1(10 pts). Find the curl and divergence of each of the given vector fields.

a. $\mathbf{F} = \langle x^2, e^z, -1 \rangle$

b. $\mathbf{G} = \langle x, yz^2, y^2z \rangle$

1c(2 pts). Answer the following questions based on your work in parts 1a. and 1b. above.

If $C$ is a closed curve in $\mathbb{R}^3$, must $\int_C \mathbf{F} \cdot d\mathbf{r} = 0$? (circle one): Yes No

If $C$ is a closed curve in $\mathbb{R}^3$, must $\int_C \mathbf{G} \cdot d\mathbf{r} = 0$? (circle one): Yes No

2(8 pts). Evaluate the line integral $\int_C \nabla f \cdot d\mathbf{r}$, where $f(x, y) = xy(x - y)$ and $C$ is the curve parametrized by $\mathbf{r}(t) = \langle t^2 + 1, t^3 + t - 1 \rangle$ for $0 \leq t \leq 1$.

3(10 pts). Determine whether the vector field $\mathbf{F} = \langle ze^{-y}, -ze^{-y}, 1 + e^{-y} \rangle$ is conservative, and, if it is, find its potential function. (Recall that the function $f$ is a “potential” for the vector field $\mathbf{F}$ if $\mathbf{F} = \nabla f$.)

4(12 pts). Use Green’s theorem to evaluate the line integral $\int_C \sin y \, dx + (xy + x \cos y) \, dy$, where $C$ is the triangular path from $(0, 0)$ to $(4, 0)$ to $(0, 2)$ to $(0, 0)$.

5(11 pts). Find the area of the surface parametrized by $x = u + 2v$, $y = u - v + 1$, $z = u - 3$ for $-1 \leq u \leq 1$, $0 \leq v \leq 2$.

6(12 pts). Find the flux of $\mathbf{F} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$ across the surface in Problem 5, oriented upward.

7(19 pts). Use Stokes’s theorem to evaluate the line integral $\int_C y \, dx + z \, dy - x \, dz$, where $C$ is the triangular path with vertices $(6, 0, 0)$, $(0, 3, 0)$, $(0, 0, 2)$, traveled in the positive direction when viewed from above.

8(16 pts). Use the Divergence theorem to evaluate the flux of $\mathbf{F} = \langle x^3, 3yz^2, 3y^2z \rangle$ across the sphere (oriented outward) centered at the origin having radius 2.
1ab. (Source: 16.5.1,6) Note that \( \text{div} \mathbf{F} \) is a scalar, and \( \text{curl} \mathbf{F} \) is a vector.

\[ \text{div} \mathbf{F} = \nabla \cdot \mathbf{F} = \left( \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right) \cdot (x^2, e^z, -1) = 2x + 0 + 0 = 2x, \text{ and} \]

\[ \text{curl} \mathbf{F} = \nabla \times \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x^2 & e^z & -1 \end{vmatrix} = ( -e^z, 0, 0 ). \]

\[ \text{div} \mathbf{G} = \nabla \cdot \mathbf{G} = \left( \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right) \cdot (x, yz^2, y^2z) = 1 + z^2 + y^2, \text{ and} \]

\[ \text{curl} \mathbf{G} = \nabla \times \mathbf{G} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x & yz^2 & y^2z \end{vmatrix} = (0, 0, 0). \]

1c. (Source: 16.5.13-18) Since \( \text{curl} \mathbf{F} \neq \mathbf{0} \) and \( \text{curl} \mathbf{G} = \mathbf{0} \) are defined on all of \( \mathbb{R}^3 \), \( \mathbf{G} \) is conservative and \( \mathbf{F} \) is not. Therefore, if \( C \) is a closed curve in \( \mathbb{R}^3 \), then \( \int_C \mathbf{G} \cdot d\mathbf{r} \) must equal zero, but \( \int_C \mathbf{F} \cdot d\mathbf{r} \) need not equal zero.

2. (Source: 16.3.1) By the Fundamental Theorem of Calculus for line integrals,

\[ \int_C \nabla f \cdot d\mathbf{r} = \left. f(x, y) \right|^{(1)}_{(0)} = xy(x - y) \]

\[ = 2 - (-2) = 4. \]

3. (Source: 16.3.3,14,17, 16.5.17) We could determine that \( \mathbf{F} \) is conservative by calculating that \( \text{curl} \mathbf{F} = \mathbf{0} \), but to find its potential, we pretty must proceed as follows. Starting with \( f_x \), we integrate with respect to \( x \), the differentiate with respect to \( y \).

\[ f_x(x, y, z) = ze^{x-y} \]
\[ f(x, y, z) = ze^{x-y} + C(y, z) \]
\[ f_y(x, y, z) = -ze^{x-y} + C_y(y, z) = -ze^{x-y} \]

from which we conclude \( C_y = 0 \), and therefore \( C = C(z) \). Now differentiate \( f \) with respect to \( z \):

\[ f(x, y, z) = ze^{x-y} + C(z) \]
\[ f_z(x, y, z) = e^{x-y} + C_z(z) = 1 + e^{x-y} \]

from which we conclude \( C_z(z) = 1 \), and therefore \( C(z) = z + K \) for some constant \( K \), and the potential function must be \( f(x, y, z) = ze^{x-y} + z + K \).

