align*}
0 &= \langle 0, 0, 0 \rangle \\
i &= \langle 1, 0, 0 \rangle \\
j &= \langle 0, 1, 0 \rangle \\
k &= \langle 0, 0, 1 \rangle
\end{align*}
\]

in \( \mathbb{R}^2 \):

\[
\begin{align*}
0 &= \langle 0, 0 \rangle \\
i &= \langle 1, 0 \rangle \\
j &= \langle 0, 1 \rangle
\end{align*}
\]

Always be careful to distinguish between the vector \( 0 \) and the scalar 0. Every vector can be written uniquely as \( ai + bj + ck \) for some scalars \( a, b, c \).

12.2.re5. Express the given vector in terms of \( i, j, \) and \( k \).

a. \( \langle 2, 3, -1 \rangle \)

b. \( \langle -1, 0, 1 \rangle \)

c. \( \langle 3, \pi \rangle \)

Magnitude

If \( \mathbf{u} = \langle u_1, u_2, u_3 \rangle \), the magnitude or length of \( \mathbf{u} \) is

\[
|\mathbf{u}| = \sqrt{u_1^2 + u_2^2 + u_3^2}
\]

If \( \mathbf{u} \) is a vector and \( s \) is a scalar, then

\[
|s\mathbf{u}| = |s||\mathbf{u}|
\]

A unit vector is a vector of length 1. If \( \mathbf{u} \neq 0 \), the unit vector

\[
\frac{1}{|\mathbf{u}|}\mathbf{u}
\]

is the normalization of \( \mathbf{u} \).

12.2.re6. Let

\[
\begin{align*}
\mathbf{u} &= \langle 2, 1, -2 \rangle \\
\mathbf{v} &= \langle 5, 2, -1 \rangle \\
\mathbf{w} &= \langle 3, -4 \rangle \\
\mathbf{p} &= 5i - 4j - 2k \\
\mathbf{q} &= -2i + 4j - k
\end{align*}
\]

and find the following, if they exist.

a. \( \mathbf{u} - 2\mathbf{v} \)

b. \( 3\mathbf{u} - \mathbf{v} + 2\mathbf{p} \)

c. \( 3\mathbf{u} - 2\mathbf{w} \)

d. \( \mathbf{u} - 2 \)

e. \( \mathbf{v} - 2i \)

f. \( |\mathbf{v}| \)

g. \( |2\mathbf{u}| \)

h. \( |-2\mathbf{w}| + |\mathbf{w}| \)

i. the normalization of \( \mathbf{u} \)

j. a vector of length 3 in the opposite direction of \( \mathbf{w} \)

Vectors in Physics

Vectors are used to model things that have magnitude and direction.

12.2.re7. A ship sails due west with speed 4 knots. Taking the magnitude of velocity to be speed, and assuming \( j \) and \( i \) represent the directions north and east, resp., find a vector representing the velocity of the ship.

12.2.re8. Find the vector representing a force with magnitude 3 Newtons in the direction of \( \langle 7, -4, -4 \rangle \).

Answers

12.2.re4. \( \mathbf{u} - \mathbf{v} \) 12.2.re5a. \( 2i + 3j - k \) 12.2.re5b. \(-i + k \) 12.2.re5c. \( 3i + \pi j \) 12.2.re6a. \(-8, -3, 0 \) 12.2.re6b. \( \langle 11, -7, -9 \rangle \) 12.2.re6c. dne 12.2.re6d. dne 12.2.re6e. \( 3i + 2j - k \) 12.2.re6f. \sqrt{30} 12.2.re6g. 6 12.2.re6h. 15 12.2.re6i. \( \langle \frac{7}{3}, \frac{1}{3}, -\frac{4}{3} \rangle \) 12.2.re6j. \( \langle -\frac{2}{3}, \frac{4}{3} \rangle \) 12.2.re7. \(-4i \) 12.2.re8. \( \langle \frac{2}{3}, -\frac{4}{3}, \frac{4}{3} \rangle \)
12.3: The Dot Product

**Definition.** The dot product of the vectors \( \mathbf{u} = \langle u_1, u_2, u_3 \rangle \) and \( \mathbf{w} = \langle w_1, w_2, w_3 \rangle \) is

\[
\mathbf{u} \cdot \mathbf{w} = u_1 w_1 + u_2 w_2 + u_3 w_3.
\]

The dot product is sometimes called the *inner product* or *scalar product*.

12.3.rel. a. \( \langle 4, 3 \rangle \cdot \langle 1, 5 \rangle = 4 \cdot 1 + 3 \cdot 5 = 19 \).

b. \( \langle 2, -4, 3 \rangle \cdot \langle 3, 3, 2 \rangle = 2 \cdot 3 - 4 \cdot 3 + 3 \cdot 2 = 0 \).

If \( \mathbf{u}, \mathbf{v}, \mathbf{w} \) are vectors and \( s \) is a scalar and \( \theta \) is the angle between \( \mathbf{u} \) and \( \mathbf{v} \), then

a. \( 0 \cdot \mathbf{u} = 0 \)  
 b. \( \mathbf{u} \cdot \mathbf{v} = \mathbf{v} \cdot \mathbf{u} \)  
 c. \( \mathbf{u} \cdot (\mathbf{v} + \mathbf{w}) = \mathbf{u} \cdot \mathbf{v} + \mathbf{u} \cdot \mathbf{w} \)  
 d. \( (su) \cdot \mathbf{v} = s(\mathbf{u} \cdot \mathbf{v}) = \mathbf{u} \cdot (sv) \)  
 e. \( \mathbf{u} \cdot \mathbf{u} = |\mathbf{u}|^2 \)  
 f. \( \mathbf{u} \cdot \mathbf{v} = |\mathbf{u}||\mathbf{v}| \cos \theta \)  
 g. \( \mathbf{u} \perp \mathbf{v} \) iff \( \mathbf{u} \cdot \mathbf{v} = 0 \)  
 h. \( \mathbf{u} \cdot \mathbf{v} > 0 \) iff \( 0 \leq \theta < \pi/2 \)  
 i. \( \mathbf{u} \cdot \mathbf{v} < 0 \) iff \( \pi/2 < \theta \leq \pi \)

**Orthogonal Projection**

If \( \mathbf{u} \) and \( \mathbf{v} \) are vectors, and \( \mathbf{v} \neq \mathbf{0} \), then the vector

\[
\text{proj}_v \mathbf{u} = \frac{\mathbf{u} \cdot \mathbf{v}}{\mathbf{v} \cdot \mathbf{v}} \mathbf{v}
\]

is called the *orthogonal projection* or *vector projection* of \( \mathbf{u} \) onto \( \mathbf{v} \), and the scalar

\[
\text{comp}_v \mathbf{u} = \frac{\mathbf{u} \cdot \mathbf{v}}{|\mathbf{v}|}
\]

is the called the *scalar projection* of \( \mathbf{u} \) onto \( \mathbf{v} \), or the *component* of \( \mathbf{u} \) in the direction of \( \mathbf{v} \).

The vector projection \( \text{proj}_v \mathbf{u} \) is the “shadow” of \( \mathbf{u} \) cast on the line spanned by \( \mathbf{v} \) by a ray of light orthogonal to \( \mathbf{v} \). The scalar projection \( \text{comp}_v \mathbf{u} \) is the signed length of \( \text{proj}_v \mathbf{u} \).

\[
\text{comp}_v \mathbf{u} > 0 \quad \text{iff} \quad \mathbf{u} \cdot \mathbf{v} > 0 \quad \text{iff} \quad \text{proj}_v \mathbf{u} \text{ is in same direction as } \mathbf{v}
\]

\[
\text{comp}_v \mathbf{u} < 0 \quad \text{iff} \quad \mathbf{u} \cdot \mathbf{v} < 0 \quad \text{iff} \quad \text{proj}_v \mathbf{u} \text{ is in opposite direction as } \mathbf{v}
\]
12.3.re2. Let
\[ u = \langle 1, 2, 2 \rangle \quad v = \langle 3, 0, -2 \rangle \quad w = \langle 2, 1, -1 \rangle \quad p = \langle -2, 1, 0 \rangle \]
and find the following, if they exist.

a. \( \text{proj}_u v \)  

b. \( \text{comp}_u v \)  

c. \( \text{proj}_v u \)  

d. \( \text{comp}_v u \)  

e. \( \text{proj}_u w \)  

f. \( \text{proj}_u p \)  

g. \( u \cdot (w - \text{proj}_u w) \)  

h. \( w \cdot (w - \text{comp}_u w) \)

12.3.re3. Use orthogonal projection to find the point on the line closest to the given point.

a. \((2, 3), y = x\)  
b. \((1, -1), 2y + 3x = 0\)

Work

The work done by a constant force \( F \) moving an object along a straight line is

\[ W = F \cdot D \]

where \( D \) is the change in position, or \textbf{displacement}, of the object.

12.3.re4. Find the work done by a force of magnitude 3 N in the direction of \( \langle 8, -4, 1 \rangle \) in moving an object in a straight line from the point \( (1, 0, 1) \) to the point \( (3, 1, -1) \). (Assume coordinates are measured in meters).

Answers

12.3.re2a. \((\frac{1}{9}, -\frac{2}{9}, -\frac{2}{9})\).  
12.3.re2b. \(-\frac{1}{9}\).  
12.3.re2c. \((\frac{1}{13}, 0, \frac{2}{13})\).  
12.3.re2d. \(-\frac{1}{\sqrt{13}}\).  
12.3.re2e. \((\frac{2}{3}, \frac{4}{3}, \frac{1}{3})\).  
12.3.re2f. 0.  
12.3.re2g. 0.  
12.3.re2h. dne.  
12.3.re3a. \((\frac{5}{7}, \frac{2}{7})\).  
12.3.re3b. \((\frac{10}{13}, -\frac{15}{13})\).  
12.3.re4. \( F = \frac{3}{9} \langle 8, -4, 1 \rangle \). \( D = (2, 1, -2) \). Work = \( \frac{10}{3} \) ft-lbs.
12.4: The Cross Product

Definition. The cross product of the vectors \( \mathbf{u} = \langle u_1, u_2, u_3 \rangle \) and \( \mathbf{w} = \langle w_1, w_2, w_3 \rangle \) is the determinant

\[
\mathbf{u} \times \mathbf{w} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ u_1 & u_2 & u_3 \\ w_1 & w_2 & w_3 \end{vmatrix},
\]

which can be calculated by expansion along the top row:

\[
= \mathbf{i} \begin{vmatrix} u_2 & u_3 \\ w_2 & w_3 \end{vmatrix} - \mathbf{j} \begin{vmatrix} u_1 & u_3 \\ w_1 & w_3 \end{vmatrix} + \mathbf{k} \begin{vmatrix} u_1 & u_2 \\ w_1 & w_2 \end{vmatrix}
\]

12.4.re1. \((2\mathbf{i} + \mathbf{j} - 3\mathbf{k}) \times (-\mathbf{i} - \mathbf{j} + \mathbf{k}) =
\[
\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & 1 & -3 \\ -1 & -1 & 1 \end{vmatrix} = \mathbf{i} \begin{vmatrix} 1 & -3 \\ -1 & 1 \end{vmatrix} - \mathbf{j} \begin{vmatrix} 2 & -3 \\ -1 & 1 \end{vmatrix} + \mathbf{k} \begin{vmatrix} 2 & 1 \\ -1 & -1 \end{vmatrix}
\]

\[
= \mathbf{i}(1 \cdot 1 - (-1)(-3)) - \mathbf{j}(2 \cdot 1 - (-1)(-3)) + \mathbf{k}(2(-1) - (-1)1)
\]

\[
= \langle -2, 1, -1 \rangle
\]

12.4.re2. Find the following, if they exist.

a. \(\langle 2, -4, 1 \rangle \times \langle 1, 0, 1 \rangle\)  
b. \(\langle 1, 0, 1 \rangle \times \langle 2, -4, 1 \rangle\)  
c. \(\langle 2, -4, 1 \rangle \times \langle -4, 8, -2 \rangle\)  
d. \(\langle 2, -4, 1 \rangle \times \mathbf{k}\)  
e. \(\langle 2, -4, 1 \rangle \times \mathbf{i}\)  
f. \(\mathbf{i} \times \mathbf{j}\)

If \(\mathbf{u}, \mathbf{v}, \mathbf{w}\) are vectors and \(s\) is a scalar and \(\theta\) is the angle between \(\mathbf{u}\) and \(\mathbf{v}\), then

a. \(\mathbf{u} \cdot (\mathbf{u} \times \mathbf{v}) = \mathbf{v} \cdot (\mathbf{u} \times \mathbf{v}) = 0\)  
d. \((s\mathbf{u}) \times \mathbf{v} = s(\mathbf{u} \times \mathbf{v}) = \mathbf{u} \times (s\mathbf{v})\)

b. \(|\mathbf{u} \times \mathbf{v}| = |\mathbf{u}||\mathbf{v}|\sin \theta\)  
e. \(\mathbf{u} \times (\mathbf{v} + \mathbf{w}) = \mathbf{u} \times \mathbf{v} + \mathbf{u} \times \mathbf{w}\)

c. \(\mathbf{u} \times \mathbf{v} = -(\mathbf{v} \times \mathbf{u})\)  
f. \(\mathbf{u}, \mathbf{v}, \mathbf{u} \times \mathbf{v}\) is a right-handed system,

a. means that \(\mathbf{u} \times \mathbf{v}\) is orthogonal to both \(\mathbf{u}\) and \(\mathbf{v}\). b. implies that \(|\mathbf{u} \times \mathbf{v}|\) = the area of the parallelogram with sides \(\mathbf{u}\) and \(\mathbf{v}\), and that \(\mathbf{u} \times \mathbf{v} = 0\) iff \(\mathbf{u} = 0, \mathbf{v} = 0, \text{ or } \mathbf{u} \parallel \mathbf{v}\).

f. means that when you point to \(\mathbf{u}\) with your open right hand, and then curl your fingers closed in the direction of \(\mathbf{v}\), your thumb points in the direction of \(\mathbf{u} \times \mathbf{v}\). For instance, \(\mathbf{i}, \mathbf{j}, \mathbf{k}\) is a right-handed system.

The triple product of the vectors \(\mathbf{u} = \langle u_1, u_2, u_3 \rangle, \mathbf{v} = \langle v_1, v_2, v_3 \rangle, \text{ and } \mathbf{w} = \langle w_1, w_2, w_3 \rangle\) is

\[
\mathbf{u} \cdot (\mathbf{v} \times \mathbf{w}) = \begin{vmatrix} u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{vmatrix}
\]

\(|\mathbf{u} \cdot (\mathbf{v} \times \mathbf{w})|\) is the volume of the parallelepiped with sides \(\mathbf{u}, \mathbf{v}, \mathbf{w}\).
12.4.re3. Find the area of the parallelogram with the given vertices.

a. (0, 0, 0), (3, 1, 2), (2, −1, 4), (5, 0, 6)

b. (7, −1), (12, 2), (3, 1), (8, 4)

12.4.re4. Find the volume of the parallelepiped with edges $u$, $v$, $w$.

\[ u = i + j \quad v = j - k \quad w = i - 2j + k \]

12.4.re5. Let

\[ u = \langle 1, 2, 2 \rangle \quad v = \langle 3, 0, -2 \rangle \quad w = \langle 2, 1, -1 \rangle \]

and find the following, if they exist.

a. $u \times v$

b. $v \times w$

c. $v \times (w \times w)$

d. $u \cdot (v \times w)$

e. $u \cdot (v \times u)$

f. $(u \cdot v) \cdot (v \cdot w)$

g. $(u \times v) \times (v \times w)$

h. $\text{comp}_{v \times w} u$

**Torque**

The torque of a force vector $\mathbf{F}$ and position vector $\mathbf{r}$ is defined to be $\vec{\tau} = \mathbf{r} \times \mathbf{F}$. Torque can be thought of as the magnitude and direction of a turning force acting on a (right-handed) bolt at the origin when the force $\mathbf{F}$ is applied to a wrench $\mathbf{r}$.

12.4.re6. A wrench 0.5 m long lies along the line $y = x$ in quadrant I in the $xy$-plane and grips a bolt at the origin. A force of magnitude 2 N in the direction $3i + 4j$ applied to the end of the wrench. Find the magnitude of the torque applied to the bolt.

**Answers**

12.4.re2a. $(-4, -1, 4)$

12.4.re2b. $(4, 1, -4)$

12.4.re2c. $\mathbf{0}$

12.4.re2d. $(-4, -2, 0)$

12.4.re2e. $(0, 1, 4)$

12.4.re2f. $\mathbf{k}$

12.4.re3a. $5\sqrt{5}$

12.4.re3b. 22

12.4.re4. 2

12.4.re5a. $(-4, 8, -6)$

12.4.re5b. $(2, -1, 3)$

12.4.re5c. $\mathbf{0}$

12.4.re5d. 6

12.4.re5e. 0

12.4.re5f. dne

12.4.re5g. $(18, 0, -12)$

12.4.re5h. $6/\sqrt{14}$

12.4.re6. $\frac{1}{5\sqrt{2}}$ Nm.
12.5: Equations of lines and planes

Lines

A line is determined by a point on the line a vector parallel the line.

12.5.re1. Find the equation(s) of the line passing through \((2, 1, -1)\) parallel to \(\langle 10, 9, 8 \rangle\).

Solution one: a point \((x, y, z)\) lies on the line iff the vector from \((2, 1, -1)\) to \((x, y, z)\) is parallel to \(\langle 10, 9, 8 \rangle\):

\[
\langle x, y, z \rangle - \langle 2, 1, -1 \rangle = t\langle 10, 9, 8 \rangle
\]

\[
\langle x, y, z \rangle = \langle 2, 1, -1 \rangle + t\langle 10, 9, 8 \rangle
\]

for some value of \(t\), so the vector-valued function

\[
r(t) = \langle 2, 1, -1 \rangle + t\langle 10, 9, 8 \rangle,
\]

or \(\langle 2 + 10t, 1 + 9t, -1 + 8t \rangle\), will trace out the line as \(t\) goes from \(-\infty\) to \(\infty\). This is called a **parametric vector form** of the line.

Solution two: setting \(x\), \(y\), and \(z\) equal the components of \(r\) gives

\[
x = 2 + 10t \quad y = 1 + 9t \quad z = -1 + 8t
\]

Together, these three equations are called a **parametric form** of the line.

Solution three: solving for \(t\) in terms of \(x\), \(y\), and \(z\) and setting these expressions equal gives

\[
\frac{x - 2}{10} = \frac{y - 1}{9} = \frac{z + 1}{8}
\]

These two equations are called a **symmetric form** of the line.

12.5.re2. Find an equation of the given line.

a. Through \((2, 0, 3)\), parallel \(\langle 1, 2, 3 \rangle\).  
b. Through the points \((2, 0, 3)\) and \((9, 1, 5)\).  
c. Through \((2, 8, 3)\), parallel \(\langle 0, 4, 5 \rangle\).

12.5.re3. Find a point on and a vector parallel to the given line. (There are many correct answers.)

a. \(x = 2 - t, y = 10 - t, z = 2 + 5t\).  
b. \(\frac{x - 3}{4} = y - 2 = 2z + 5\)  
c. \(r = \langle 2 + 3t, 3 - t, 8 \rangle\).
Planes

A plane is determined by a point on the plane a vector orthogonal to the plane (called a normal vector).

12.5.re4. Find an equation of the plane passing through \((7, 8, -9)\) normal to \(\langle 2, 3, 4 \rangle\).

Solution: a point \((x, y, z)\) lies on the plane iff the vector from \((7, 8, -9)\) to \((x, y, z)\) is orthogonal to \(\langle 2, 3, 4 \rangle\):

\[
\langle 2, 3, 4 \rangle \perp \langle x - 7, y - 8, z + 9 \rangle,
\]

which is true iff their dot product is zero. Therefore, the plane is the solution set to the equation

\[2(x - 7) + 3(y - 8) + 4(z + 9) = 0.\]

There are other equations for the same plane, obtainable from this one by some algebra, e.g.

\[2x + 3y + 4z = 2.\]

12.5.re5. Find an equation of the given plane.

a. Through the point \((2, 1, 0)\) and normal to \(\langle 1, 2, 3 \rangle\).
b. Through the points \((2, 1, 0), (3, 2, 1)\), and \((9, 1, 5)\).
c. Through the points \((2, 1, 0)\) and parallel the plane \(x - 5z = 10\).
d. The plane containing \((0, 1, -1)\) and the line \(x = 1 + t\), \(y = 2t - 1\), \(z = 3t\).

12.5.re6. Find a point on and a vector normal to the given plane. (There are many correct answers.)

a. \(3x + 10y + 5z = 6\).
b. \(2x + 1 = 4y - z\).

12.5.re7. Does the given equation(s) describe a line or a plane?

a. \(x = 4t\), \(y = 2 - t\), \(z = 5 - 6t\)  
b. \(x - \frac{2}{3} = 2y - 2 = \frac{x+1}{5}\)  
c. \(4x - y - 6z = 0\)

12.5.re8. Find the point of intersection, if there is one.

a. The line \(x = 1 - t\), \(y = 2t - 1\), \(z = 3 + 2t\) and the plane \(2x + 3y - z = -6\)
b. The lines \(x = 1 + 3t\), \(y = 2 - 4t\), \(z = 4 + t\) and \(x = 3 + 2s\) \(y = -s + 1\), \(z = -2s + 1\).
c. The lines \(x = 7 + 3t\), \(y = 6 - 4t\), \(z = 6 + t\) and \(x = -1 + 2s\) \(y = s + 1\), \(z = 3s + 1\).

Answers

12.5.re2a. vector form: \(\mathbf{r} = \langle 2 + t, 2t, 3 + 3t \rangle\). symmetric form: \(x - 2 = \frac{t}{3} = \frac{z-3}{3}\).
12.5.re2b. vector form: \(\mathbf{r} = \langle 2 + 7t, t, 3 + 2t \rangle\). symmetric form: \(\frac{x-2}{7} = \frac{y-3}{1} = \frac{z-2}{2}\).
12.5.re2c. vector form: \(\mathbf{r} = \langle 2, 8 + 4t, 3 + 5t \rangle\). symmetric form: \(x = 2; \frac{y-8}{4} = \frac{z-3}{5}\).
12.5.re3a. \(\langle 2, 10, 2 \rangle\), \(\langle -1, -1, 5 \rangle\).
12.5.re3b. \(\langle 3, 2, - \frac{2}{3} \rangle\), \(\langle 4, 1, \frac{1}{2} \rangle\).
12.5.re3c. \((5, 2, 8)\) (when \(t = 1\)), \(\langle 3, -1, 0 \rangle\).
12.5.re5a. \(x-2 + 2(y-1) + 3z = 0\), or \(x + 2y + 3z = 4\).
12.5.re5b. \(5x+2y-7z = 12\).
12.5.re5c. \(x-5z = 2\).
12.5.re5d. \(4x + y - 2z = 3\).
12.5.re5a. \(\langle 2, 0, 0 \rangle, \langle 3, 10, 5 \rangle\).
12.5.re6a. \(\langle 1, 0, -3 \rangle, \langle 2, -4, 1 \rangle\).
12.5.re7a. line (in parametric form). 12.5.re7b. line (in symmetric form).
12.5.re7c. plane. 12.5.re8a. \((x, y, z) = \langle 2, 3, 1 \rangle\) (at \(t = -1\)). 12.5.re8b. none. 12.5.re8c. both \(= \langle 1, 2, 4 \rangle\) at \(t = -2\), \(s = 1\).
12.6: Cylinders and Quadratic Surfaces

See http://kunklet.people.cofc.edu/MATH221/transformations221.pdf for a review of how changes to an equation change the corresponding graph.

Elementary conic sections in the \(xy\)-plane

1. Parabolas

\[
y = kx^2 \quad \text{for } k > 0 \\
\]

\[
x = ky^2 \quad \text{for } k > 0 \\
\]

2. Ellipses

\[
\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1
\]

semimajor axis = \(\max\{a, b\}\)
semiminor axis = \(\min\{a, b\}\)

3. Hyperbolas

\[
\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1 \\
\]

\[
\frac{y^2}{b^2} - \frac{x^2}{a^2} = 1
\]

For practice with conics, see http://kunklet.people.cofc.edu/MATH221/stew1005prob.pdf.
Cylinders
A cylinder is a surface obtained by dragging a planar curve in the direction perpendicular to its plane. Any equation in $x, y,$ or $x, z,$ or $y, z$ generates a cylinder in $xyz$ space.

12.6.re1. The graph of $x^2 + y^2 = 1$ is a circle in the $xy$ plane, where $z = 0$. Since the equation is independent of $z$, its graph is the (right circular) cylinder made up of copies of the same circle at all other $z$-values. (below left)

12.6.re2. The graph of $z = 1 - y^2$ is a parabola in the $yz$-plane $x = 0$. Dragging this curve in the $x$-direction generates the graph of the equation in $xyz$-space. (above right)

12.6.re3. Sketch the graph of the given equation.
   a. $z = \sin y$           b. $xy = -1$           c. $y^2 - z^2 = 4$           d. $x = 2z - z^2$
Quadratic surfaces

12.6.re4. $z = x^2 + y^2$

<table>
<thead>
<tr>
<th>When</th>
<th>equation is</th>
<th>cross-section is</th>
</tr>
</thead>
<tbody>
<tr>
<td>$x = 0$</td>
<td>$z = y^2$</td>
<td>a parabola;</td>
</tr>
<tr>
<td>$y = 0$</td>
<td>$z = x^2$</td>
<td>a parabola;</td>
</tr>
<tr>
<td>$z = \text{const.}$</td>
<td>$x^2 + y^2 = \text{const.}$</td>
<td>a circle.</td>
</tr>
</tbody>
</table>

elliptic paraboloid

12.6.re5. $x^2 + y^2 = z^2$

<table>
<thead>
<tr>
<th>When</th>
<th>equation is</th>
<th>cross-section is</th>
</tr>
</thead>
<tbody>
<tr>
<td>$x = 0$</td>
<td>$y^2 = z^2$</td>
<td>a pair of lines ($z = \pm y$);</td>
</tr>
<tr>
<td>$y = 0$</td>
<td>$x^2 = z^2$</td>
<td>a pair of lines ($z = \pm x$);</td>
</tr>
<tr>
<td>$z = \text{const.}$</td>
<td>$x^2 + y^2 = \text{const.}$</td>
<td>a circle.</td>
</tr>
</tbody>
</table>

elliptical cone

12.6.re6. $x^2 + y^2 + \frac{4}{9}z^2 = 1$

<table>
<thead>
<tr>
<th>When</th>
<th>equation is</th>
<th>cross-section is</th>
</tr>
</thead>
<tbody>
<tr>
<td>$x = 0$</td>
<td>$y^2 + \frac{4}{9}z^2 = 1$</td>
<td>an ellipse;</td>
</tr>
<tr>
<td>$y = 0$</td>
<td>$x^2 + \frac{4}{9}z^2 = 1$</td>
<td>an ellipse;</td>
</tr>
<tr>
<td>$z = \text{const.}$</td>
<td>$x^2 + y^2 = \text{const.}$</td>
<td>a circle.</td>
</tr>
</tbody>
</table>

Tip: graph of $x^2 + y^2 + \left(\frac{z}{3/2}\right)^2 = 1$ is obtained from unit sphere $x^2 = y^2 = z^2 = 1$ by scaling in $z$-direction by a factor of $3/2$. 

ellipsoid
12.6.re7. \( x^2 + y^2 - z^2 = 1 \)

<table>
<thead>
<tr>
<th>When</th>
<th>equation is</th>
<th>cross-section is</th>
</tr>
</thead>
<tbody>
<tr>
<td>( x = 0 )</td>
<td>( y^2 - z^2 = 1 )</td>
<td>a hyperbola ((y \neq 0));</td>
</tr>
<tr>
<td>( y = 0 )</td>
<td>( x^2 - z^2 = 1 )</td>
<td>a hyperbola ((x \neq 0));</td>
</tr>
<tr>
<td>( z = \text{const.} )</td>
<td>( x^2 + y^2 = \text{const.} )</td>
<td>a circle.</td>
</tr>
</tbody>
</table>

hyperboloid of one sheet

12.6.re8. \( -x^2 - y^2 + z^2 = 1 \)

<table>
<thead>
<tr>
<th>When</th>
<th>equation is</th>
<th>cross-section is</th>
</tr>
</thead>
<tbody>
<tr>
<td>( x = 0 )</td>
<td>( -y^2 + z^2 = 1 )</td>
<td>a hyperbola ((z \neq 0));</td>
</tr>
<tr>
<td>( y = 0 )</td>
<td>( -x^2 + z^2 = 1 )</td>
<td>a hyperbola ((z \neq 0));</td>
</tr>
<tr>
<td>( z = \text{const.} )</td>
<td>( x^2 + y^2 = \text{const.} )</td>
<td>a circle.</td>
</tr>
</tbody>
</table>

hyperboloid of two sheets

12.6.re9. \( z = y^2 - x^2 \)

<table>
<thead>
<tr>
<th>When</th>
<th>equation is</th>
<th>cross-section is</th>
</tr>
</thead>
<tbody>
<tr>
<td>( x = \text{const.} )</td>
<td>( z = y^2 + \text{const.} )</td>
<td>a parabola;</td>
</tr>
<tr>
<td>( y = \text{const.} )</td>
<td>( z = -x^2 + \text{const.} )</td>
<td>a parabola;</td>
</tr>
<tr>
<td>( z = \text{const.} )</td>
<td>( -x^2 + y^2 = \text{const.} )</td>
<td>a hyperbola.</td>
</tr>
</tbody>
</table>

hyperbolic paraboloid
See the Table 1 in 12.6 for a summary of quadratic surfaces in general. These six quadratic surfaces can always be drawn in the positions shown above after shifting (possibly by completing the square) and rotating axes (as in example 12.1.re5).

12.6.re10. Describe and sketch the graph of the given equation.

a. \( x^2 + 4y^2 + 8y + z^2 - 2z = 4 \)  
b. \( 4x^2 - z^2 = y^2 \)  
c. \( -x^2 + y^2 - z^2 = 9 \)  
d. \( -x^2 + y^2 + 2y - z^2 = 0 \)  
e. \( x^2 + 4y^2 = 4 \)  
f. \( -x^2 + y^2 + z^2 = 9 \)  
g. \( -x^2 + 4x + y^2 - z^2 = 3 \)  
h. \( x^2 = z^2 - y \)  
i. \( 2 = -x + y^2 + 4z^2 \)  
j. \( 0 = x^2 - 4y^2 + 4z^2 \)  
k. \( 0 = x^2 - 4z^2 \)

Answers

12.6.re3. a.,b.,c.,d.:
13.1: Vector-valued functions and the representation of curves by equations

Vector-valued functions
When \( x = x(t), y = y(t), \) and \( z = z(t) \) are scalar-valued functions of the scalar variable \( t \), then \( \mathbf{r}(t) = \langle x(t), y(t), z(t) \rangle \) is a **vector-valued function** of \( t \). Its domain is the set of \( t \)-values at which \( x(t), y(t), \) and \( z(t) \) are all well-defined.

13.1.re1. Find the domain of vector-valued function.
   a. \( \langle \sqrt{t} - 2, \ln(5 - t) \rangle \)
   b. \( \langle e^t, \frac{t+1}{t-1}, \sqrt{t} \rangle \)
   c. \( \langle \frac{1}{\sqrt{2t-3}}, \sin(t^2), \frac{2t-1}{t^2-5t+6} \rangle \)

Limits of vector-valued functions are computed component-wise.

13.1.re2. \( \lim_{t \to 1} \langle e^{t-1}, \frac{t^2-1}{t-1}, \ln t \rangle = \langle \lim_{t \to 1} e^{t-1}, \lim_{t \to 1} \frac{t^2-1}{t-1}, \lim_{t \to 1} \ln t \rangle \).

The first of these three limits equals 1 by continuity. The second is the same as \( \lim_{t \to 1} \frac{(t+1)(t-1)}{t-1} = 2 \). The third, by l’Hospital’s Rule, is \( \lim_{t \to 1} \frac{t-1}{2t} = \frac{1}{2} \). Therefore, the limit of \( \mathbf{r}(t) \) is \( \langle 1, 2, \frac{1}{2} \rangle \).

13.1.re3. Calculate the limit
   a. \( \lim_{t \to 0} \langle \frac{\sin t}{t}, \frac{1-\cos t}{t}, \frac{1-\cos t}{t^2} \rangle \)
   b. \( \lim_{t \to 4} \langle \sin(t\pi), \cos(\left(\frac{t+1}{2}\right)\pi), \frac{2-\sqrt{t}}{t-4} \rangle \)
   c. \( \lim_{h \to 0} \langle \frac{(t+h)^3-t^3}{h}, \frac{\ln(t+h)-\ln t}{h}, \frac{e^{-t-h}-e^{-t}}{h} \rangle \)

(Hint for part c: what is \( \lim_{h \to 0} \frac{f(t+h)-f(t)}{h} \)?)

Representations of curves by equations

<table>
<thead>
<tr>
<th>( \mathbb{R}^2 )</th>
<th>PARAMETRIC EQUATIONS</th>
<th>IMPLICIT EQUATIONS</th>
</tr>
</thead>
<tbody>
<tr>
<td>( x = x(t) )</td>
<td>( f(x, y) = 0 )</td>
<td></td>
</tr>
<tr>
<td>( y = y(t) )</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>( \mathbb{R}^3 )</th>
<th>PARAMETRIC EQUATIONS</th>
<th>IMPLICIT EQUATIONS</th>
</tr>
</thead>
<tbody>
<tr>
<td>( x = x(t) )</td>
<td>( f(x, y, z) = 0 )</td>
<td></td>
</tr>
<tr>
<td>( y = y(t) )</td>
<td>( g(x, y, z) = 0 )</td>
<td></td>
</tr>
<tr>
<td>( z = z(t) )</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

13.1.re4. The unit circle in \( \mathbb{R}^2 \) can be expressed implicitly by \( x^2+y^2 = 1 \) and parametrically by \( x = \cos t, y = \sin t \).
We sometimes express the parametric functions for \( x, y, \) and \( z \) along a curve in a single vector-valued function \( \mathbf{r}(t) \), as in the next example.

**13.1.re5.** The line in \( \mathbb{R}^3 \) passing through the point \((0, 1, -2)\) parallel to \( \langle 3, -4, 5 \rangle \) can be expressed parametrically by

\[
x = 3t, \quad y = 1 - 4t, \quad z = -2 + 5t,
\]

or

\[
\mathbf{r}(t) = \langle 3t, 1 - 4t, -2 + 5t \rangle.
\]

The same line can be expressed implicitly by the two equations of its symmetric form

\[
\frac{x}{3} = \frac{y - 1}{-4} = \frac{z + 2}{5},
\]

or, if you prefer, \( \frac{x}{3} + \frac{y - 1}{4} = 0, \frac{y - 1}{4} + \frac{z + 2}{5} = 0 \).

**Sketching curves in space**

Sketching curves in space by hand is a worthwhile exercise (though, in practice, best left to machines). It often helps to identify the equation of a surface to which the curve belongs, that is, one of the equations of its implicit representation.

**13.1.re6.** Sketch the curve given parametrically by \( \langle t, t^2, t^3 \rangle \).

It’s difficult to capture the shape of this curve in a single drawing. It might help to eliminate the parameter \( t \) to obtain an \( xy \) equation, an \( xz \) equation, and a \( yz \) equation. Then draw these curves in the three coordinate planes. These are views of the curve from the positive \( z \)-, negative \( y \)- and positive \( x \)-axes.
Based on these, we produce a sketch like the graph below left. To make the drawing clearer, include the cylinder $y = x^2$ on which the curve lies, below right.

### 13.1.re7
Describe and sketch the given parametrically by $\mathbf{r}(t)$.

- a. $\langle t, \sin t, -t \rangle$
- b. $\langle \sin t, \sin t, -\cos t \rangle$
- c. $\langle t, 1 - t^2, 1 \rangle$
- d. $\langle 2 \cos t, t, \sin t \rangle$
- e. $\langle t \sin t, t \cos t, t \rangle$
- f. $\langle t, 1 - t^2, t^2 \rangle$

### 13.1.re8
Find a parametric representation of the curve given implicitly by the system of equations.

- a. $x + y = 1$, $x^2 - y^2 = z$
- b. $z = (x - 1)^2 + y^2$, $x^2 + y^2 = 1$
- c. $xy = 1$, $z = e^{(x+y)^2}$
- d. $(x - 3)^2 + z^2 = 1$, $x^2 - y^2 + z^2 = 2$, $y > 0$

### Answers

- 13.1.re1a. $[2, 5)$
- 13.1.re1b. $[0, 1) \cup (1, \infty)$
- 13.1.re1c. $(\frac{3}{2}, 2) \cup (2, 3) \cup (3, \infty)$
- 13.1.re3a. $(1, 0, \frac{1}{2})$
- 13.1.re3b. $[0, 0, -\frac{1}{4})$
- 13.1.re3c. $[3t^2, \frac{1}{4}, -e^{-t}]
- 13.1.re7a. A sinusoidal curve along the line $x + z = 0$; $y = 0$
- 13.1.re7b. An ellipse in the plane $x = y$ whose shadow in the $xz$-plane is the unit circle.
- 13.1.re7c. The parabola $y = 1 - x^2$ in the plane $z = 1$
- 13.1.re7d. A helix on the elliptical cylinder $\frac{1}{4}x^2 + z^2 = 1$
- 13.1.re7e. A helix on the cone $x^2 + y^2 = z^2$
- 13.1.re7f. The parabola in the plane $y + z = 1$

Graphs a-f below.

13.1.re8a. $\langle t, 1 - t, 2t - 1 \rangle$
- 13.1.re8b. $\langle \cos t, \sin t, 2 - 2 \cos t \rangle$
- 13.1.re8c. $\langle t, t^{-1}, e^{t^2 + 2t + t^{-2}} \rangle$ $(t \neq 0)$
- 13.1.re8d. $\langle 3 + \cos t, \sqrt{8 + 6 \cos t}, \sin t \rangle$
13.2: Calculus on vector-valued functions

Differentiation

If \( \mathbf{r}(t) = \langle x(t), y(t), z(t) \rangle \), then \( \frac{d\mathbf{r}}{dt} = \langle \frac{dx}{dt}, \frac{dy}{dt}, \frac{dz}{dt} \rangle \).

13.2.re1. \( \frac{d}{dt} \langle e^t, \sec t, \tan^{-1} t \rangle = \langle e^t, \sec t \tan t, \frac{1}{t^2 + 1} \rangle \)

See page 858 of the text for important rules of vector differentiation.

13.2.re2. Find the derivative of vector-valued function.

a. \( \langle \sqrt{t - 2}, \ln |t - 5| \rangle \)  
b. \( \langle e^t, \frac{t+1}{t-1}, \sqrt{t^2 + 1} \rangle \)  
c. \( \langle -\frac{1}{\sqrt{3t-1}}, \sinh(t^2), \cosh^2 t \rangle \)

Integration

Like differentiation, integration of vector-valued functions is performed component-wise. The Fundamental Theorem of Calculus for vector-valued functions says that

\[
\int_a^b \frac{d\mathbf{r}}{dt} \, dt = \mathbf{r}(b) - \mathbf{r}(a),
\]

provided \( \frac{d\mathbf{r}}{dt} \) is continuous.

13.2.re3.

a. \( \int (t^2 + 1, \sec t) \, dt = \langle \frac{1}{3}t^3 + t + C_1, \ln |\sec t + \tan t| + C_2 \rangle \), or \( \langle \frac{1}{3}t^3 + t, \ln |\sec t + \tan t| \rangle + C \), where \( C \) is a constant vector in \( \mathbb{R}^2 \).

b. \( \int_{-1}^1 (t^2 + t, 2te^{-t^2}) \, dt = \langle \frac{1}{3}t^3 + \frac{1}{2}t^2, -e^{-t^2} \rangle \bigg|_{-1}^1 = \langle \frac{1}{3} + \frac{1}{2}, -e^{-1} \rangle - \langle -\frac{1}{3} + \frac{1}{2}, -e^{-1} \rangle = \langle \frac{2}{3}, 0 \rangle \)

13.2.re4. Integrate.

a. \( \int \langle \frac{t}{t^2 - 4}, \sin t \cos^3 t, te^{t} \rangle \, dt \)  
b. \( \int \langle \frac{1}{t^2 - 4}, e^x \sec^2(e^x), \tan t \rangle \, dt \)  
c. \( \int_0^1 \langle \frac{1}{t^2 + 1}, \sin(\pi t), (t + 1)^4 \rangle \, dt \)
Tangent & unit tangent vectors

If \( \frac{d\mathbf{r}}{dt} \neq 0 \), then \( \frac{d\mathbf{r}}{dt} \) is a tangent vector to the curve parametrized by \( \mathbf{r} \), and

\[
\mathbf{T} = \frac{\mathbf{r}'(t)}{|\mathbf{r}'(t)|}
\]

is the unit tangent vector to the curve.

The vector-valued function \( \mathbf{r} \) is said to be smooth if \( \frac{d\mathbf{r}}{dt} \) is continuous and never equal 0.

13.2.re5. For the given \( \mathbf{r} \), find \( \frac{d\mathbf{r}}{dt} \), \( \mathbf{T} \), and the line tangent to the curve parametrized by \( \mathbf{r} \) at the point corresponding to the given time.

a. \( \mathbf{r} = \langle t^2 + 1, t^3 - t, t \rangle \), \( t = 1 \)
   
   b. \( \mathbf{r} = \langle \frac{1}{2} t^2, \ln |t| \rangle \), \( t = -1 \)
   
   c. \( \mathbf{r} = e^t \mathbf{i} + e^t \sin t \mathbf{j} + e^t \cos t \mathbf{k} \), \( t = 0 \)

Answers

13.2.re2a. \( \langle \frac{1}{2} (t - 2)^{-1/2}, (t - 5)^{-1} \rangle \). 13.2.re2b. \( \langle e^t, -2(t - 1)^{-2}, t(t^2 + 1)^{-1/2} \rangle \). 13.2.re2c. \( -\frac{1}{(3t - 1)^{4/3}}, 
   \frac{1}{3} \ln |t - 4|, -\frac{1}{4} \cos^4 t, t e^t - e^t + \mathbf{C} \). 13.2.re4a. \( \langle \frac{1}{2} (\ln |t^2 - 4| - \frac{1}{4} \cos^4 t, t e^t - e^t + \mathbf{C} \). 13.2.re4b. \( \langle \frac{1}{4} (\ln |t - 2| - \ln |t + 2|), \tan(e^t), \ln |\sec t| + \mathbf{C} \). 13.2.re4c. \( \langle \frac{5}{2}, \frac{1}{2}, \frac{3}{4} \rangle \). 13.2.re5a. \( \frac{d\mathbf{r}}{dt} = \langle 2t, 3t^2 - 1, 1 \rangle \). \( \mathbf{T} = \frac{1}{\sqrt{1 + 3t^2 - (3t^2 - 1)^2}} \langle 2t, 3t^2 - 1, 1 \rangle \). Line is \( \langle 2, 0, 1 \rangle + t \langle 2, 2, 1 \rangle \).
   
   13.2.re5b. \( \frac{d\mathbf{r}}{dt} = \langle t, t^{-1} \rangle \). \( \mathbf{T} = \frac{\frac{1}{\sqrt{t^2 + 2}}}{\sqrt{t^2 + 2}} \langle t, t^{-1} \rangle \). Line is \( \langle \frac{1}{2}, 0 \rangle - t \langle 1, 1 \rangle \).
   
   13.2.re5c. \( \frac{d\mathbf{r}}{dt} = e^t \langle 1, \sin t + \cos t, \cos t - \sin t \rangle \). \( \mathbf{T} = \frac{1}{\sqrt{3}} \langle 1, \sin t + \cos t, \cos t - \sin t \rangle \). Line is \( \langle 1, 0, 1 \rangle + t \langle 1, 1, 1 \rangle \).
13.3: Arc length, curvature, and the TNB frame

Arc length
The total length of the curve parametrized by \( \mathbf{r}(t) = (x(t), y(t), z(t)) \) from \( t = a \) to \( t = b \) is

\[
\int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2} \, dt = \int_a^b \left| \frac{d\mathbf{r}}{dt} \right| \, dt
\]

The length of a curve is also called its **arc length**, but arc length can also refer to a variable \( s \) that increases from 0 at the beginning of the curve to its total length at the end of the curve. If you drove a car along the curve, and if you set the trip odometer to zero at start of the curve, then the odometer will display the value of \( s \) as you travel. At time \( t \), the current value of \( s \) is

\[
s = \int_a^t \left| \frac{d\mathbf{r}}{dt} \right| \, dt^*
\]

(where \( t^* \) is dummy variable of integration). As a consequence, the particle’s speed

\[
\frac{ds}{dt} = \left| \frac{d\mathbf{r}}{dt} \right|
\]

The direction of \( \frac{d\mathbf{r}}{dt} \) = the direction of the particle’s motion, and

the magnitude of \( \frac{d\mathbf{r}}{dt} \) = the speed of the particle.

13.3.re1. Find the length of the helix parametrized by \( \mathbf{r} = (\sin t, \cos t, t) \) for \( 0 \leq t \leq \pi \).
Solution: \( \frac{ds}{dt} = |(\cos t, -\sin t, 1)| = \sqrt{\cos^2 t + \sin^2 t + 1} = \sqrt{2} \). Therefore, the arc length equals \( \int_0^\pi \sqrt{2} \, dt = \sqrt{2}\pi \).

13.3.re2. Find the length of the given curve.
   a. \( \mathbf{r} = (\frac{1}{2} t^2, \frac{4}{3} t^{3/2}, 2t) \) \( 0 < t < 1 \)
   b. \( \mathbf{r} = (\ln t, \frac{1}{2} t^2, \sqrt{2}t) \) \( 1 < t < e \)
   c. \( y = x^2, z = \frac{2}{3} x^3 \), from \((-1, 1, -\frac{2}{3})\) to \((1, 1, \frac{2}{3})\)
The **Unit Tangent**, **Unit Normal**, and **Binormal** are three mutually orthogonal unit vectors given by

\[
T = \frac{\frac{dr}{dt}}{\left| \frac{dr}{dt} \right|} \quad N = \frac{\frac{dT}{dt}}{\left| \frac{dT}{dt} \right|} \quad B = T \times N
\]

(as long as \(\frac{dr}{dt} \neq 0\)). \(T, N, B\) is a right-handed system. You can think of them as a set of coordinate axes that travels along the curve, twisting so that \(T\) is always tangent to the curve and \(N\) always points in the direction the curve is turning. There’s nice animation of the **TNB** “frame” moving along a curve in space at [https://youtu.be/JZGFcwipHYY](https://youtu.be/JZGFcwipHYY).

**Curvature** is the scalar given by

\[
\kappa = \frac{\left| \frac{dT}{ds} \right|}{\left| \frac{dr}{dt} \right|} = \frac{\left| \frac{dT}{dt} \right| / ds}{dt} = \frac{|r'(t) \times r''(t)|}{|r'(t)|^3}
\]

\(\kappa\) is the speed at which \(T\) turns when we travel along the curve with the constant speed 1. \(\kappa\) is small here. \(\kappa\) is large here.

The curvature of a straight line is zero, and the curvature of a circle is the reciprocal of its radius. Along most curves, curvature is not constant.

While \(r\) and its derivatives depend on the motion of the particle tracing out the curve, \(T, N, B,\) and \(\kappa\) are geometric properties of the curve itself.
13.3. re3. Find $\mathbf{T}$, $\mathbf{N}$, $\mathbf{B}$, and $\kappa$ along the curve given by $\mathbf{r} = \langle \frac{1}{2}t^2, \frac{4}{3}t^{3/2}, 2t \rangle$ ($t > 0$).

Two tips in calculations such as these:

If $c$ is a scalar and $\mathbf{u}$ a vector, then $\frac{d}{dt}(c\mathbf{u}) = \frac{dc}{dt}\mathbf{u} + c\frac{d\mathbf{u}}{dt}$.

If $c$ is positive, then $\mathbf{u}$ and $c\mathbf{u}$ have the same normalization.

Solution. $\frac{d\mathbf{r}}{dt} = \langle t, 2t^{1/2}, 2 \rangle$, and $\frac{d\mathbf{r}}{dt}$, the length of $\frac{d\mathbf{r}}{dt}$, is $\sqrt{t^2 + 4t + 4} = \sqrt{(t + 2)^2} = t + 2$ (since $t + 2 > 0$). Normalize $\frac{d\mathbf{r}}{dt}$ to obtain

$$\mathbf{T} = (t + 2)^{-1} \langle t, 2t^{1/2}, 2 \rangle$$

Differentiate using the product rule:

$$\frac{d\mathbf{T}}{dt} = (t + 2)^{-1} \langle 1, t^{-1/2}, 0 \rangle - (t + 2)^{-2} \langle t, 2t^{1/2}, 2 \rangle$$

We can obtain $\mathbf{N}$ by normalizing

$$(t + 2)^2 \frac{d\mathbf{T}}{dt} = (t + 2) \langle 1, t^{-1/2}, 0 \rangle - \langle t, 2t^{1/2}, 2 \rangle = \langle 2, 2t^{-1/2} - t^{1/2}, -2 \rangle,$$

the magnitude of which is

$$\sqrt{4 + (2t^{-1/2} - t^{1/2})^2 + 4} = \sqrt{4 + (4t^{-1} - 4t + t) + 4}$$

$$= \sqrt{4t^{-1} + 4t} = 2t^{-1/2} + t^{1/2}$$

Therefore,

$$\mathbf{N} = (2t^{-1/2} + t^{1/2})^{-1} \langle 2, 2t^{-1/2} - t^{1/2}, -2 \rangle$$

Now $\mathbf{B} = \mathbf{T} \times \mathbf{N} =

(t + 2)^{-1} (2t^{-1/2} + t^{1/2})^{-1} \left( \langle t, 2t^{1/2}, 2 \rangle \times \langle 2, 2t^{-1/2} - t^{1/2}, -2 \rangle \right)

= (t^{-1/2}(t + 2)^2)^{-1} \begin{vmatrix} i & j & k \\ t & 2t^{1/2} & 2 \\ 2 & 2t^{-1/2} - t^{1/2} & -2 \end{vmatrix}

= \frac{t^{1/2}}{(t + 2)^2} \left( i \begin{vmatrix} 2t^{1/2} & 2 \\ 2t^{-1/2} - t^{1/2} & -2 \end{vmatrix} - j \begin{vmatrix} t & 2 \\ 2 & -2 \end{vmatrix} + k \begin{vmatrix} t & 2t^{1/2} \\ 2 & 2t^{-1/2} - t^{1/2} \end{vmatrix} \right)

= \frac{t^{1/2}}{(t + 2)^2} \left( -(2t^{1/2} + 4t^{-1/2})i + (2t + 4)j - (2t^{1/2} + t^{3/2})k \right)
or, remarkably,
\[
\left\langle \frac{-2}{t + 2} \cdot \frac{2t^{1/2}}{t + 2}, \frac{-t}{t + 2} \right\rangle.
\]

Finally, we can calculate \( \kappa \) either from
\[
\left| \frac{dT}{dt} \right| \div \frac{ds}{dt} = \left( \frac{2t^{-1/2} + t^{1/2}}{(t + 2)^2} \right) \div (t + 2) = \frac{t^{-1/2}}{(t + 2)^2}
\]
or by calculating
\[
\mathbf{r}' \times \mathbf{r}'' = \begin{vmatrix}
i & j & k \\
t & 2t^{1/2} & 2 \\
1 & t^{-1/2} & 0
\end{vmatrix} = \langle -2t^{-1/2}, 2, -t^{1/2} \rangle = t^{-1/2} \langle -2, 2t^{1/2} - t \rangle,
\]
the length of which is \( t^{-1/2}(t + 2) \), which we divide by \( (\frac{ds}{dt})^3 \) to obtain again
\[
\kappa = t^{-1/2}(t + 2) \div (t + 2)^3 = \frac{t^{-1/2}}{(t + 2)^2}.
\]

Note that
\[
\mathbf{B} = (t + 2)^{-1} \langle -2, 2t^{1/2}, -t \rangle
\]
is a positive-scalar multiple of \( \mathbf{r}' \times \mathbf{r}'' \) and so could be obtained by normalizing this cross product. We’ll see in section 13.4 that this is always the case.

13.3.\textbf{re4.} Find \( \mathbf{T}, \mathbf{N}, \mathbf{B}, \) and \( \kappa \) along the given curve.

a. \( \mathbf{r} = \langle 4\sin t, 3t, -4\cos t \rangle \)

b. \( \mathbf{r} = \langle \ln t, \frac{1}{2} t^2, \sqrt{2}t \rangle \)

c. \( \mathbf{r} = \langle \cos^3 t, \sin^3 t, 0 \rangle \quad 0 < t < \pi/2 \)

d. \( \mathbf{r} = \langle 3\sin t, 4\sin t, 5\cos t \rangle \)
The normal and osculating planes

As \( \mathbf{TNB} \) travel along the curve, two planes travel along with them. At any point on the curve, the **normal plane** passes through that point and is orthogonal to \( \mathbf{T} \), and the **osculating plane** passes through that point and is orthogonal to \( \mathbf{B} \).

It is not necessary to compute \( \mathbf{T} \) to find the normal plane, since \( \mathbf{r}' \) is also orthogonal to this plane. It is also unnecessary to compute \( \mathbf{B} \) to find the osculating plane, since, as we’ll see in 13.4, \( \mathbf{r}'(t) \times \mathbf{r}''(t) \) is orthogonal to this plane.

13.3.re5. Find the normal and osculating planes to the curve at the given point.

- a. \( \mathbf{r} = \langle t, t \sin t, t \cos t \rangle \), at \((2\pi, 0, 2\pi)\)
- b. \( \mathbf{r} = \langle t^2, t^2 \sin t, t^2 \cos t \rangle \), at \((2\pi, 0, 2\pi)\)
- c. \( y = z^2 \) and \( xy = 1 \), at \((\frac{1}{4}, 4, -2)\)

Answers

13.3.re2a. \( \frac{5}{2} \) 13.3.re2b. \( \frac{2+1}{3} \) 13.3.re2c. \( \frac{10}{3} \) 13.3.re4a. \( \mathbf{T} = \frac{1}{5}(4 \cos t, 3, 4 \sin t) \), \( \mathbf{N} = \langle -\sin t, 0, \cos t \rangle \), \( \mathbf{B} = \frac{1}{5}(3 \cos t, 4 \sin t) \), \( \kappa = \frac{4}{25} \). 13.3.re4b. \( \mathbf{T} = (t^2 + 1)^{-1}(1, t^2, \sqrt{2}t) \), \( \mathbf{N} = (t^2 + 1)^{-1}(\sqrt{2}t, \sqrt{2}t, 1 - t^2) \), \( \mathbf{B} = (t^2 + 1)^{-2}( -t^2, -1, \sqrt{2}t) \), \( \kappa = 2^{1/2}t^{-2}(t^2 + 1)^{-2} \). 13.3.re4c. \( \mathbf{T} = \langle -\cos t, \sin t, 0 \rangle \), \( \mathbf{N} = \langle \sin t, \cos t, 0 \rangle \), \( \mathbf{B} = -\mathbf{k} \), \( \kappa = \frac{1}{4} \) sec \( t \) csc \( t \). 13.3.re4d. \( \mathbf{T} = \frac{1}{\sqrt{5}}(3 \cos t, 4 \cos t, -5 \sin t) \), \( \mathbf{N} = -\frac{1}{\sqrt{5}}(3 \sin t, 4 \sin t, \cos t) \), \( \mathbf{B} = \langle -4/5, 3/5, 0 \rangle \), \( \kappa = \frac{1}{5} \). 13.3.re5a. n.p.: \( x + 2\pi y + z = 4\pi \); o.p.: \( -2\pi^2 - 1)(x - 2\pi) + \pi y + z - 2\pi = 0 \). 13.3.re5b. same as in a. 13.3.re5c. n.p.: \( \frac{1}{3}(x - \frac{1}{3}) + 4(y - 4) + (z + 2) = 0 \); o.p.: \( -2(x - \frac{1}{3}) + \frac{4}{3}(y - 4) - (z + 2) = 0 \).
13.4: Velocity and acceleration

It \( \mathbf{r}(t) \) represents the position of an object at time \( t \), then its first two derivatives are named velocity and acceleration. The magnitude of velocity is speed. We sometimes “suppress the \( t \)” when writing these functions, e.g., when we write \( \mathbf{r} \) instead of \( \mathbf{r}(t) \).

\[
\begin{align*}
\mathbf{r} &= \text{position (vector)} \\
\mathbf{v} &= \frac{d\mathbf{r}}{dt} = \text{velocity (vector)} \\
\mathbf{a} &= \frac{d\mathbf{v}}{dt} = \text{acceleration (vector)} \\
\frac{ds}{dt} &= |\mathbf{v}| = \text{speed (scalar)}
\end{align*}
\]

13.4.re1. Find velocity, acceleration, and speed for the given position.

a. \( \mathbf{r} = \langle 4 \sin t, 3t, -4 \cos t \rangle \)  
   b. \( \mathbf{r} = \langle \ln t, \frac{1}{2} t^2, \sqrt{2t} \rangle \)  
   c. \( \mathbf{r} = \langle \cos^3 t, \sin^3 t, t \rangle \)

Initial value problems

13.4.re2. Find position \( \mathbf{r} \) if \( \mathbf{a}(t) = \langle te^t, 2t, 1 \rangle \), \( \mathbf{v}(0) = \langle -1, 1, 0 \rangle \), and \( \mathbf{r}(0) = \langle -1, 0, -1 \rangle \).

Solution: Integrate once to find \( \mathbf{v} \) and again to find \( \mathbf{r} \). Use the given values of \( \mathbf{v} \) and \( \mathbf{r} \) to solve for constants of integration. (Integrate \( te^t \) by parts.)

\[
\begin{align*}
\mathbf{v} &= \langle te^t - e^t, t^2, t \rangle + \mathbf{C} \\
\langle -1, 1, 0 \rangle &= \langle -1, 0, 0 \rangle + \mathbf{C} \quad \Rightarrow \quad \mathbf{C} = \langle 0, 1, 0 \rangle \\
\mathbf{v} &= \langle te^t - e^t, t^2, t \rangle + \langle 0, 1, 0 \rangle \\
&= \langle te^t - e^t, t^2 + 1, t \rangle \\
\mathbf{r} &= \langle te^t - 2e^t + \frac{1}{3} t^3 + t, \frac{1}{2} t^2 \rangle + \mathbf{D} \\
\langle -1, 0, -1 \rangle &= \langle -2, 0, 0 \rangle + \mathbf{D} \quad \Rightarrow \quad \mathbf{D} = \langle 1, 0, -1 \rangle \\
\mathbf{r} &= \langle te^t - 2e^t + \frac{1}{3} t^3 + t, \frac{1}{2} t^2 \rangle + \langle 1, 0, -1 \rangle \\
&= \langle te^t - 2e^t + 1, \frac{1}{3} t^3 + t, \frac{1}{2} t^2 - 1 \rangle 
\end{align*}
\]

13.4.re3. A constant force of magnitude 15 in the direction of \(-3\mathbf{i} + 4\mathbf{k}\) acts on an object of mass 1/2. If, at time 0, the object’s position and velocity are \(2\mathbf{i} - \mathbf{k}\) and \(\mathbf{j} - \mathbf{i}\) respectively, find the object’s position at time \( t \). Hint: Newton’s second law of motion states that force = mass \( \times \) acceleration.

13.4.re4. An acrobat is at to be shot from a cannon with speed \(32\sqrt{2}\) ft/sec at an upward angle \(\pi/4\) radians. So that we may correctly position the net to catch her, find the (horizontal) distance from the cannon at which the acrobat will descend to altitude of 12 ft. Assume the acrobat is launched from altitude zero and that, due to gravity, her acceleration is \(-32\) ft/sec\(^2\) downward.

Tip: place the cannon at the origin in the \(xy\)-plane, firing into the first quadrant. At what \( x \) will \( y = 12 \)?
Tangential and normal components of acceleration

The tangential and normal components of \( \mathbf{a} \) are scalars \( a_T \) and \( a_N \) for which

\[
(13.4.1) \quad \mathbf{a} = a_T \mathbf{T} + a_N \mathbf{N}.
\]

Consequently, \( \mathbf{v} \) and \( \mathbf{a} \) lie in the same plane as \( \mathbf{T} \) and \( \mathbf{N} \), the osculating plane (as seen in figure 13.3.1). \( a_T \) and \( a_N \) are the components of \( \mathbf{a} \) in the directions \( \mathbf{T} \) and \( \mathbf{N} \) seen in section 12.3 and can be calculated any of these formulas

\[
a_T = \frac{d^2 s}{dt^2} = |\mathbf{a}| \cos \theta = \frac{\mathbf{v} \cdot \mathbf{a}}{|\mathbf{v}|} = |\mathbf{a}| \sin \theta = \frac{\mathbf{v} \times \mathbf{a}}{|\mathbf{v}|} = \sqrt{|\mathbf{a}|^2 - a_T^2},
\]

Note that \( a_T \) is positive [negative] if the object is speeding up [slowing down]. Unless speed or curvature is zero, \( a_N \) is positive: \( \mathbf{a} \) and \( \mathbf{N} \) lie on the same side of the line containing \( \mathbf{v} \) in the osculating plane.

13.4.re5. Find \( a_T \) and \( a_N \) for the given \( \mathbf{r} \).

a. \( \mathbf{r} = \langle 4 \sin t, 3t, -4 \cos t \rangle \)  b. \( \mathbf{r} = \langle \ln t, \frac{1}{2} t^2, \sqrt{2} t \rangle \)  c. \( \mathbf{r} = \langle \frac{1}{2} t^2, \frac{4}{3} t^{3/2}, 2t \rangle \) (see 13.3.re3)
Calculating the TNB frame from $r'$ and $r''$

It’s possible to find $N$ and $B$ without differentiating $T$ (which is easily found by normalizing $r'$). Since $r''$ lies in the osculating plane on the same side of $T$ as $N$, one can find $B$ by normalizing $r' \times r''$. And since TNB form a right-handed system, $N$ must equal $B \times T$:

$$T = \frac{r'}{|r'|} \quad N = B \times T \quad B = \frac{r' \times r''}{|r' \times r''|}$$

13.4.re6. Suppose that, at a particular time, $v = \langle -2, 1, 2 \rangle$ and $a = \langle 1, 1, -1 \rangle$. Find the following at that time.

a. $a_T$  
b. $a_N$  
c. $T$  
d. $N$  
e. $B$

Answers

13.4.re1a. $v = \langle 4 \cos t, 3, 4 \sin t \rangle$, $a = \langle -4 \sin t, 0, 4 \cos t \rangle$, $\frac{dv}{dt} = 5$  
13.4.re1b. $v = \langle t^{-1}, t, \sqrt{2} \rangle$, $a = \langle -t^{-2}, 1, 0 \rangle$, $\frac{da}{dt} = t^{-1} + t$.  
13.4.re1c. $v = \langle -3 \cos^2 t \sin t, 3 \sin^2 t \cos t, 1 \rangle$, $a = \langle 6 \sin^2 t \cos t - 3 \cos^3 t, 6 \cos^2 t \sin t - 3 \sin^3 t, 0 \rangle$, $\frac{dv}{dt} = \sqrt{1 + 9 \cos^2 t \sin^2 t}$.  
13.4.re3. $a = 6 \langle -3, 0, 4 \rangle$, $v = 6 \langle -3t, 0, 4t \rangle + \langle -1, 1, 0 \rangle$, $r = \langle -9t^2 - t + 2, t, 12t^2 - 1 \rangle$.  
13.4.re4. 48 ft (at time $t = 3/2$).  
13.4.re5a. $a_T = 0$, $a_N = 4$  
13.4.re5b. $a_T = \frac{t - t^3}{t + t^2 + 1}$, $a_N = \frac{\sqrt{2} - 2}{t}$.  
13.4.re5c. $a_T = 1$, $a_N = t^{-1/2}$.  
13.4.re6a. $-1$.  
13.4.re6b. $\sqrt{2}$.  
13.4.re6c. $\frac{1}{3}(2, 1, -2)$.  
13.4.re6d. $\frac{1}{\sqrt{2}}(1, -4, -1)$.  
13.4.re6e. $\frac{1}{\sqrt{2}}(-1, 0, -1)$.  

14.1: Real-valued functions of several real variables

Recall that a function is a rule that assigns to each elements of a set, called its domain, a unique element of another set, called its range. We may write

\[ f : U \rightarrow V \]

to indicate that \( f \) maps elements of the set \( U \) to elements of the set \( V \).

In Calculus III, we consider functions with domain \( \subseteq \mathbb{R}^m \) and range \( \subseteq \mathbb{R}^n \) for some \( m \) and \( n \). Generally, a function’s domain is easier to determine than its range.

14.1.re1.

<table>
<thead>
<tr>
<th>function ( f )</th>
<th>domain</th>
<th>range</th>
</tr>
</thead>
<tbody>
<tr>
<td>( f(x) = \frac{1}{x-2} )</td>
<td>( (-\infty, 2) \cup (2, \infty) \subseteq \mathbb{R}^1 )</td>
<td>( (-\infty, 0) \cup (0, \infty) \subseteq \mathbb{R}^1 )</td>
</tr>
<tr>
<td>( g(x) = \ln x )</td>
<td>( (0, \infty) \subseteq \mathbb{R}^1 )</td>
<td>( \mathbb{R}^1 )</td>
</tr>
<tr>
<td>( u(t) = ((t-1)^2, \sin t, t) )</td>
<td>( \mathbb{R}^1 )</td>
<td>( ) a curve ( \subseteq \mathbb{R}^3 )</td>
</tr>
<tr>
<td>( v(x, y) = \frac{x^2 - y^2}{x + 2y - 1} )</td>
<td>( {(x, y) \mid x + 2y \neq 1} \subseteq \mathbb{R}^2 )</td>
<td>( \mathbb{R}^1 )</td>
</tr>
<tr>
<td>( w(x, y, z) = \sqrt{1 - x^2 - 4z^2} )</td>
<td>( {(x, y, z) \mid x^2 + 4z^2 \leq 1} \subseteq \mathbb{R}^3 )</td>
<td>([0, 1] \subseteq \mathbb{R}^1 )</td>
</tr>
</tbody>
</table>

14.1.re2. Sketch and describe the domain of the given function.

a. \( u(x, y) = \frac{x^2-y^2}{x+2y-1} \)

b. \( v(x, y, z) = \sqrt{1 - x^2 - 4z^2} \)

Solutions.

a. The plane minus the line \( x + 2y = 1 \).

b. The cylinder \( x^2 - 4z^2 = 1 \) and its interior.

14.1.re3. Describe and sketch the domain of the given function.

a. \( \nu(x, y) = \sqrt{y-x + \sin^{-1}(x+y)} \)

b. \( \omega(x, y) = \ln((x-y)(y-2)) \)

c. \( \alpha(x, y, z) = \frac{x^2 + xz + z^2}{(x-y)(z-x^2-y^2)} \)

d. \( \beta(x, y, z) = \sqrt{1 - y^2 - z^2} + \frac{1}{\sqrt{x^2 + y^2 + z^2} - 4} \)
Graphs of functions on $\mathbb{R}$ and $\mathbb{R}^2$

The **graph of an equation** $g(x, y) = 0$ [or $g(x, y, z) = 0$] is its **solution set**: the set of all points in the $xy$-plane [or in $xyz$-space] whose coordinates satisfy the equation. The graph of a function $f(x)$ [or $f(x, y)$] is the graph of the equation $y = f(x)$ [or $z = f(x, y)$], that is, the set of all the points $(x, f(x))$ in the plane [or the set of all points $(x, y, f(x, y))$ in space].

14.1.re4. The graph of $f(x) = x^2$ is the parabola $y = x^2$, below left. The graph of $g(x, y) = x^2 - y^2$ is the hyperbolic paraboloid $z = x^2 - y^2$, below right.

![Graphs of functions](image)

14.1.re5. Sketch the graph of the given function.

a. $f(x) = \frac{x^2 - 1}{x}$

b. $g(x, y) = -4x^2 - 8x - y^2$

c. $h(x, y) = 1 - \sqrt{x^2 + y^2}$

d. $k(x, y) = 4 - 2x - 3y$

**Level curves and surfaces; contour maps**

The **level curves** of a function $f(x, y)$ are the graphs of equations of the form $f(x, y) = k$, where $k$ is a constant. That is, level curves are the curves along which $f(x, y)$ is constant. The **level surfaces** of a function $f(x, y, z)$ are the graphs of equations $f(x, y, z) = k$, that is, the surfaces along which $f(x, y, z)$ is constant.

Each point in the domain of $f(x, y)$ lies on exactly one level curve of $f$. Consequently, the level curves of $f(x, y)$ are non-overlapping and completely fill the domain of $f$. Likewise, the level surfaces of $f(x, y, z)$ are nonoverlapping and completely fill the domain of $f(x, y, z)$.

A **contour map** for a function is a graph of a representative sample of its level curves. Typically, a contour map displays the curves

$$f(x, y) = k_0, \quad f(x, y) = k_1, \quad f(x, y) = k_2, \quad \ldots f(x, y) = k_n$$

for some equally spaced numbers $k_0, k_1, \ldots k_n$. 
14.1.re6. The domain of \( f(x, y) = \sqrt{x^2 + y^2} \) is the entire \( xy \)-plane, and its level curves are circles centered at the origin. Each point in the plane lies on exactly one of the level curves of \( f \). The graph of \( f(x, y) \) appears below left. A contour map for \( f(x, y) \) for \( f(x, y) = 0, 1, 2, \ldots, 8 \) appears below right.

If the contour map displays the level curves for equally spaced values of \( f \), then we judge where and in which directions the function is increasing rapidly per unit

14.1.re7. More graphs and contour maps:

- Graph of \( x^2 + y^2 \)
- Contour map of \( x^2 + y^2 \)
- Graph of \( x - \frac{1}{12} x^3 - \frac{1}{4} y^2 + \frac{1}{2} \)
- Contour map of \( x - \frac{1}{12} x^3 - \frac{1}{4} y^2 + \frac{1}{2} \)
14.1.re8. We can’t display the graph of $x^2 + y^2 - z^2$ in three dimensions, but we can still see a contour map:

Answers
14.1.re3a. $y \geq x$ and $-1 \leq x + y \leq 1$. The region in the plane between $x + y = -1$ and $x + y = 1$ and above $y = x$. 14.1.re3b. $(x - y)(y - 2) > 0$. The region above $y = 2$ and below $y = x$ plus the region below $y = 2$ and above $y = x$ (excluding those lines). 14.1.re3c. $z \neq x^2 + y^2$ and $x \neq y$. All of $\mathbb{R}^3$, minus the circular paraboloid $z = x^2 + y^2$ and the plane $x = y$. 14.1.re3d. $y^2 + z^2 \leq 1$ and $x^2 + y^2 + z^2 > 4$. The part of the cylinder $y^2 + z^2 = 1$ (and its interior) that lies outside the sphere $x^2 + y^2 + z^2 = 4$.
14.1.re5. Top row = \{a, b\}. Bottom row = \{c, d\}.
14.2: Limits and Continuity of functions of several variable

Showing that a limit fails to exist

It is fortunate that most limits we have to take in Calc III are limits on one variable only, because genuine limits of functions of two or more variables can be difficult to evaluate.

14.2.re1. Show that the limit does not exist.

a. \( \lim \limits_{(x,y) \to (0,0)} \frac{x^2 + 2y^2}{x^2 + y^2} \)  

b. \( \lim \limits_{(x,y) \to (0,0)} \frac{xy}{(x + y)^2} \)

Solutions:

a. If it exists, the limit of \( \frac{x^2 + 2y^2}{x^2 - y^2} \) should be the same no matter how \((x, y)\) approaches \((0, 0)\). Consider the limits when we let \((x, y)\) approach \((0, 0)\) along the lines \(x = 0\) and \(y = 0\) (which go through the origin).

   along the \(x\)-axis: \( \lim \limits_{(x,0) \to (0,0)} \frac{x^2 + 2y^2}{x^2 - y^2} = \lim \limits_{x \to 0} \frac{x^2}{x^2} = 1 \)

   along the \(y\)-axis: \( \lim \limits_{(0,y) \to (0,0)} \frac{x^2 + 2y^2}{x^2 - y^2} = \lim \limits_{y \to 0} \frac{2y^2}{-y^2} = -2 \)

Since these are not the same, the limit \( \lim \limits_{(x,y) \to (0,0)} \frac{x^2 + 2y^2}{x^2 - y^2} \) cannot exist.

b. Compute the limit along these paths that pass through the origin.

   along the \(x\)-axis: \( \lim \limits_{(x,0) \to (0,0)} \frac{xy}{(x + y)^2} = \lim \limits_{x \to 0} \frac{0}{y^2} = 0 \)

   along the \(y\)-axis: \( \lim \limits_{(0,y) \to (0,0)} \frac{xy}{(x + y)^2} = \lim \limits_{y \to 0} \frac{0}{x^2} = 0 \)

   along the line \(x = y\): \( \lim \limits_{(x,x) \to (0,0)} \frac{xx}{(x + x)^2} = \lim \limits_{x \to 0} \frac{x^2}{4x^2} = \frac{1}{4} \)

Since we found two paths with two different limits, the limit does not exist.

14.2.re2. Show that \( \lim_{(x,y) \to (0,0)} \) of the given function does not exist.

a. \( \frac{x^4 - 3y^4}{(x^2 + y^2)^2} \)  
b. \( \frac{xy}{(x - y)^2} \)  
c. \( \frac{x^2y^{1/3}}{x^3 + y} \)

If the limit

\[(14.2.1) \quad \lim \limits_{(x,y) \to (a,b)} f(x,y) \]

is known to exist, then

\[ \lim \limits_{(x,y) \to (a,b)} f(x,y) = \lim \limits_{x \to a} \lim \limits_{y \to b} f(x,y) = \lim \limits_{y \to b} \lim \limits_{x \to a} f(x,y). \]

But example 14.2.re1b demonstrates that

\[ \lim \limits_{x \to a} \lim \limits_{y \to b} f(x,y) = \lim \limits_{y \to b} \lim \limits_{x \to a} f(x,y) \]

is no guarantee that \((14.2.1)\) exists.
Continuity
A function \( f(x, y) \) is said to be \textbf{continuous at the point} \((a, b)\) if
\[
\lim_{{(x, y) \to (a, b)}} f(x, y) = f(a, b)
\]
A function is said to be \textbf{continuous} if it is continuous at every point in its domain.
(Continuity for functions of more than two variables is defined similarly.)

A \textbf{monomial} in \( x, y \) is a function of the form \( x^m y^n \) for nonnegative integers \( m \) and \( n \)

A \textbf{polynomial} is a sum of scalar multiples of monomials.

A \textbf{rational function} is the ratio of two polynomials.
(Monomials, polynomials, and rational functions of more than two variables are defined similarly.)

**14.2.re3.** \( x^0, x^2y^{13}, \) and \( x^7y^2z^9 \) are monomials. \( x^2y^{3/2} \) is not.
\( x^2 + 2xy - y^2 + 2y - 1 \) and \( (x + y - 2z)^4 \) are polynomials; \( e^{xy} \) is not.
\[
\frac{x^2 - y^2 + y}{1 + 2x - 3y + z^3}
\]
is a rational function; \( \frac{\sin x + \cos y}{\sqrt{x + 1}} \) is not.

**Fact:** Polynomials and rational functions are continuous.

**Fact** (from calculus I): These functions (of one variable) are all continuous:

<table>
<thead>
<tr>
<th>Function Type</th>
<th>Example</th>
</tr>
</thead>
<tbody>
<tr>
<td>Power functions ( x^p )</td>
<td>( x^2 )</td>
</tr>
<tr>
<td>Absolute value</td>
<td>( \text{abs}(x) )</td>
</tr>
<tr>
<td>Trig functions</td>
<td>( \sin(x) )</td>
</tr>
<tr>
<td>Inverse trig functions</td>
<td>( x^{-1} )</td>
</tr>
<tr>
<td>Exponential functions</td>
<td>( e^x )</td>
</tr>
<tr>
<td>Logarithms</td>
<td>( \log(x) )</td>
</tr>
</tbody>
</table>

**Fact:** If \( f \) and \( g \) are continuous, then so are
\[
 f + g \quad f - g \quad f \cdot g \quad f \div g \quad f \circ g.
\]

Consequently, other than piecewise defined functions, most functions we can write down are automatically continuous, meaning that their limits can be computed simply by evaluating the function.
**Tools to evaluate limits in several variables**

Other than continuity, we have only these two facts to allow to take limits in two or more variables.

**Fact:** if \( f(x, y) = g(x, y) \) for all \((x, y)\) other than \((a, b)\), then

\[
\lim_{(x, y) \to (a, b)} f(x, y) = \lim_{(x, y) \to (a, b)} g(x, y),
\]

meaning that either both exist are equal or both fail to exist.

**Squeeze Theorem:** If \( g(x, y) \leq f(x, y) \leq h(x, y) \) for all \((x, y)\) other than \((a, b)\), and if

\[
\lim_{(x, y) \to (a, b)} g(x, y) = \lim_{(x, y) \to (a, b)} h(x, y) = L,
\]

then

\[
\lim_{(x, y) \to (a, b)} f(x, y)
\]

must also exist and equal \(L\).

---

**14.2.re4.** Evaluate the limit as \((x, y) \to (0, 0)\), or determine that it does not exist.

- a. \[
\frac{x^2 - 4xy + 4y^2}{(xy + 1)(x - 2y)}
\]
- b. \[
\frac{e^{x+y+1}}{1 + \cos(2x)}
\]
- c. \[
\frac{2x^2 + 7xy + 3y^2}{x^2 + 2xy - 3y^2}
\]
- d. \[
\frac{2x^2 + 4x - xy - 2y}{2xy - y^2 - 2x + y}
\]
- e. \[
\left| x - y \right| \sin \left( \frac{x}{y} \right)
\]
- f. \[
\frac{x^2 + 5xy + 6y^2}{x^2 + 3xy - x - 3y}
\]

**Answers**

14.2.re2a. Compare limits along \(x = 0\) and \(y = 0\). 14.2.re2b. Compare limits at \((0,0)\) along \(y = 0\) and \(y = -x\). 14.2.re2c. Compare along \(x = 0\) and \(y = x^3\). 14.2.re4a. 0. 14.2.re4b. \(e/2\), by continuity.
14.2.re4c. **DNE.** 14.2.re4d. **−2.** 14.2.re4e. 0, by Squeeze. 14.2.re4f. 0.
14.3: Partial Derivatives

The **partial derivative** of \( f(x, y, z) \) with respect to \( x \) is the limit

\[
f_x(x, y, z) = \lim_{h \to 0} \frac{f(x + h, y, z) - f(x, y, z)}{h}.
\]

The partial derivatives (or, more simply, “partials”) of \( f \) with respect to \( y \) and \( z \) are defined similarly.

\[
f_y(x, y, z) = \lim_{h \to 0} \frac{f(x, y + h, z) - f(x, y, z)}{h},
\]

\[
f_z(x, y, z) = \lim_{h \to 0} \frac{f(x, y, z + h) - f(x, y, z)}{h}.
\]

Because of its similarity to the derivative seen in calc I, \( f_x \) (or \( f_y \) or \( f_z \)) can be calculated by treating all variables other than \( x \) (or \( y \) or \( z \)) as constants and differentiating with the familiar rules.

### 14.3.re1.

If \( f(x, y, z) = x^3y^2z + 2x + 3y + x\sin(yz) \), then

\[
f_x = 3x^2y^2z + 2 + \sin(yz) \\
f_y = 2x^3yz + 3 + xz \cos(yz) \\
f_z = x^3y^2 + xy \cos(yz)
\]

Higher order partials are indicated with more subscripts:

\[
f_{xx} = (f_x)_x = 6xy^2z \\
f_{yx} = (f_y)_x = 6x^2yz + z \cos(yz) \\
f_{zy} = (f_z)_y = 2x^3y + x \cos(yz) - xyz \sin(yz)
\]

For most functions we see, the order of differentiation isn’t important:

**Clairaut’s Theorem on the equality of mixed partials:** if \( f_{xy}(x, y) \) and \( f_{yx}(x, y) \) are continuous on an open disk in the plane, then \( f_{xy} = f_{yx} \) on that disk.

### 14.3.re1, continued.

\( f_{xy} = (f_x)_y = (3x^2y^2z + 2 + \sin(yz))_y = 6x^2yz + z \cos(yz) = f_{yx} \) found above.

### 14.3.re2.

Find all first and second order partials of the given function.

a. \( f = (x + y)(y - 2z) \)  
   b. \( g = e^{x+2y} - \cos(xz) + \ln(y^2z^2) \)  
   c. \( h = \tan^{-1}(x + 2y) \)
**Geometric interpretation of the partial derivative**

$f_x(x_0, y_0)$ tells us the slope (change in $z$ over change in $x$) of the cross-section to the graph of $f(x, y)$ at $y = y_0$ at the point $(x_0, y_0)$. Similarly, $f_y(x_0, y_0)$ tells us the slope of the cross-section to the graph of $f(x, y)$ at $x = x_0$ at the point $(x_0, y_0)$.

**14.3.re3.** To write the line tangent to the the curve

$$z = x^2 + 3y^2, \quad y = 1$$

at the point $(2, 1, 7)$, we calculate $z_x = 2x$, which equals 4 at $x = 2$. The tangent line is given implicitly by the equations

$$z - 7 = 4(x - 2), \quad y = 1.$$ 

To parametric equations for the line, set $t = z - 7 = 4(x - 2)$ and solve for $x$ and $z$:

$$x = \frac{1}{4}t + 2, \quad z = 7 + t, \quad y = 1.$$ 

**14.3.re4.**

a. Find implicit and parametric equations of line tangent to the curve $z = e^{x^2}y^{-1}$, $y = 2$ at the point $(1, 2, e)$.

b. Repeat for the curve $z = e^{x^2}y^{-1}$, $x = 2$ at the point $(2, \frac{1}{4}, 1)$.

**Alternate notation for partial derivatives**

$$f_x = D_x f = \frac{\partial f}{\partial x}$$

$$f_{xx} = D_x^2 f = \frac{\partial}{\partial y} \frac{\partial f}{\partial x} = \frac{\partial^2 f}{\partial x^2}$$

$$f_{xy} = D_y D_x f = \frac{\partial}{\partial y} \frac{\partial f}{\partial x} = \frac{\partial^2 f}{\partial y \partial x}$$
Implicit differentiation

Along a surface given implicitly by the equation $f(x, y, z) = 0$ we can express $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$ in terms of $x$, $y$, and $z$ by implicit differentiation as in calculus I.

14.3.re5. Find $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$ along the surface $3x + e^{xz} = y^2 z$.

Solution. To find $z_x$, differentiate both sides of the equation as though $y$ was constant and $z$ a function of $x$:

$$3 + e^{xz}(xz)_x = y^2 z_x$$

$$3 + e^{xz}(1 + xz_x) = y^2 z_x$$

Now solve for $z_x$:

$$3 + z e^{xz} + xz_x e^{xz} = y^2 z_x$$

$$3 + z e^{xz} = y^2 z_x - xz_x e^{xz}$$

$$3 + z e^{xz} = z_x(y^2 - x e^{xz})$$

$$3 + z e^{xz} = z_x$$

Find $z_y$ similarly: differentiate as though $x$ was constant and $z$ a function of $y$:

$$xz_y e^{xz} = 2yz + y^2 z_y$$

$$xz_y e^{xz} - y^2 z_y = 2yz$$

$$z_y = 2yz$$

$$z_y = \frac{2yz}{x e^{xz} - y^2}$$

14.3.re6. Find $z_x$ and $z_y$ along the graph of the given equation.

a. $e^{x+2y+3z} = x^2 + y^2 + z^2$  

b. $x^2 y - y^3 z + x \ln z = y$  

c. $x^2 + 2xz - y^2 - xz^3 = y$

Answers

14.3.re2a. $f_x = y - 2z$, $f_y = x + 2y - 2z$, $f_z = -2x - 2y$. $f_{xx} = 0$. $f_{xy} = 1$, $f_{xz} = -2$. $f_{yy} = 2$. $f_{yz} = -2$. $f_{zz} = 0$.  
14.3.re2b. $g_x = e^{x+2y} + z \sin(xz)$. $g_y = e^{x+2y} + y^{-1}$. $g_z = x \sin(xz) + 2z^{-1}$. $g_{xx} = e^{x+2y} + z^2 \cos(xz)$. $g_{xy} = 2e^{x+2y}$. $g_{xz} = \sin(xz) + xz \cos(xz)$. $g_{yy} = 4e^{x+2y} - y^{-2}$. $g_{yz} = 0$. $g_{zz} = x^2 \cos(xz) - 2z^{-2}$.  
14.3.re2c. $h_x = (1 + (x + 2y)^2)^{-1}$. $h_y = (1 + (x + 2y)^2)^{-1}$. $h_{xx} = -2(x + 2y)(1 + (x + 2y)^2)^{-2}$. $h_{yy} = -4(x + 2y)(1 + (x + 2y)^2)^{-2}$. $h_{xy} = -8(x + 2y)(1 + (x + 2y)^2)^{-2}$.  
14.3.re4a. $\text{imp: } z - e = 4e(x - 1)$, $y = 2$. para: $x = \frac{1}{2}l + 1$, $y = 2$, $z = e + t$. b. imp: $z - 1 = 4(y - \frac{1}{2})$, $x = 2$. para: $x = 2$, $y = \frac{4l + 1}{2}$, $z = 1 + t$.  
14.3.re6a. $z_x = \frac{2x + e^{x+2y+3z}}{2z - 3e^{x+2y+3z}}$, $z_y = -\frac{2y + 2e^{x+2y+3z}}{2z - 3e^{x+2y+3z}}$.  
14.3.re6b. $z_x = \frac{2xy + \ln z}{y^2 - xz^3}$, $z_y = \frac{x^2 + 2xz - y^2}{y^2 - xz^3}$.  
14.3.re6c. $z_x = \frac{2x + e^{x+2y+3z}}{2z - 3e^{x+2y+3z}}$, $z_y = \frac{3y^2 - xz^3}{2x - 3e^{x+2y+3z}}$.  

14.4: Tangent planes, linear approximation, and differentiability

Tangent planes

The plane tangent to the graph of \( f(x, y) \) at the point \((a, b, f(a, b))\) is given by

\[
z - f(a, b) = f_x(a, b)(x - a) + f_y(a, b)(y - b).
\]

Note the similarity to the equation of the tangent line from calculus I:

\[
y - f(a) = f'(a)(x - a).
\]

14.4.re1. To find the plane tangent to the graph of \( r(x, y) = e^{x+2y} - \sin(\pi xy) \) at \((x, y) = (2, 1)\), evaluate \( r \) and its first partials at \((2, 1)\):

\[
\begin{align*}
  r(x, y) &= e^{x+2y} - \sin(\pi xy) \\
  r_x(x, y) &= e^{x+2y} - \pi y \cos(\pi xy) \\
  r_y(x, y) &= 2e^{x+2y} - \pi x \cos(\pi xy)
\end{align*}
\]

\[
\begin{align*}
  r(2, 1) &= e^4 \\
  r_x(2, 1) &= e^4 - \pi \\
  r_y(2, 1) &= 2e^4 - 2\pi
\end{align*}
\]

The tangent plane is given by the equation

\[
z - e^4 = (e^4 - \pi)(x - 2) + (2e^4 - 2\pi)(y - 1)
\]

14.4.re2. Find an equation of the plane tangent to the graph of the given function at the given point \((x, y)\).

a. \((x + y + 1)(y - 2x - 1)\) at \((1, 0)\)

b. \(e^{x+2y} - \ln(x^2 y)\) at \((-1, 2)\)

c. \(\frac{x + y}{x - 2y}\) at \((-2, 2)\).
Linear approximation of functions of one variable
In calculus I, if \( f \) is a function of one variable whose derivative exists at \( x = a \), we say \( f \) is differentiable at \( x = a \). When that occurs,

\[
f(x) \approx f(a) + f'(a)(x - a)
\]

for \( x \) is near \( a \). The precise meaning of \( \approx \) is given elsewhere, but this means that, near \( x = a \), \( f(x) \) can be well approximated by its linearization

\[
L(x) = f(a) + f'(a)(x - a).
\]

Linear approximation of functions of two or more variables
The linearization of \( f(x, y) \) at the point \( (a, b) \) is defined to be the function

\[
L(x, y) = f(a, b) + f_x(a, b)(x - a) + f_y(a, b)(y - b).
\]

When we say that \( f(x, y) \) is differentiable at \( (a, b) \), we mean not only that \( f_x \) and \( f_y \) exist at that point, but that \( L(x, y) \) provides a good approximation to \( f(x, y) \):

\[
f(x, y) \approx f(a, b) + f_x(a, b)(x - a) + f_y(a, b)(y - b) \text{ if } (x, y) \text{ is near } (a, b).
\]

While they may appear different, this definition of differentiability and our earlier use of that word in calc I coincide when \( f \) is a function of only one variable.

Not every function with partial derivatives is differentiable. For instance, here’s the graph of a continuous function \( f \) which fails to be differentiable at \((0,0)\), even though both \( f_x \) and \( f_y \) exist there.

But, if a function possess continuous partial derivatives, it must be differentiable:

**Theorem:** If \( f_x \) and \( f_y \) exist in a disk about the point \((a, b)\) and are continuous at \((a, b)\), then \( f \) is differentiable at \((a, b)\)

More generally,

**Theorem:** If \( f \) is a function of several real variables and if its first partial derivatives are continuous in some neighborhood about the point \( a \) and are continuous at \( a \), then \( f \) is differentiable at \( a \).
14.4.re3. Explain why the function is differentiable on its entire domain, and find its linearization at the given point.

a. \( f = \frac{x - y}{x + y} \) at \((2, -1)\)

b. \( h = \tan^{-1}(\sin x + \cos y) \) at \((\frac{\pi}{6}, \frac{\pi}{3})\).

c. \( g = e^{x+2y-3z} \) at \((1, 1, 1)\).

Answers

14.4.re2a. \( z + 6 = -7(x - 1) - y \). 14.4.re2b. \( z - e^3 + \ln 2 = (2 + e^3)(x + 1) + (2e^3 - \frac{1}{2})(y - 2) \). 14.4.re2c. \( 6z + x + y = 0 \). 14.4.re3a. \( f \) and its partial derivatives are rational functions, which are known to be continuous. \( L(x, y) = 3 - 2(x - 2) - 4(y + 1) \). 14.4.re3b. Trig functions, inverse trig functions, and rational functions are continuous. The partials of \( h \) are combinations of these. \( L(x, y) = \frac{\pi}{4} + \frac{\sqrt{3}}{4}(x - \frac{\pi}{6}) - \frac{\sqrt{3}}{4}(y - \frac{\pi}{3}) \).

14.4.re3c. Exponentials, polynomials, and any combinations of these are continuous, so \( g \) has continuous partials on all of \( \mathbb{R}^3 \). \( L(x, y, z) = 1 + (x - 1) + 2(y - 1) - 3(z - 1) \), or \( 1 + x + 2y - 3z \).
14.5: The Chain Rule

If \( f \) is a differentiable function of \( x \) and \( y \), and \( x \) and \( y \) are differentiable functions of \( t \), then \( f \) is a differentiable function of \( t \), and

\[
\frac{df}{dt} = \frac{\partial f}{\partial x} \frac{dx}{dt} + \frac{\partial f}{\partial y} \frac{dy}{dt}
\]

(In derivatives, we use \( d \) for functions of one variable and \( \partial \) for several variables.)

14.5.re1. If \( f(x, y) = xy - \cos x + \ln |\sec y| \) and \( x \) and \( y \) are differentiable functions of \( t \), then

\[
\frac{df}{dt} = (y + \sin x) \frac{dx}{dt} + (x + \tan y) \frac{dy}{dt}
\]

Furthermore, if \( x = e^{2t} \) and \( y = t^3 \), then

\[
\frac{df}{dt} = (t^3 + \sin e^{2t})2e^{2t} + (e^{2t} + \tan t^3)3t^2
\]

\[
= 2e^{2t}t^3 + e^{2t}3t^2 + 2e^{2t} \sin e^{2t} + 3t^2 \tan t^3,
\]

\[
= \frac{d}{dt}(e^{2t}t^3 - \cos e^{2t} + \ln |\sec t^3|)
\]

The chain rule works the same regardless of the number of variables involved. For instance, if \( f = f(x, y, z) \), then

\[
\frac{df}{dt} = \frac{\partial f}{\partial x} \frac{dx}{dt} + \frac{\partial f}{\partial y} \frac{dy}{dt} + \frac{\partial f}{\partial z} \frac{dz}{dt}
\]

If \( x, y, \) and \( z \) are functions of \( s \) and \( t \), then

\[
\frac{df}{ds} = \frac{\partial f}{\partial x} \frac{dx}{\partial s} + \frac{\partial f}{\partial y} \frac{dy}{\partial s} + \frac{\partial f}{\partial z} \frac{dz}{\partial s}
\]

\[
\frac{df}{dt} = \frac{\partial f}{\partial x} \frac{dx}{\partial t} + \frac{\partial f}{\partial y} \frac{dy}{\partial t} + \frac{\partial f}{\partial z} \frac{dz}{\partial t}
\]
14.5.re2. Suppose \( f(x, y), x(u, v), \) and \( y(u, v) \) are differentiable. Use the table of values to find \( f_u \) and \( f_v \) at \( u = 3, v = 1 \):

\[
\begin{array}{c|ccc|ccccccc}
  x & y & f & f_x & f_y \mid u & v & x & y & x_u & x_v & y_u & y_v \\
  1 & -1 & 10 & -6 & 8 & 3 & 1 & 1 & -1 & 7 & 5 & 3 & 9
\end{array}
\]

Remember that \( f(x, y) = f(x(u, v), y(u, v)) \), so \( f \) and its derivatives are evaluated at functions of \( x \) and \( y \). At \( u = 3 \) and \( v = 1 \) (where \( x = 1, y = -1 \)),

\[
\begin{align*}
  f_u &= f_x x_u + f_y y_u \\
  &= -6 \cdot 7 + 8 \cdot 3 = -18 \\
  f_v &= f_x x_v + f_y y_v \\
  &= -6 \cdot 5 + 8 \cdot 9 = 42
\end{align*}
\]

14.5.re3. The temperature at the point \((x, y)\) is \( T(x, y) = x^2 + 2xy \) degrees. Suppose that, at one moment, a particle traveling in the plane is at the point \((-3, 2)\), moving in the positive \( x \) direction 0.5 units/second and in the negative \( y \) direction 1.5 units/second. At what rate is the temperature at the particle’s position changing at that moment? Is the temperature increasing or decreasing then?

If the particle is at position \((x(t), y(t))\) at time \( t \), then the temperature at its position is \( T(x(t), y(t)) \). At the moment in question,

\[
\begin{align*}
  T_x &= 2x + 2y = -6 + 4 = -2 \\
  T_y &= 2x = -6 \\
  \frac{dT}{dt} &= T_x x_t + T_y y_t = -2 \cdot \frac{1}{2} + -6 \cdot \frac{-3}{2} = -1 + 9 = 8
\end{align*}
\]

That is, temperature experienced by the particle is increasing 8 degrees per second.
14.5.re4. Find $f_u$ and $f_v$.
   
a. $f = (x - y)(2x + 3y), \quad x = \ln(u - v), \quad y = e^u v$
   
b. $f = \cos(p + 2q), \quad p = uv, \quad q = uv^{-1}$.
   
c. $f = a(b - c)^2, \quad a = \sqrt{u + v}, \quad b = \cos v, \quad c = \sin u$

14.5.re5. The dimensions of a rectangular box are changing with time. At one particular moment, the length, width, and height of the box are 2 cm, 3 cm 4 cm, resp., and are changing 0.4 cm/sec, -0.3 cm/sec, 0.2 cm/sec. At that time, at what rates are the following changing?

   a. the volume of the box
   
   b. the surface area of the box

14.5.re6. Suppose $f(x, y), \ x(u, v)$, and $y(u, v)$ are differentiable. Use the first two tables to complete the third.

<table>
<thead>
<tr>
<th>$x$</th>
<th>$y$</th>
<th>$f$</th>
<th>$f_x$</th>
<th>$f_y$</th>
<th>$u$</th>
<th>$v$</th>
<th>$x$</th>
<th>$y$</th>
<th>$x_u$</th>
<th>$x_v$</th>
<th>$y_u$</th>
<th>$y_v$</th>
</tr>
</thead>
<tbody>
<tr>
<td>2</td>
<td>1</td>
<td>$\pi$</td>
<td>$-2$</td>
<td>$3$</td>
<td>3</td>
<td>1</td>
<td>2</td>
<td>1</td>
<td>9</td>
<td>8</td>
<td>7</td>
<td>6</td>
</tr>
<tr>
<td>0</td>
<td>4</td>
<td>2</td>
<td>$1$</td>
<td>$-4$</td>
<td>8</td>
<td>9</td>
<td>0</td>
<td>4</td>
<td>3</td>
<td>2</td>
<td>1</td>
<td>$-1$</td>
</tr>
</tbody>
</table>

Answers

14.5.re4a. $f_u = (4x + y) \left( \frac{1}{u-v} \right) + (x - 6y)ve^{uv}$. $f_v = (4x + y) \left( \frac{1}{u-v} \right) + (x - 6y)ue^{uv}$. 14.5.re4b. $f_u = -\sin(p + 2q)v - 2\sin(p + 2q)v^{-1}$. $f_v = -\sin(p + 2q)u + 2\sin(p + 2q)uv^{-2}$. 14.5.re4c. $f_u = \frac{1}{2}(b - c)^2(u + v)^{-1/2} - 2a(b - c)\cos u$. $f_v = \frac{1}{2}(b - c)^2(u + v)^{-1/2} - 2a(b - c)\sin v$. 14.5.re5a. 3.6 cm$^3$/sec

<table>
<thead>
<tr>
<th>$u$</th>
<th>$v$</th>
<th>$f$</th>
<th>$f_u$</th>
<th>$f_v$</th>
</tr>
</thead>
<tbody>
<tr>
<td>3</td>
<td>1</td>
<td>$\pi$</td>
<td>0.3</td>
<td>0.2</td>
</tr>
</tbody>
</table>

14.5.re5b. 4.0 cm$^2$/sec 14.5.re6.

<table>
<thead>
<tr>
<th>$u$</th>
<th>$v$</th>
<th>$f$</th>
<th>$f_u$</th>
<th>$f_v$</th>
</tr>
</thead>
<tbody>
<tr>
<td>8</td>
<td>9</td>
<td>2</td>
<td>$-0.1$</td>
<td>0.6</td>
</tr>
</tbody>
</table>
14.6: The Gradient

The gradient of a differentiable function $f(x, y)$ of two variables is the vector

$$\nabla f(x, y) = (f_x(x, y), f_y(x, y)).$$

The gradient of a differentiable function $f(x, y, z)$ of three variables is the vector

$$\nabla f(x, y, z) = (f_x(x, y, z), f_y(x, y, z), f_z(x, y, z)).$$

While $f$ is a scalar-valued function, $\nabla f$ is a vector-valued function having the same number of variables as $f$ and the same number of components as variables. That is,

$$f : \mathbb{R}^n \to \mathbb{R}^1 \quad \nabla f : \mathbb{R}^n \to \mathbb{R}^n.$$

$\nabla f$ is an example of a vector field: a (vector-valued) function from $\mathbb{R}^n$ into $\mathbb{R}^n$. (§16.1)

14.6.2. If $f(x, y) = y - x^2$, the its gradient is

$$\nabla f = \langle -2x, 1 \rangle.$$

Here's a plot of the values of $\nabla f$ at some select points in the plane.

---

Directional derivatives and the gradient

If $f$ is a function of two or three variables, if $\mathbf{r}$ denotes either $\langle x, y \rangle$ or $\langle x, y, z \rangle$, and if $\mathbf{u}$ is a unit vector, the derivative of $f$ in the direction $\mathbf{u}$ is the limit

$$D_uf(\mathbf{r}) = \lim_{h \to 0} \frac{f(\mathbf{r} + h\mathbf{u}) - f(\mathbf{r})}{h}.$$

For instance, if $f = f(x, y)$ and $\mathbf{u} = \langle \frac{3}{5}, -\frac{4}{5} \rangle$, then

$$D_uf(x, y) = \lim_{h \to 0} \frac{f(x + \frac{3}{5}h, y - \frac{4}{5}h) - f(x, y)}{h}.$$

The directional derivative $D_uf(x, y, z)$ is the rate of increase of $f$ at the point $(x, y, z)$ when we move away from that point in the direction of $\mathbf{u}$. The partial derivatives we studied earlier are directional derivatives.
14.6.re2. Identify the three unit vectors \( \mathbf{u}, \mathbf{v}, \mathbf{w} \) in \( \mathbb{R}^3 \) for which
\[
\frac{\partial}{\partial x} = D_\mathbf{u} \quad \frac{\partial}{\partial y} = D_\mathbf{v} \quad \frac{\partial}{\partial z} = D_\mathbf{w}.
\]

In practice, we can calculate directional derivatives without using their definition as limits:

**Fact.** If \( f \) is differentiable at the point \( \mathbf{p} \) and \( \mathbf{u} \) is a unit vector, then \( D_\mathbf{u} f(\mathbf{p}) = \mathbf{u} \cdot \nabla f(\mathbf{p}) \).

14.6.re3. Find \( \nabla f \) for the given \( f \). Then calculate \( D_\mathbf{u} f \) and \( D_\mathbf{v} f \) at the given point \( \mathbf{p} \) for the given unit vector \( \mathbf{u} \) and \( \mathbf{v} \).

a. \( f = y^2 - x^3; \quad \mathbf{p} = (1, -1); \quad \mathbf{u} = \frac{1}{5}(-4, 3); \quad \mathbf{v} = \frac{1}{\sqrt{13}}(2, -3) \)

b. \( f = xy - \ln(2x - y); \quad \mathbf{p} = (1, -1); \quad \mathbf{u} = \frac{1}{\sqrt{2}}(1, 1), \quad \mathbf{v} = -\mathbf{u} \)

c. \( f = \frac{x+y}{z}; \quad \mathbf{p} = (1, -1, 2); \quad \mathbf{u} = \frac{1}{3}(2, -1, -2); \quad \mathbf{v} = \frac{1}{2\sqrt{5}}(0, 1, 2) \)

An interpretation of the gradient

**Fact:** If \( f \) is differentiable at the point \( \mathbf{p} \), then \( \pm|\nabla f(\mathbf{p})| \) are the greatest and least directional derivatives of \( f \) at \( \mathbf{p} \), and they occur in the directions \( \pm \nabla f(\mathbf{p}) \).

14.6.re4. The temperature at the point \((x, y, z)\) is \( T(x, y, z) = xyz + 2xy - yz + 3xz \). I’m standing at the point \((0, 1, 2)\) and find it too hot. In which direction should I move to see the rapid decrease in temperature. If \( T \) is measured in \( ^\circ \text{C} \) and \( x, y, \) and \( z \) are measured in meters, at what rate will temperature decrease when I move away from \((0, 1, 2)\) in that direction?

14.6.re5. Find the direction of greatest and least rate of change of \( f \) at the given point, and the derivative of \( f \) in that direction at \( p \). Give the direction at a unit vector.

a. \( f = y^2 - x^3; \quad \mathbf{p} = (1, -1). \)

b. \( f = xy - \ln(2x - y); \quad \mathbf{p} = (1, -1). \)

c. \( f = \frac{x+y}{z}; \quad \mathbf{p} = (1, -1, 2). \)
Level curves/surfaces and the gradient

**Fact:** If \( f \) is differentiable, then \( \nabla f(p) \) is orthogonal at \( p \) to the level curve or surface of \( f \) that passes through \( p \).

14.6.re1, continued. Here’s the same plot of \( \nabla (y - x^2) \) at selected points in the plane superimposed on a contour map of level curves of \( y - x^2 \). Note the orthogonality of the gradient to the level curves.

14.6.re6. Find an equation to the plane tangent to the surface \( x^2 - y^3 + 4z^2 = -3 \) at the point \((1, 2, -1)\).

Solution: The surface is orthogonal to \( \nabla (x^2 - y^3 + 4z^2) = \langle 2x, -3y^2, 8z \rangle = \langle 2, -12, -8 \rangle \) at \((1, 2, -1)\). Plane is \( 2(x - 1) - 12(y - 2) - 8(z + 1) = 0 \), or \( x - 6y - 4z = -7 \).

14.6.re7. Find an equation to the plane tangent to the given surface at the given point.

a. \( e^{x+y-2z} - e^{2x-y+z} = 0 \); \((0, 0, 0)\).

b. \( \sin x - \cos(x + z) - y = 0 \); \((\pi, 0, \pi/2)\).

c. \( 1 + \tan^{-1}(x + y) = e^z + \frac{\pi}{4} \); \((\frac{3}{2}, -\frac{1}{2}, 0)\)

Answers

14.6.re2. \( \frac{\partial}{\partial x} = D_i \) \( \frac{\partial}{\partial y} = D_j \) \( \frac{\partial}{\partial z} = D_k \). 14.6.re3a. \( \nabla f = \langle -3x^2, 2y \rangle \); \( D_u f(P) = \frac{\delta}{\delta x} \) \( D_v f(P) = 0 \). 14.6.re3b. \( \nabla f = \langle y - \frac{xz}{x^2+y}, x + \frac{xz}{x^2+y} \rangle \); \( D_u f(P) = -\frac{1}{\sqrt{2}} \) \( D_v f(P) = \frac{1}{\sqrt{2}} \). 14.6.re3c. \( \nabla f(x, y, z) = \langle z^{-1}, z^{-1}, -(x + y)z^{-2} \rangle \); \( D_u f(P) = \frac{1}{6} \) \( D_v f(P) = \frac{1}{2\sqrt{3}} \). 14.6.re4. Most rapid decrease in temperature occurs in the direction \( \frac{1}{\sqrt{105}} \langle -10, 2, 1 \rangle \). (This is the normalization of \( -\nabla T \) at \((0, 1, 2)\).) In that direction, temperature decreases \( \sqrt{105} \) C/m. 14.6.re5a. Maximum and minimum derivatives are \( \pm \sqrt{13} \). Max occurs in the direction \( \frac{1}{\sqrt{13}} \langle 3, 2 \rangle \) and the minimum in the direction \( \frac{1}{\sqrt{13}} \langle -3, 2 \rangle \). 14.6.re5b. Maximum and minimum derivatives are \( \pm \frac{1}{\sqrt{41}} \). Max occurs in the direction \( \frac{1}{\sqrt{41}} \langle -5, 4 \rangle \) and the minimum in the direction \( \frac{1}{\sqrt{41}} \langle 5, -4 \rangle \). 14.6.re5c. Maximum and minimum derivatives are \( \pm \frac{1}{\sqrt{2}} \). Max occurs in the direction \( \frac{1}{\sqrt{2}} \langle 1, 1, 0 \rangle \) and the minimum in the opposite direction. 14.6.re7a. \( -x + 2y = 3z \).

14.6.re7b. \( 2x + y + z = 5\pi/2 \) 14.6.re7c. \( x + y - 2z = 1 \)
14.7: Extrema of multivariate functions

An extremum of a function is a maximum or a minimum value of a function. The plural of extremum is extrema. There are two types of extrema: local and absolute.

Local extrema

These are function values that are the maximum or minimum of nearby values of the function.

As we saw in calculus I, the local extrema of a \( f(x) \) can occur only at its critical points: points \( p \) in the domain where \( f'(p) \) is zero or undefined. But, not every critical point is the location of an extremum. The second derivative test can sometimes tell whether a local extremum occurs at a given critical point.

For functions of two variables, much the same thing is true:

**Fact:** The local extrema of \( f(x, y) \) can occur only at its **critical points**: points \( p \) in its domain at which both \( f_x(p) \) and \( f_y(p) \) are either zero or undefined.

**The Second Derivative Test:** Suppose \( f_x(p) = f_y(p) = 0 \) for some point \( p = (x, y) \), and define \( D \) to be the determinant

\[
| f_{xx} & f_{xy} \\ f_{yx} & f_{yy} |
= f_{xx} f_{yy} - f_{xy}^2.
\]

If \( D(p) < 0 \), then \( f \) has a saddle point at \( p \)

If \( D(p) > 0 \), then \( f \) has a local extremum at \( p \):

- a local min if \( f_{xx}(p) \) or \( f_{yy}(p) > 0 \), and
- a local max if \( f_{xx}(p) \) or \( f_{yy}(p) < 0 \).
14.7.re1. Find the locations of all local extrema and saddle points of $f(x, y) = x^2 + 2y^2 - x^2y$.

Search for critical points by setting $f_x$ and $f_y$ equal 0:

$$f_x(x, y) = 2x - 2xy = 0$$
$$= 2x(1 - y) = 0 \Rightarrow x = 0 \text{ or } y = 1.$$  

$$f_y(x, y) = 4y - x^2 = 0$$
$$x = 0 \Rightarrow y = 0$$
$$y = 1 \Rightarrow x = \pm 2$$

So, the critical points are (0, 0), (2, 1), and (−2, 1).

Now use the Second Derivative Test at the critical points. You can simplify your calculations by factor a 2 out of both rows.

$$D = \begin{vmatrix} f_{xx} & f_{xy} \\ f_{xy} & f_{yy} \end{vmatrix} = \begin{vmatrix} 2 - 2y & -2x \\ -2x & 4 \end{vmatrix} = 2 \cdot 2 \begin{vmatrix} 1 - y & -x \\ -x & 2 \end{vmatrix} = 4(2(1 - y) - x^2)$$

<table>
<thead>
<tr>
<th>critical point</th>
<th>$D$</th>
<th>$f_{xx}$</th>
<th>conclusion</th>
</tr>
</thead>
<tbody>
<tr>
<td>(0, 0)</td>
<td>8 &gt; 0</td>
<td>2 &gt; 0</td>
<td>Local Minimum</td>
</tr>
<tr>
<td>(2, 1)</td>
<td>−16 &lt; 0</td>
<td>irrelevant</td>
<td>Saddle Point</td>
</tr>
<tr>
<td>(−2, 1)</td>
<td>−16 &lt; 0</td>
<td>irrelevant</td>
<td>Saddle Point</td>
</tr>
</tbody>
</table>

14.7.re2. Find the locations of all local extrema and saddle points.

a. $x\left(\frac{4}{3}x^2 - 1\right) - (x + y)^2$  
b. $(x + y)^3 - (x - 2)^2 - 3x - 3y$  
c. $xye^{x-y}$
Absolute extrema on a domain

The absolute max and min of a function on a region $R$ in its domain are its largest and smallest values on $R$. Just as we saw in calculus I, the absolute extrema of a function on a region must occur at a boundary point or at a critical point interior to the region. The second derivative test isn’t relevant in such a problem.

14.7.re3. Find the absolute extrema of $f(x, y) = (x - y)e^{2xy}$ on the square bounded by the lines $x = 0$, $x = -1$, $y = 0$, $y = 1$.

The solution is to identify all the places in the interior or on the boundary at which the extrema might occur.

First look for interior critical points. When setting partials equal zero, it helps to factor.

\[
\begin{align*}
    f_x &= e^{2xy} + 2(x - y)y e^{2xy} = e^{2xy}(1 + 2(x - y)y) = 0 \\
    f_y &= -e^{2xy} + 2(x - y)x e^{2xy} = e^{2xy}(-1 + 2(x - y)x) = 0
\end{align*}
\]

Remembering that $e^{2xy}$ cannot equal zero, the system of equations becomes

\[
\begin{align*}
    1 + 2(x - y)y &= 0 \\
    -1 + 2(x - y)x &= 0
\end{align*}
\]

Notice that $x - y$ can’t equal zero, we can divide by it to solve for $x$ and $y$ in each equation.

\[
\begin{align*}
    y &= \frac{-1}{2(x - y)} \\
    x &= \frac{1}{2(x - y)} = -y
\end{align*}
\]

Set $x = -y$ and the first equation becomes $y = -\frac{1}{4y}$ which implies $y = \pm \frac{1}{2}$ and therefore $x = \mp \frac{1}{2}$. Of the critical points we found, only $(-\frac{1}{2}, \frac{1}{2})$ is in the interior of the region in question.
Now we’ll find where along the boundary \( f \) could take its extrema. Remember that the max of a function of one variable on a closed interval can occur only at critical points interior to the interval or at the endpoints of the interval.

Let’s analyze each line segment separately.

Segment 1: \( x = -1, \ 0 \leq y \leq 1 \): Along this segment, \( f(-1, y) = (-1-y)e^{-2y} \), the derivative of which is (after some calculations, which you should double-check) \( e^{-2y}(1+2y) \). This equals zero only at \( y = -\frac{1}{2} \), which is outside the interval \( 0 \leq y \leq 1 \). So, the extrema on this segment can only occur at the endpoints \((-1, 0)\) and \((-1, 1)\).

Segment 2: \( x = 0, \ 0 \leq y \leq 1 \): Along this segment, \( f(0, y) = -y \), the derivative of which is never zero. So, the extrema on this segment can only occur at the endpoints \((0, 0)\) and \((0, 1)\).

Segment 3: \( y = 1, \ -1 \leq x \leq 0 \): Along this segment, \( f(x, 1) = (x-1)e^{2x} \), the derivative of which is \( e^{2x}(2x-1) \). This equals zero only at \( x = \frac{1}{2} \), which is outside the interval \(-1 \leq x \leq 0 \). So, the extrema on this segment can only occur at the endpoints \((-1, 1)\) and \((0, 1)\), which we identified earlier.

Segment 4: \( y = 0, \ -1 \leq x \leq 0 \): Along this segment, \( f(x, 0) = x \), the derivative of which is never zero. So, the extrema on this segment can only occur at the endpoints \((-1, 0)\) and \((0, 0)\).

We now know that the absolute extrema of \( f \) can only occur at one of the points of interest identified above. When evaluate \( f \) at these five points, its absolute max and min must be in the list of calculated values.

<table>
<thead>
<tr>
<th>((x, y))</th>
<th>(f(x, y))</th>
</tr>
</thead>
<tbody>
<tr>
<td>((-\frac{1}{2}, \frac{1}{2}))</td>
<td>-0.61</td>
</tr>
<tr>
<td>((0, 0))</td>
<td>0</td>
</tr>
<tr>
<td>((0, 1))</td>
<td>-1</td>
</tr>
<tr>
<td>((-1, 0))</td>
<td>-1</td>
</tr>
<tr>
<td>((-1, 1))</td>
<td>-0.27</td>
</tr>
</tbody>
</table>

Conclusion: the absolute max of \( f \) on the square is 0, and its absolute min is -1.

14.7.re4. Find the absolute extrema of the function on the region bounded by the given curves.

a. \( 2x^2 - 4x + y^2 - 4y + 1; \ x = 0, \ y = 2, \ y = 2x. \)

b. \( x^2 - xy + y^2 - 3x; \ x = 0, \ x = 3, \ y = 0, \ y = 2. \)

c. \( -\frac{1}{2}x^2 + \frac{1}{2}y^2 - x; \ x^2 + y^2 = 1 \)

In addition to the problems in our text, you’ll find more of these at https://kunklet.people.cofc.edu/MATH221/stew1407prob.pdf

Answers

14.7.re2a. Saddle at \((\frac{1}{2}, -\frac{1}{2})\). Loc max at \((-\frac{1}{2}, \frac{1}{2})\). 14.7.re2b. Local max at \((2, -3)\). Saddle point at \((2, -1)\). 14.7.re2c. Local min at \((-1, 1)\). Saddle point at \((0, 0)\). 14.7.re4a. Abs max is 1 at \((0, 0)\). Abs min is -5 at \((1, 2)\). 14.7.re4b. Abs max is 4 at \((0, 2)\). Abs min is \(-\frac{3}{4}\) at \((\frac{3}{2}, 0), (3, \frac{3}{2}), (\frac{7}{2}, 2)\). 14.7.re4c. Abs max is \(\frac{3}{4}\) at \((-\frac{1}{2}, \pm \sqrt{\frac{3}{2}})\). Abs min is \(-\frac{1}{2}\) at \((1, 0)\).
14.8: Lagrange multipliers

In this section we search for the extrema of a function subject to a constraint equation.

The method of Lagrange multipliers with one constraint equation

Fact 14.8.1: If \( f \) and \( g \) are functions of 2 or 3 variables, then the extrema of \( f \) subject to the constraint

\[ g = \text{a constant} \]

can only occur at points on the constraint where

\[ \nabla f \times \nabla g = 0. \tag{14.8.2} \]

14.8.1. Find the extreme values of \( ye^x \) subject to the constraint \( x^2 + 2y^2 = 2 \).

Solution: Set \( f(x, y) = ye^x \) and \( g(x, y) = x^2 + 2y^2 \). The extrema of \( f \) along \( g = 2 \) can only occur at those points where

\[ \nabla f \times \nabla g = \langle ye^x, e^x, 0 \rangle \times \langle 2x, 4y, 0 \rangle \]

is the zero vector. Therefore, the critical points are the solutions to the system

\[ 2e^x(2y^2 - x) = 0 \]
\[ x^2 + 2y^2 = 2 \]

Since \( e^x \) is never zero, \( 2y^2 = x \) at the critical points. Substituting this into \( g = 2 \) gives \( x^2 + x = 2 \), whose solutions are \( x = -2 \) and \( x = 1 \). Since \( x = 2y^2 \) must be nonnegative, we ignore \( x = -2 \). Substitute \( x = 1 \) into \( 2y^2 = x \) to find \( y = \pm \sqrt{\frac{1}{2}} \). Therefore, \( f \) can take its max and min over \( g = 2 \) at the points \( x = 1, y = \pm \sqrt{\frac{1}{2}} \).

Evaluate \( f \) at these two points.

<table>
<thead>
<tr>
<th>((x, y))</th>
<th>( ye^x )</th>
<th>conclusion</th>
</tr>
</thead>
<tbody>
<tr>
<td>( (1, \frac{1}{\sqrt{2}}) )</td>
<td>( \frac{e}{\sqrt{2}} )</td>
<td>absolute max</td>
</tr>
<tr>
<td>( (1, -\frac{1}{\sqrt{2}}) )</td>
<td>( -\frac{e}{\sqrt{2}} )</td>
<td>absolute min</td>
</tr>
</tbody>
</table>

Therefore the absolute max and min of \( f \) along the curve \( g = 2 \) are \( \frac{e}{\sqrt{2}} \) and \( -\frac{e}{\sqrt{2}} \), respectively.
14.8.re2. Find the points on \( x^2 + y^2 + z^2 = 6 \) that are nearest to and farthest from the point \((-1, 1, 1)\).

**Tip:** Distance is maximized (or minimized) exactly when distance-squared is maximized (or minimized).

Solution: We wish to maximize and minimize distance-squared from points \((x, y, z)\) to \((-1, 1, 1)\), so set \(f(x, y, z) = (x + 1)^2 + (y - 1)^2 + (z - 1)^2\). Set \(g(x, y, z) = x^2 + y^2 + z^2\), so that the constraint is \(g = 6\).

\(f\) can take its max and min along the constraint only at points at which \(\nabla f \times \nabla g = 0\):

\[
\begin{vmatrix}
i & j & k \\
2(x + 1) & 2(y - 1) & 2(z - 1) \\
2x & 2y & 2z
\end{vmatrix} = 4(y - z, -x - z, x + y) = 0.
\]

Setting these vectors equal means

\[
y - z = 0 \quad -x - z = 0 \quad x + y = 0
\]

which tells us that \(x = -y = -z\). Substitute this into the constraint equation

\[
x^2 + y^2 + z^2 = 6,
\]

which implies \(3z^2 = 6\), and \(z = \pm\sqrt{2} = y = -x\). Now evaluate \(f\) at the two critical points:

<table>
<thead>
<tr>
<th>critical point</th>
<th>((x + 1)^2 + (y - 1)^2 + (z - 1)^2)</th>
<th>conclusion</th>
</tr>
</thead>
<tbody>
<tr>
<td>((\sqrt{2}, -\sqrt{2}, -\sqrt{2}))</td>
<td>(3(\sqrt{2} + 1)^2)</td>
<td>absolute maximum</td>
</tr>
<tr>
<td>((-\sqrt{2}, \sqrt{2}, \sqrt{2}))</td>
<td>(3(\sqrt{2} - 1)^2)</td>
<td>absolute minimum</td>
</tr>
</tbody>
</table>

To answer the question, the point closest to \((-1, 1, 1)\) is \((-\sqrt{2}, \sqrt{2}, \sqrt{2})\) and the farthest is \((\sqrt{2}, -\sqrt{2}, -\sqrt{2})\). Had we been asked to report those distances, we’d take square root to obtain \(\sqrt{3}(\sqrt{2} \pm 1)\).
The method of Lagrange multipliers with two constraint equations

**Fact 14.8.3:** If \( f, g, \) and \( h \) are functions of 3 variables, then the extrema of \( f \) subject to the constraints

\[
g(x, y, z) = \text{a constant} \\
h(x, y, z) = \text{a constant}
\]

can occur only at points where

\[
(14.8.4) \quad \nabla f \cdot (\nabla g \times \nabla h) = 0.
\]

14.8.re3. Find the minimum distance between the origin and the intersection of the planes \( x + 2y = 12 \) and \( y + z = 6 \).

Solution: The constraints are level surfaces of the functions \( x + 2y \) and \( y + z \), the gradients of which are

\[
\nabla(x + 2y) = \langle 1, 2, 0 \rangle \quad \nabla(y + z) = \langle 0, 1, 1 \rangle.
\]

To minimize distance-squared-to-origin, set \( f = x^2 + y^2 + z^2 \), calculate its gradient \( \nabla f = \langle 2x, 2y, 2z \rangle \), and search for the point(s) on the two constraints at which the triple product

\[
\langle 2x, 2y, 2z \rangle \cdot (\langle 1, 2, 0 \rangle \times \langle 0, 1, 1 \rangle) = \begin{vmatrix} 2x & 2y & 2z \\ 1 & 2 & 0 \\ 0 & 1 & 1 \end{vmatrix}
\]

is zero. You can take out common factors from any row or column. In particular, factor 2 out of the top row:

\[
= 2 \begin{vmatrix} x & y & z \\ 1 & 2 & 0 \\ 0 & 1 & 1 \end{vmatrix} = 2 \begin{vmatrix} 2 & 0 \\ 1 & 1 \end{vmatrix} y + z \begin{vmatrix} 1 & 2 \\ 0 & 1 \end{vmatrix}
\]

\[
= 2(2x - y + z).
\]

The critical point is the solution to the system of equations

\[
x + 2y = 12 \\
y + z = 6 \\
2x - y + z = 0.
\]

It’s not too hard to find that the sole solution to this system is \( x = 2, y = 5, z = 1 \), and since we know some point on the constraint line is closest to the origin, this must be that point. Its distance to the origin is \( \sqrt{f(2, 5, 1)} = \sqrt{30} \).
14.8.re4. Find the max and min of the function subject to the constraint.

a. \(x^2 + 4y^3; x^2 + 2y^2 = 1\).
b. \(3x^2 + 2y^2 - 4y + 1; x^2 + y^2 = 16\)
c. \(x + 2y - z; x^2 + 3y^2 + z^2 = \frac{5}{2}\).

For the record, the critical point condition in Lagrange multipliers is traditionally stated as

\[
\nabla f = \lambda \nabla g \quad \text{(for some } \lambda) \quad \text{or} \quad \nabla g = 0,
\]

which is equivalent to (14.8.2). In case of two constraint equations, you’re more likely to see

\[
\nabla f = \lambda_1 \nabla g + \lambda_2 \nabla h \quad \text{(for some } \lambda_1 \text{ and } \lambda_2) \quad \text{or} \quad \nabla g = 0 \quad \text{or} \quad \nabla h = 0
\]

than the simpler (14.8.4). The method is named for the “multipliers” \(\lambda, \lambda_1, \lambda_2\). Equations (14.8.2) and (14.8.4) are generally simpler to solve than these, since they avoid the unnecessary \(\lambda\)s.

In addition to the problems in our text, you’ll find more of these at https://kunklet.people.cofc.edu/MATH221/stew1408prob.pdf

**Answers**

14.8.re4a. max = \(\sqrt{2}\) at \((0, \frac{1}{\sqrt{2}})\). min = \(-\sqrt{2}\) at \((0, -\frac{1}{\sqrt{2}})\). 14.8.re4b. max = 53 at the two points \((\pm \sqrt{12}, -2)\). min = 17 at \((0, 4)\). 14.8.re4c. max is \(\frac{4}{3} \sqrt{3}\) at \((\frac{\sqrt{3}}{2}, \frac{\sqrt{3}}{2}, -\frac{\sqrt{3}}{2})\). min is \(-\frac{4}{3} \sqrt{3}\) at \((-\frac{\sqrt{3}}{2}, -\frac{\sqrt{3}}{3}, \frac{\sqrt{3}}{2})\).
15.1: Double integrals over rectangles

Review of Riemann sums and the definite integral from calc I

Suppose \( f(x) \) is a function defined on the interval \([a, b]\). Divide \([a, b]\) into \(m\) subintervals of equal length \(\Delta x\), and choose “sample points” \(x_1^*, x_2^*, \ldots, x_m^*\), one from each subinterval. Then

\[
f(x_1^*)\Delta x + f(x_2^*)\Delta x + \cdots + f(x_m^*)\Delta x = \sum_{i=1}^{m} f(x_i^*)\Delta x
\]

is a Riemann sum for \( f(x) \) on \([a, b]\). The definite integral of \( f(x) \) on \([a, b]\) is defined to be the limit of its Riemann sums:

\[
\int_a^b f(x)\,dx = \lim_{m \to \infty} \sum_{i=1}^{m} f(x_i^*)\Delta x
\]

15.1.re1. Below, a Riemann sum is calculated for a function \( f(x) \) using 6 subintervals and their midpoints. Because \( f(x) > 0 \) on \([a, b]\), this Riemann sum is an approximation to the area under the graph of \( f \) over this interval on the \(x\)-axis.

Riemann sums in two variables over rectangles

The rectangle

\[
\{(x, y) \mid a \leq x \leq b, c \leq y \leq d\}
\]

in the \(xy\)-plane is denoted \([a, b] \times [c, d]\).

If we divide \([a, b]\) into \(m\) subintervals and \([c, d]\) into \(n\) subintervals, and choose a sample point \((x_{i,j}^*, y_{i,j}^*)\) in each of the resulting \(\Delta x \times \Delta y\) subrectangles of \([a, b] \times [c, d]\), then

\[
\sum_{i=1}^{m} \sum_{j=1}^{n} f(x_{i,j}^*, y_{i,j}^*) \Delta x \Delta y
\]

is a Riemann sum for \( f(x, y) \) over \([a, b] \times [c, d]\).

The double integral of \( f(x, y) \) over this rectangle is defined as the limit of its Riemann sums as both \(m\) and \(n\) go to \(\infty\):

\[
\int \int_{[a,b] \times [c,d]} f(x, y)\,dA = \lim_{n,m \to \infty} \sum_{i=1}^{m} \sum_{j=1}^{n} f(x_{i,j}^*, y_{i,j}^*) \Delta x \Delta y
\]
15.1.re2. Below, a Riemann sum is calculated for a function \( f(x, y) \) on the rectangle \([0, 2] \times [0, 2]\) using 3 subintervals in the \( x \)-direction and 4 in the \( y \)-direction. Sample points are taken to be the midpoints of the subrectangles. Because \( f(x, y) > 0 \) on \([0, 2] \times [0, 2]\), this Riemann sum is an approximation to the area under the graph of \( f \) over this rectangle in the \( xy \)-plane.

15.1.re3. Calculate a Riemann sum for the function \( e^{2x+3y} \) on \([-1, 1] \times [0, 1] \) using \( m = 2 \) and \( n = 3 \) subintervals in the \( x \)- and \( y \)-directions, respectively. Use the midpoints of the lower side of each subrectangle for sample points.

Solution: It helps to picture the subdivision of the rectangle:
The sample points (\( \bullet \)) are at \( x = \pm \frac{1}{2} \) and \( y = 0, \frac{1}{3}, \frac{2}{3} \). Dimensions of the subrectangles are \( \Delta x = 1 \) by \( \Delta y = \frac{1}{3} \), and so the Riemann sum is

\[
1 \cdot \frac{1}{3} \left( e^{-2 \cdot \frac{1}{2} + 3 \cdot 0} + e^{-2 \cdot \frac{1}{2} + 3 \cdot \frac{1}{3}} + e^{-2 \cdot \frac{1}{2} + 3 \cdot \frac{2}{3}} + e^{2 \cdot \frac{1}{2} + 3 \cdot 0} + e^{2 \cdot \frac{1}{2} + 3 \cdot \frac{1}{3}} + e^{2 \cdot \frac{1}{2} + 3 \cdot \frac{2}{3}} \right)
\]

\[
= \frac{1}{3} \left( e^{-1} + e^0 + e^1 + e^2 + e^3 \right).
\]

15.1.re4. Calculate the Riemann sum for the given function, rectangle, \( m, n \), and sample points.

a. \( xy + x - y - 1; [0, 2] \times [1, 3]; n = 2; m = 2; \) upper right corners
b. \( \cos(x + y); [0, \pi] \times [-\frac{\pi}{2}, \frac{\pi}{2}]; m = 3; n = 2; \) midpoints.
c. \( x^2 + y^2; [-1, 1] \times [0, 2]; m = 4; n = 2; \) points closest to (0, 0).
Iterated integration

We can calculate double integrals by integrating iteratively: first with respect to one variable, then with respect to the other. **Fubini's Theorem** promises that the value of the integral is independent of the order of integration, provided the integrand is piecewise continuous.

15.1.re5. Demonstrate that the value of the double integral of \( x^2 + y \) over the rectangle \([-1, 1] \times [0, 2]\) is the same regardless of the order of integration.

Let’s first integrate with respect to \( x \), then \( y \):

\[
\int_0^2 \int_{-1}^1 (x^2 + y) \, dx \, dy = \int_0^2 \left( \frac{1}{3}x^3 + xy \right|_{-1}^1 \right) \, dy \\
= \int_0^2 \left( \frac{1}{3} + y - \left( -\frac{1}{3} - y \right) \right) \, dy = \int_0^2 \left( \frac{2}{3} + 2y \right) \, dy \\
= \left( \frac{2}{3}y + y^2 \right|_0^2 = \frac{16}{3}.
\]

This time, integrate first with respect to \( y \):

\[
\int_{-1}^1 \int_0^2 (x^2 + y) \, dy \, dx = \int_{-1}^1 \left( yx^2 + \frac{1}{2}y^2 \right|_0^2 \right) \, dx \\
= \int_{-1}^1 \left( 2x^2 + 2 \right) \, dx = \left( \frac{2}{3}x^3 + 2x \right|_{-1}^1 = \frac{16}{3}.
\]

15.1.re6. Evaluate the double integral by iterated integration.

a. \( \int \int_{[1,2] \times [1,3]} (xy + x - y - 1) \, dA \)

b. \( \int \int_{[0,\pi/4] \times [-\pi/2,\pi/2]} (\cos(x + y)) \, dA \)

c. \( \int \int_{[-2,2] \times [-1,1]} (2x + y^2) \, dA \)
Average value

The average value of a function $f(x, y)$ over a region $R$ in the $xy$-plane is defined as

$$f_{\text{ave}} = \frac{1}{\text{area}(R)} \int \int_R f(x, y) \, dA.$$ 

Consequently,

$$\text{area}(R) \cdot f_{\text{ave}} = \int \int_R f(x, y) \, dA.$$ 

That is, $f_{\text{ave}}$ is the constant function having the same integral over $R$ as $f$.

15.1.re7. Calculate the average value over the given rectangle of the given function.

a. $[1, 2] \times [1, 3]; xy + x - y - 1$

b. $[0, \frac{\pi}{4}] \times [-\frac{\pi}{2}, \frac{\pi}{2}]; \cos(x + y)$

c. $[-2, 2] \times [-1, 1]; 2x + y^2$

Answers

15.1.re4a. 7. 15.1.re4b. $\frac{\pi}{\pi^2} \left( \cos(-\frac{\pi}{4}) + \cos(\frac{\pi}{4}) + \cos(\frac{\pi}{4}) + \cos(\frac{\pi}{4}) + \cos(\frac{\pi}{4}) \right)$. 15.1.re4c. 5/2.
15.1.re6a. 3. 15.1.re6b. $\sqrt{2}$. 15.1.re6c. $\frac{8}{\pi}$. 15.1.re7a. 3/2. 15.1.re7b. $4\sqrt{2}/\pi^2$. 15.1.re7c. $\frac{1}{4}$. 

15.2: Double integrals over general regions and properties of the double integral.

Integration over non-rectangular regions

We can integrate iteratively over regions other than rectangles by letting the limits of the inner integral depend on the variable of the outer integrals, e.g.

\[
\int_a^b \int_{g(x)}^{h(x)} f(x, y) \, dy \, dx \quad \text{or} \quad \int_a^b \int_{g(y)}^{h(y)} f(x, y) \, dx \, dy.
\]

That is, limits on the outer integral must be constant, but limits on the inner integral can depend on the outer variable of integration.

15.2.re1. Calculate \( \iint_R (y - 1) \, dA \), where \( R \) is the region in quadrant one bounded by \( y = 4, x = 0, \) and \( y = x^2 \), twice, using the two different orders of integration.

Evaluation by \( \int \int dy \, dx \):

\[
\int_0^2 \int_{x^2}^4 (y - 1) \, dy \, dx = \int_0^2 \left( \frac{1}{2} y^2 - y \right)^4_{x^2} \, dx
\]

\[
= \int_0^2 x \left[ (8 - 4) - \left( \frac{1}{2} x^4 - x^2 \right) \right] \, dx = \int_0^2 x(4 + x^2 - \frac{1}{2} x^4) \, dx
\]

\[
= \int_0^2 \left( 4x + x^3 - \frac{1}{2} x^5 \right) \, dx = \left. \left( 2x^2 + \frac{1}{4} x^4 - \frac{1}{12} x^6 \right) \right|_0^2 = \frac{20}{3}
\]

Evaluation by \( \int \int dx \, dy \):

\[
\int_0^4 \left( \int_0^{\sqrt{y}} (y - 1) \, dx \right) \, dy = \int_0^4 \left( \frac{1}{2} y^2(y - 1) \right|_0^{\sqrt{y}} \, dy
\]

\[
= \int_0^4 \frac{1}{2} y(y - 1) \, dy = \frac{1}{2} \int_0^4 (y^2 - y) \, dy
\]

\[
= \frac{1}{2} \left( \frac{1}{3} y^3 - \frac{1}{12} y^2 \right) \bigg|_0^4 = \frac{20}{3}
\]

In some cases, we choose the order of integration to make the problem easier.

15.2.re2. Let \( D \) be the region bounded by \( y = 0, y = 2x \) and \( y = (x - 4)^2 \) shown here. Write \( \iint_D f(x, y) \, dA \) as an iterated integral in both possible orders.

15.2.re3. Evaluate \( \int_0^1 \int_{-\sqrt{x}}^{\sqrt{x}} \sqrt{y^3 + 1} \, dy \, dx \).
15.2.re4. Evaluate the double integral, using whichever order of integration is easiest.
   a. \( \iint_D (y + 1)(x - 1) \, dA, \quad D = \{(x, y) \mid x^2 - 1 \leq y \leq 1 - x^2\} \)
   b. \( \iint_E xy \, dA, \quad E = \text{the triangle with vertices } (0, 0), (1, 0), (2, 2) \).
   c. \( \iint_F y \, dA; \quad F = \text{the upper half of the circle centered at } (0, 0) \text{ of radius 2} \).

15.2.re5. Find the average value of the function on the given region.
   a. \( (y + 1)(x - 1), \text{ on } \{(x, y) \mid x^2 - 1 \leq y \leq x^2 + 1\} \)
   b. \( y, \text{ on the upper half of the circle centered at } (0, 0) \text{ of radius 2} \).

**Properties of the double integral**

**Fact.** If \( b \) and \( c \) are constants, and \( R \) and \( S \) are regions in the plane on which both \( f(x, y) \) and \( g(x, y) \) exist, then

1. \( \iint_R (bf(x, y) + cg(x, y)) \, dA = b \iint_R f(x, y) \, dA + c \iint_R g(x, y) \, dA \).
2. \( \iint_R c \, dA = c \cdot \text{area}(R) \).
3a. \( \iint_{R \cup S} f(x, y) \, dA = \iint_R f(x, y) \, dA + \iint_S f(x, y) \, dA - \iint_{R \cap S} f(x, y) \, dA \), and so
3b. \( \iint_{R \cup S} f(x, y) \, dA = \iint_R f(x, y) \, dA + \iint_S f(x, y) \, dA \), if \( R \cap S \) are zero area.
15.2.re6. Find the volume of the region in $xyz$-space bounded by the planes $x = 0$, $y = 0$, $3x + 2y = 6$, $z = 0$, and the surface $z = x^2 + y^2$.

Answers

15.2.re2. $\int_0^4 \int_{y/2}^2 f(x, y) \, dy \, dx$, and $\int_0^2 \int_0^{2x} f(x, y) \, dy \, dx + \int_0^4 \int_0^{(x-4)^2} f(x, y) \, dy \, dx$  
15.2.re3. To avoid integration of $\sqrt{y^3 + 1} \, dy$, change order of integration to $\int_{-1}^0 \int_0^{\sqrt{y^3 + 1}} \, dx \, dy$ and evaluate. Result: $\frac{4}{3}$.
15.2.re4a. $-8/3$. 15.2.re4b. 1. 15.2.re4c. $16/3$. 15.2.re5a. $-1$. 15.2.re5b. $\frac{8}{27}$. 15.2.re6. $\frac{13}{2}$. 

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15.3: Double integrals in polar coordinates.

The polar coordinate system is coordinate system for the plane. For a review of polar coordinates, see the “Polar Coordinates” section (especially the first page and half) of my MATH 220 review notes at https://kunklet.people.cofc.edu/MATH220/220review.pdf.

Differential area $dA$ in polar coordinates.

All the lines along which $x$ or $y$ are constant divide the plane into infinitesimally small rectangles, whose area is $dA = dx\, dy$. Likewise, the circles along which $r$ is positive and constant and the rays along with $\theta$ is constant divide the plane into infinitesimally small rectangles with dimensions $dr$ by $r\, d\theta$, and so $dA = r\, dr\, d\theta$.

Warning: Since $dA$ must be positive, the formula

$$dA = r\, dr\, d\theta$$

requires $r > 0$. In general, $dA = |r|\, dr\, d\theta$.

It may be useful to write

$$\int\int_D f(x, y)\, dA$$

in polar coordinates if either the region $D$ or the integrand $f$ is easier to write in polar coordinates than in rectangular.

Tip: practice writing the following four relationships between $(r, \theta)$ and $(x, y)$ with the help of this picture (drawn as if $r$ were positive and $\theta$ were acute):

- $x = r\cos \theta$
- $y = r\sin \theta$
- $r^2 = x^2 + y^2$
- $\tan \theta = y/x$

15.3.re1. Integrate $\int\int_D x\, dA$, where $D$ is the region pictured here, bounded by the circles centered at $(0, 0)$ of radii 1 and 2, the negative $x$ axis, and the line $y = -x$.

Solution: $D$ is easily described in polar coordinates: $-\frac{\pi}{4} \leq \theta \leq \pi$, $1 \leq r \leq 2$. In polar coordinates, $x\, dA = (r\cos \theta)\, r\, dr\, d\theta$, so the integral is
\[
\int_{-\pi/4}^{\pi} \int_1^{2+\cos \theta} r^2 \cos \theta \, dr \, d\theta = \int_{-\pi/4}^{\pi} \frac{1}{3} r^3 \bigg|_1 \cos \theta \, d\theta = \frac{7}{3} \int_{-\pi/4}^{\pi} \cos \theta \, d\theta = \frac{7}{3} \sin \theta \bigg|_0^\pi = \frac{7}{3} \sin \pi - \sin \left(-\frac{\pi}{4}\right) = \frac{7}{3\sqrt{2}}.
\]

15.3.re2. Find the area inside the polar curve \( r = 2 + \cos \theta \).
To integrate \( \cos^2 \theta \), rewrite it using the half angle identity:
\[
\cos^2 \theta = \frac{1}{2}(1 + \cos(2\theta)).
\]

Area \( (D) = \int\limits_D dA = \int_0^{2\pi} \int_0^{2+\cos \theta} r \, dr \, d\theta = \frac{1}{2} \int_0^{2\pi} (2 + \cos \theta)^2 \, d\theta = \frac{1}{2} \int_0^{2\pi} (4 + 4 \cos \theta + \cos^2 \theta) \, d\theta = \frac{1}{2} \left[ \frac{9}{2} \theta + 4 \sin \theta + \frac{1}{4} \sin(2\theta) \right]_0^{2\pi} = \frac{9}{2} \pi.
\]

15.3.re3. Find the area of one loop of the rose \( r = \sin(2\theta) \).

15.3.re4. Evaluate the double integral
a. \( \int\int_D (x-y) \, dA \), \( D \) is the portion of the disk \( x^2 + y^2 \leq 4 \) in quadrants I, II, and III.
b. \( \int\int_E \cos(x^2 + y^2) \, dA \), \( E \) is the portion of the disk \( x^2 + y^2 \leq 4 \) in quadrant 1 below \( y = x \).
c. \( \int\int_F y \, dA \), \( F = \{(x, y) | 1 \leq x^2 + y^2 \leq 2, y \geq 0 \} \).
d. \( \int_0^{\sqrt{2}} \int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} (x^2 + y^2)^{3/2} \, dy \, dx \)

15.3.re5. Find the volume of the given solid.
a. Inside \( x^2 + y^2 = 1 \), between \( z = x^2 + y^2 \) and \( z = 0 \).
b. Inside both the cylinder \( x^2 + y^2 = \frac{1}{4} \) and the sphere \( x^2 + y^2 + z^2 = 1 \).
c. Under the cone \( z = 2 - \sqrt{x^2 + y^2} \) and above the annulus \( 1 \leq x^2 + y^2 \leq 4 \) in the plane.

Answers
15.3.re3. \( \pi/8 \) 15.3.re4a. \( -\frac{16}{5} \) 15.3.re4b. \( \frac{\pi}{8} \sin 4 \) 15.3.re4c. \( \frac{2}{3} (2^{3/2} - 1) \) 15.3.re4d. \( 32 \frac{\pi}{5} \).
15.3.re5a. \( \pi/2 \) 15.3.re5b. \( \left( \frac{2}{5} - \frac{\sqrt{3}}{4} \right) \pi \) 15.3.re5c. \( \frac{4\pi}{5} \).
15.5: Surface area of \( z = f(x, y) \).

The area of the surface \( z = f(x, y) \) above the region \( D \) in the \( xy \)-plane is

\[
\iint_D \sqrt{1 + f_x^2 + f_y^2} \, dA.
\]

Note the similarity of this formula to the arc length of the curve \( y = f(x) \) over the interval \([a, b] \):

\[
\int_a^b \sqrt{1 + f'^2} \, dx
\]

that we learned in calculus II.

15.5.re1. Find the area of the surface \( z = 5 - x^2 - y^2 \) above \( z = 1 \).

The region of integration in the \( xy \)-plane is where \( 5 - x^2 - y^2 \geq 1 \), or \( 4 \geq x^2 + y^2 \), the interior of the circle of radius 2 centered at the origin. Both this and the integrand \( \sqrt{1 + z_x^2 + z_y^2} = \sqrt{1 + 4x^2 + 4y^2} \) suggest that we switch to polar coordinates:

\[
\int_0^{2\pi} \int_0^2 r \sqrt{1 + 4r^2} \, dr \, d\theta = 2\pi \cdot \frac{\pi}{6} (17^{3/2} - 1).
\]

15.5.re2. Find the area of the graph of the given surface.

a. \( z = 1 - 2x + 3y \), between \( y = x^2 \) and \( y = 2 - x^2 \)

b. \( z = 3x + y^2 \), above the triangle in the plane with vertices \((0, 0)\), \((0, 1)\), and \((2, 1)\).

c. \( z = xy \), inside the cylinder \( x^2 + y^2 = 3 \).

d. the upper half of the sphere \( x^2 + y^2 + z^2 = 4 \), inside \( x^2 + y^2 = 3 \).

e. the upper half of the sphere \( x^2 + y^2 + z^2 = 4 \), inside \( x^2 + y^2 = 2x \).

f. the upper half of the cylinder \( x^2 + z^2 = 1 \), above the rectangle \([0, 1] \times [-1, 1] \).

Answers

15.5.re2a. \( \frac{8}{3} \sqrt{14} \). 15.5.re2b. \( \frac{1}{4} (7\sqrt{14} - 5\sqrt{10}) \). 15.5.re2c. \( \frac{4\pi}{3} \). 15.5.re2d. \( 4\pi \). 15.5.re2e. \( 4\pi - 8 \). 15.5.re2f. \( \pi \).
15.6: Triple integrals

Like a double integral, the triple integral of a function $f(x, y, z)$ over a region $D$ in $\mathbb{R}^3$

$$\iiint_D f(x, y, z) \, dV$$

is defined as a limit of Riemann sums and evaluated by iterated integration.

15.6.re1. Evaluate $\int_{-2}^{2} \int_{0}^{4-x^2} \int_{0}^{z} 3y \, dy \, dz \, dx$.

Solution:

$$\int_{-2}^{2} \int_{0}^{4-x^2} \left[ \frac{3}{2} y^2 \right]_0^z \, dz \, dx = \int_{-2}^{2} \int_{0}^{4-x^2} \frac{3}{2} z^2 \, dz \, dx$$

$$= \int_{-2}^{2} \frac{1}{2} z^3 \bigg|_0^{4-x^2} \, dx = \int_{-2}^{2} \frac{1}{2} (4 - x^2)^3 \, dx$$

Since $(4 - x^2)^3$ is even, this equals

$$\int_0^{2} (4 - x^2)^3 \, dx = \int_0^{2} (4^3 - 3 \cdot 4^2 x^2 + 3 \cdot 4x^4 - x^6) \, dx$$

$$= \left(64x - 16x^3 + \frac{12}{5} x^5 - \frac{1}{7} x^7\right) \bigg|_0^2 = \frac{12}{5} 2^5 - \frac{1}{7} 2^7.$$
15.6.re4. Evaluate the triple integral.

a. \[ \int_0^1 \int_0^{1-z} \int_1^{4-3x-3z} e^y \, dy \, dx \, dz \]
b. \[ \int_0^1 \int_{2-x}^{1-x^2} \int_0^{\frac{x}{3z}} dz \, dy \, dx \]
c. \[ \int_0^2 \int_0^{4-x^2} \int_0^{\sqrt{z}} x \, dy \, dz \, dx \]

15.6.re5. Rewrite the integrals in the given order of integration.

a. \[ \int_0^1 \int_0^{4-3x} \int_0^{1-z} e^y \, dy \, dx \, dz \to \int \int \int \, dz \, dy \, dx \]
b. \[ \int_0^2 \int_0^{\sqrt{z}} \int_0^{4-x^2} dy \, dz \, dx \to \int \int \int \, dx \, dz \, dy \]
c. \[ \int_{-2}^2 \int_0^{4-x^2} \int_0^z 3y \, dy \, dz \, dx \to \int \int \int \, dz \, dx \, dy \]

15.6.re6. Evaluate the triple integral by first changing the order of integration.

\[ \int_{-1}^2 \int_0^{\sqrt{2-z}} \int_y^{\sqrt{2-z}} (1 + \sqrt{1 + x^2}) \, dx \, dy \, dz \]

Answers

15.6.re2. 256/15. 15.6.re4a. \(-\frac{16}{15} e + \frac{4}{15} e^4\). 15.6.re4b. \(-\frac{7}{9} + 2 \ln \frac{7}{9}\). 15.6.re4c. \(\frac{64}{18}\).

15.6.re5a. \(\int_0^1 \int_0^{4-3x} \int_0^{x-y} e^y \, dy \, dz \, dx\). 15.6.re5b. \(\int_0^2 \int_0^{\sqrt{4-z}} \int_0^{\sqrt{4-z}} dx \, dz \, dy\).

15.6.re5c. \(\int_0^4 \int_{\sqrt{4-y}}^{4-x^2} \int_0^{3y} dz \, dx \, dy\). 15.6.re6. \(\frac{323}{60}\).
15.7: Cylindrical coordinates

In the cylindrical coordinate system, a point \((x, y, z)\) in space is represented by the variables \((r, \theta, z)\), where \((r, \theta)\) are polar coordinates for the point \((x, y)\). Since \(dx\,dy = r\,dr\,d\theta\),

\[dV = dx\,dy\,dz = r\,dr\,d\theta\,dz.\]

The relations between \((r, \theta)\) and \((x, y)\) are the same as seen section 15.3, remembered with the help of this picture (drawn as if \(r\) were positive and \(\theta\) were acute):

\[
x = r\cos \theta \quad r^2 = x^2 + y^2
\]
\[
y = r\sin \theta \quad \tan \theta = y/x
\]

15.7.re1. Evaluate the integral \(\iiint_D (x+y+z)\,dV\) where \(D\) is the region \(\{(x, y, z) | x^2+y^2 \leq 1, 0 \leq z \leq y\}\).

Solution:

\[
\int_0^\pi \int_0^1 \int_0^r \sin \theta \ r(\cos \theta + r\sin \theta + z)\,dz\,dr\,d\theta
\]
\[
= \int_0^\pi \int_0^1 \int_0^r (r^2 \cos \theta + r^2 \sin \theta + rz)\,dz\,dr\,d\theta
\]
\[
= \int_0^\pi \int_0^1 (r^3 \cos \theta \sin \theta + r^3 \sin^2 \theta + \frac{1}{2}r^3 \sin^2 \theta)\,dr\,d\theta
\]
\[
= \int_0^1 r^3 \,dr \int_0^\pi \left( \cos \theta \sin \theta + \frac{3}{2} \sin^2 \theta \right)\,d\theta
\]
\[
= \frac{1}{4} \int_0^\pi \left( \cos \theta \sin \theta + \frac{3}{4} (1 - \cos(2\theta)) \right)\,d\theta
\]
\[
= \frac{1}{4} \left( \frac{1}{2} \sin^2 \theta + \frac{3}{4} \theta - \frac{3}{8} \sin(2\theta) \right) \bigg|_0^\pi = \frac{3\pi}{16}
\]

15.7.re2. Find the volume of the region below \(z = 2x\) and above \(z = x^2 + y^2\).

Solution: The surfaces intersect on the circle \(2x = x^2 + y^2\), or

\[(15.7.1) \quad 2r\cos \theta = r^2,\]

which implies that either \(r = 0\) or \(2\cos \theta = r\). The only point on the graph of \(r = 0\) is the origin, which is also on the graph of \(2\cos \theta = r\), so we can safely replace (15.7.1) by

\[r = 2\cos \theta.\]
The circle \( r = 2 \cos \theta \) is traced once when \(-\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2}\). You can check that the plane is above the paraboloid in the interior of the circle by calculating \(2x\) and \(x^2 + y^2\) at the circle’s center \((1, 0)\). And so the volume equals

\[
\int_{-\pi/2}^{\pi/2} \int_{-\pi/2}^{\pi/2} \int_{0}^{2r \cos \theta} 1 \, dz \, r \, dr \, d\theta = \int_{-\pi/2}^{\pi/2} \int_{0}^{2 \cos \theta} (2r \cos \theta - r^2) r \, dr \, d\theta
\]

\[
= \int_{-\pi/2}^{\pi/2} \int_{0}^{2 \cos \theta} (2r^2 \cos \theta - r^3) \, dr \, d\theta = \int_{-\pi/2}^{\pi/2} \left( \frac{2}{3} r^3 \cos \theta - \frac{1}{4} r^4 \right) \bigg|_{0}^{2 \cos \theta} \, d\theta = \int_{-\pi/2}^{\pi/2} \frac{4}{3} \cos^4 \theta \, d\theta.
\]

This is a challenging integral which is best evaluated using the reduction formula

\[
\int \cos^n \theta \, d\theta = \frac{1}{n} \cos^{n-1} \theta \sin \theta + \frac{n-1}{n} \int \cos^{n-2} \theta \, d\theta
\]

from calculus II. (See “Integration by parts” in my MATH 220 review notes: https://kunklet.people.cofc.edu/MATH220/220review.pdf)

\[
\frac{4}{3} \int_{-\pi/2}^{\pi/2} \cos^4 \theta \, d\theta = \frac{4}{3} \left( \frac{1}{4} \cos^3 \theta \sin \theta \bigg|_{-\pi/2}^{\pi/2} + \frac{3}{4} \int_{-\pi/2}^{\pi/2} \cos^2 \theta \, d\theta \right)
\]

\[
= \frac{4}{3} \left( 0 + \frac{3}{4} \left( \frac{1}{2} \cos \theta \sin \theta \bigg|_{-\pi/2}^{\pi/2} + \frac{1}{2} \int_{-\pi/2}^{\pi/2} d\theta \right) \right)
\]

\[
= \frac{4}{3} \left( \frac{3}{4} \left( 0 + \frac{1}{2} \pi \right) \right) = \frac{\pi}{2}
\]

15.7.re3. Find the volume of the given region.

a. \(\{(x, y, z) \mid x^2 + y^2 \leq 1, 0 \leq z \leq y\}\)

b. \(\{(x, y, z) \mid x^2 + y^2 \leq 1, 0 \leq y \leq z \leq 1\}\)

c. The region between the paraboloid \(z = x^2 + y^2\) and the cone \(z = \sqrt{x^2 + y^2}\).

15.7.re4. Evaluate the triple integral.

a. \(\int_{-1}^{1} \int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} \int_{0}^{\sqrt{x^2+y^2}} z^{1/3} \, dz \, dy \, dx\)

b. \(\iiint_D e^{x^2+y^2-z} \, dV\), where \(D\) is the region between the cylinders \(x^2 + y^2 = 1\) and \(x^2 + y^2 = 4\), above \(z = 0\) and below \(z = x^2 + y^2\).

c. \(\iiint_E \frac{x+y}{\sqrt{x^2+y^2}} \, dV\), where \(E\) is the portion of the unit sphere and its interior in the first octant.

Answers

15.7.re3a. \(\frac{2}{3}\). 15.7.re3b. \(\frac{\pi}{2} - \frac{2}{7}\). 15.7.re3c. \(\frac{\pi}{6}\). 15.7.re4a. \(\frac{9\pi}{20}\). 15.7.re4b. \(\pi(e^4 - e - 3)\). 15.7.re4c. \(\frac{2}{3}\).
15.8: Spherical coordinates

In the spherical coordinate system, a point \((x, y, z)\) in space is represented by the variables \((\rho, \phi, \theta)\), where \(\rho\) is the distance from the point to the origin, \(\phi\) is the angle from the positive \(z\)-axis to the vector \(\langle x, y, z \rangle\), and \(\theta\) is the same as in cylindrical coordinates. According to this definition, \(\rho \geq 0\) and \(0 \leq \phi \leq \pi\).

In these coordinates,

\[
dV = \rho^2 \sin \phi \, d\rho \, d\phi \, d\theta
\]

The relations between rectangular, cylindrical, and spherical coordinates can be remembered with the help of this picture:

\[
\begin{align*}
  x &= r \cos \theta \\
  y &= r \sin \theta \\
  \frac{y}{x} &= \tan \theta \\
  r^2 &= x^2 + y^2 \\
  \rho^2 &= r^2 + z^2
\end{align*}
\]

15.8.re1. Identify the graph of \(\rho = 2 \cos \phi\) by converting to rectangular coordinates.

To convert, multiply both sides by \(\rho\):

\[
\begin{align*}
  \rho^2 &= 2 \rho \cos \phi \\
  x^2 + y^2 + z^2 &= 2z
\end{align*}
\]

\[
x^2 + y^2 + z^2 - 2z = 0 \\
x^2 + y^2 + z^2 - 2z + 1 = 1 \\
x^2 + y^2 + (z - 1)^2 = 1
\]

The surface is a sphere or radius 1 centered at the point \((0, 0, 1)\).

15.8.re2. Verify that the volume of the sphere of radius \(r\) is \(\frac{4}{3}\pi r^3\).

Solution: We’ll find the volume of the sphere by using \(\text{volume}(D) = \iiint_D 1 \, dV = \iiint_D \rho^2 \sin \phi \, d\rho \, d\phi \, d\theta\):

\[
\begin{align*}
  &\int_0^{2\pi} \int_0^\pi \int_0^r \rho^2 \sin \phi \, d\rho \, d\phi \, d\theta = \int_0^{2\pi} \int_0^\pi \sin \phi \left[ \frac{1}{3} \rho^3 \right]_0^r \, d\phi \, d\theta \\
  &= \frac{1}{3} r^3 \int_0^{2\pi} (2 \cos \phi) \left[ \frac{1}{3} \rho^3 \right]_0^r \, d\phi = \frac{1}{3} r^3 \int_0^{2\pi} (2 \cos \phi) \, d\theta = \frac{1}{3} r^3 \int_0^{2\pi} (2 \cos \phi) \, d\theta \\
  &= \frac{1}{3} r^3 \cdot 2\pi \cdot 2 = \frac{4}{3}\pi r^3
\end{align*}
\]
15.8.re3. Find the average value of $\frac{1}{\rho}$ on the set \{(x, y, z) \mid 1 \leq x^2 + y^2 + z^2 \leq 4\}.
Hint: you can use 15.8.re2 to find the volume of this set.

15.8.re4. Find the volume contained by the surface $\rho = 1 + 2 \cos \phi$.
Note: by their definition, $\rho \geq 0$ and $0 \leq \phi \leq \pi$. Use these to find the correct interval of $\phi$ for the integral.

15.8.re5. Find the volume inside the surface $\rho = 1 + 2 \cos \phi$ below the cone $\phi = \frac{\pi}{3}$ and above the $xy$-plane.

15.8.re6. Compute the integral:

$$
\int_{0}^{1} \int_{-\sqrt{1-y^2}}^{\sqrt{1-y^2}} \int_{1+\sqrt{1-x^2-y^2}}^{\sqrt{x^2+y^2}} \frac{dz \, dx \, dy}{\sqrt{x^2+y^2}}
$$

Hint: convert to spherical or cylindrical coordinates.

Answers
15.8.re3. $\frac{9}{16}$ 15.8.re4. $\frac{27\pi}{4}$ 15.8.re5. $\frac{5\pi}{4}$ 15.8.re6. $\frac{\pi}{2}$
15.9: Change of variables in multiple integrals.

If we wish to write a double integral in the $xy$-plane in terms of new variables $u$ and $v$, and if we can express $x$ and $y$ in terms of $u$ and $v$,

$$x = x(u, v) \quad y = y(u, v),$$

then differential area in the plane can be written

$$dA = \left| \begin{array}{cc} x_u & x_v \\ y_u & y_v \end{array} \right| \, du \, dv. \quad (15.9.1)$$

Here, the outer $|$ denote the absolute value and the inner $|$ denote a determinant.

If we wish to write a triple integral in $xyz$-space in terms of new variables $u, v, w$, and

$$x = x(u, v, w) \quad y = y(u, v, w) \quad z = z(u, v, w),$$

then differential volume in space can be written

$$dV = \left| \begin{array}{ccc} x_u & x_v & x_w \\ y_u & y_v & y_w \\ z_u & z_v & z_w \end{array} \right| \, du \, dv \, dw. \quad (15.9.2)$$

The formulas we encountered earlier for $dA$ and $dV$ in polar, spherical, and cylindrical coordinates are all special cases of (15.9.1) and (15.9.2).

These determinants are called **Jacobians** and are denoted

$$\frac{\partial(x, y)}{\partial(u, v)} = \left| \begin{array}{cc} x_u & x_v \\ y_u & y_v \end{array} \right| \quad \frac{\partial(x, y, z)}{\partial(u, v, w)} = \left| \begin{array}{ccc} x_u & x_v & x_w \\ y_u & y_v & y_w \\ z_u & z_v & z_w \end{array} \right|. \quad (15.9.3)$$

15.9.re1. Find the Jacobian of the given change of variables.

a. $x = (u + v)^2, \quad y = u^2 - v^2$

b. $x = u \cos v, \quad y = u \sin v$

c. $x = u + vw, \quad y = uw + w^2, \quad z = \ln w$

15.9.re2. Evaluate the double integral $\iint_D e^{x^2 - 4y} \, dA$ where $D$ is the parallelogram with vertices (1, 0), (3, -1), (3, 2), and (5, 1).

Solution. The equations of the four sides of the parallelogram are

$$x - y = 1 \quad x + 2y = 1$$

$$x - y = 4 \quad x + 2y = 7$$

This suggests that we define new variables to be

$$u = x - y$$

$$v = x + 2y$$
Solve for $x$ and $y$ in terms of $u$ and $v$:

\[
\begin{align*}
2u + v &= 3x \\ v - u &= 3y
\end{align*}
\implies
\begin{align*}
x &= \frac{2}{3}u + \frac{1}{3}v \\
y &= -\frac{1}{3}u + \frac{1}{3}v
\end{align*}
\]

By (15.9.1),

\[\text{(15.9.3)} \quad dA = \begin{vmatrix} x_u & x_v \\ y_u & y_v \end{vmatrix} \, du \, dv = \begin{vmatrix} \frac{2}{3} & \frac{1}{3} \\ -\frac{1}{3} & \frac{1}{3} \end{vmatrix} \, du \, dv = \frac{1}{3} \, du \, dv \]

and the integral equals

\[
\begin{aligned}
\int_1^4 \int_1^7 e^{\left(\frac{2}{3}u + \frac{1}{3}v\right)} - 4\left(-\frac{1}{3}u + \frac{1}{3}v\right) \frac{1}{3} \, dv \, du &= \frac{1}{3} \int_1^4 \int_1^7 e^{2u - v} \, dv \, du \\
&= \frac{1}{3} \int_1^4 \int_1^7 e^{2u} \, du \int_1^7 e^{-v} \, dv = \frac{1}{6} (e^8 - e^2)(e^{-1} - e^{-7}).
\end{aligned}
\]

15.9.re3. Compute the integral $\iint_D \frac{x}{y} e^{xy} \, dA$, where $D$ is the region in the first quadrant bounded by

\[
xy = 1 \\
xy = 4 \\
\frac{x}{y} = 1 \\
\frac{x}{y} = 3.
\]

Choose the variables

\[
\begin{align*}
u &= xy \\
v &= \frac{x}{y}
\end{align*}
\implies
\begin{align*}
x &= u^{1/2}v^{1/2} \\
y &= u^{1/2}v^{-1/2}
\end{align*}
\]

By (15.9.1),

\[
\begin{aligned}
dA &= \begin{vmatrix} x_u & x_v \\ y_u & y_v \end{vmatrix} \, du \, dv \\
&= \begin{vmatrix} \frac{1}{2}u^{-1/2}v^{1/2} & \frac{1}{2}u^{1/2}v^{-1/2} \\ \frac{1}{2}u^{-1/2}v^{-1/2} & -\frac{1}{2}u^{1/2}v^{-3/2} \end{vmatrix} \, du \, dv = \left| -\frac{1}{2} v^{-1} \right| \, du \,dv = \frac{1}{2} v^{-1} \, du \, dv.
\end{aligned}
\]

(We can use $|v^{-1}| = v^{-1}$, since $v > 0$ in this integral.) The integral equals

\[
\begin{aligned}
\int_1^3 \int_1^4 ve^{u} \, \frac{1}{2} v \, du \, dv &= \frac{1}{2} \int_1^3 \int_1^4 e^{u} \, du \, dv = e^4 - e.
\end{aligned}
\]
15.9.re4. Let \( D \) be the triangle with vertices \((1, 0), (3, 2), \) and \((5, 1)\). Rewrite the integral \( \iint_D (x - y) \, dA \) in terms of the variables \( u \) and \( v \) from 15.9.re2.

Solution: Using

\[
\begin{align*}
    u &= x - y & x &= \frac{2}{3} u + \frac{1}{3} v \\
    v &= x + 2y & y &= -\frac{1}{3} u + \frac{1}{3} v
\end{align*}
\]

the edges of the triangle become

\[
\begin{align*}
    x - y &= 1 & x + 2y &= 7 & x - 4y &= 1 \\
    u &= 1 & v &= 7 & (\frac{2}{3} u + \frac{1}{3} v) - 4(-\frac{1}{3} u + \frac{1}{3} v) &= 1 \\
    & & & 2u - v &= 1
\end{align*}
\]

which look like this in the \( uv \)-plane:

Using this and (15.9.3), the integral can be written either

\[
\frac{1}{3} \int_1^4 \int_{2u-1}^7 u \, dv \, du \quad \text{or} \quad \frac{1}{3} \int_1^7 \int_{\frac{1}{4}(1+v)}^1 u \, du \, dv.
\]

15.9.re5. Evaluate the integral by making an appropriate change of variables.

a. \( \iint_E (x + y) \, dA \), \( E \) = the parallelogram with vertices \((0,0), (1,2), (4,-1), (5,1)\).

b. \( \iint_C (x + y) \, dA \), \( C \) = the triangle with vertices \((1,2), (4,-1), (5,1)\).

c. \( \iint_D (x + y)^{-2} \, dA \), \( D \) is bounded by \( \frac{x}{y} = 2, \frac{u}{v} = \frac{1}{2}, x + y = 1, \) and \( x + y = 3 \).

d. \( \iiint_P (x^2 - z^2) \, dV \), where \( P \) is given by

\[
\begin{align*}
    0 &\leq x + z \leq 1 \\
    0 &\leq x - z \leq 2 \\
    -1 &\leq x + y + z \leq 1
\end{align*}
\]

Answers

15.9.re1a. \(-4(u + v)^2\) 15.9.re1b. \( u \) 15.9.re1c. \( w^{-1} u - v \) 15.9.re5a. New variables are \( 2x - y \) and \( x + 4y \).

Integral = 27. 15.9.re5b. New variables are \( 2x - y \) and \( x + 4y \). Integral = 18. 15.9.re5c. Let \( u = \frac{x}{y}, v = x + y \). Then \( y = \frac{u}{1+u} \) and \( x = \frac{vu}{1+u} \). Jacobian = \( \frac{vu}{(1+u)^2} \) and integral = \( \frac{1}{3} \ln 3 \). 15.9.re5d. 1