4. (Source: 16.4.11) By Green’s theorem, the path integral equals a double integral over the interior \( T \) of the triangle. Specifically,

\[ \int_C \sin y \, dx + (xy + x \cos y) \, dy = \iint_T \left( (xy + x \cos y)_x - (\sin y)_y \right) \, dA \]

\[ = \iint_T \left( y + \cos y - \sin y \right) \, dA. \]
The line from \((4, 0)\) to \((0, 2)\) has the equation \(y = 2 - \frac{1}{2}x\), and so the double integral over the triangle is

\[
\int_0^4 \int_0^{2 - \frac{1}{2}x} (y + \cos y - \cos y) \, dy \, dx = \int_0^4 \int_0^{2 - \frac{1}{2}x} y \, dy \, dx
\]

\[
= \int_0^4 \frac{1}{2}(2 - \frac{1}{2}x)^2 \, dx = -\frac{1}{3}(2 - \frac{1}{2}x)^3|_0^4 = \frac{8}{3}
\]

5. (Source: 16.6.40) Let \(\mathbf{r}(u, v) = \langle u + 2v, u - v + 1, u - 3 \rangle\), and then \(dS = |\mathbf{r}_u \times \mathbf{r}_v| \, du \, dv =

\[
\left|\begin{array}{ccc}
\mathbf{i} & \mathbf{j} & \mathbf{k} \\
1 & 1 & 1 \\
2 & -1 & 0 \\
\end{array}\right| \, du \, dv = |\langle 1, 2, -3 \rangle| \, du \, dv = \sqrt{14} \, du \, dv,
\]

and therefore the area of the surface is

\[
S = \iint dS = \int_0^2 \int_{-1}^1 \sqrt{14} \, du \, dv = 4\sqrt{14}.
\]

6. (Source: 16.7.8) \(n \, dS = \pm (\mathbf{r}_u \times \mathbf{r}_v) \, du \, dv = \pm \langle 1, 2, -3 \rangle \, du \, dv\). For upward orientation, use \(n \, dS = \langle -1, -2, 3 \rangle \, du \, dv\). Dot this with \(\mathbf{F}\) and integrate over \(-1 \leq u \leq 1, 0 \leq v \leq 2:\n
\[
\int_0^2 \int_{-1}^1 \langle u + 2v, u - v + 1, u - 3 \rangle \cdot \langle -1, -2, 3 \rangle \, du \, dv = \int_0^2 \int_{-1}^1 (-11) \, du \, dv = -44
\]

7. (Source: 16.8.7) According to Stokes’s theorem, letting \(T\) denote the interior of the triangle, \(\int_C y \, dx + z \, dy - x \, dz = \iint_T \text{curl}(y, z, -x) \cdot n \, dS\).

\[
\text{curl}(y, z, -x) = \left|\begin{array}{ccc}
\frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\
y & z & -x \\
\end{array}\right| = \langle -1, 1, -1 \rangle
\]

To calculate the surface integral, find the equation of the plane passing though the three given points: \(x + 2y + 3z = 6\). The surface is then parametrized by \(\mathbf{r}(y, z) = \langle 6 - 2y - 3z, y, z \rangle\).

\[
\mathbf{n} \, dS = \pm (\mathbf{r}_y \times \mathbf{r}_z) \, dy \, dz = \pm \left|\begin{array}{ccc}
\mathbf{i} & \mathbf{j} & \mathbf{k} \\
-2 & 1 & 0 \\
-3 & 0 & 1 \\
\end{array}\right| \, dy \, dz = \pm \langle 1, 2, 3 \rangle \, dy \, dz
\]

Use the + so that the surface is oriented upward. The surface integral is then

\[
\iint \langle -1, 1, -1 \rangle \cdot \langle 1, 2, 3 \rangle \, dy \, dz = \iint \langle -2 \rangle \, dy \, dz,
\]
where the double integral is taken over the footprint of the triangle in the $yz$-plane. This footprint is itself a triangle with base 3 and height 2, so the integral equals $-2$ times the area of this triangle, or $-2 \cdot \frac{1}{2} \cdot 3 \cdot 2 = -6$.

8. (Source: 16.9.8) The Divergence Theorem tells that the flux of $\mathbf{F}$ across the boundary of the sphere equals the triple integral over the sphere of $\text{div} \mathbf{F} \ dV$, which we rewrite in spherical coordinates:

$$\iiint \nabla \cdot (x^3, 3yz^2, 3y^2z) \ dV = \iiint (3x^2 + 3z^2 + 3y^2) \ dV$$

$$= \int_0^{2\pi} \int_0^{\pi} \int_0^2 3\rho^2 \rho^2 \sin \phi \ d\rho \ d\phi \ d\theta \ = 3 \int_0^{2\pi} \int_0^{\pi} \sin \phi \int_0^2 \rho^4 \ d\rho \ d\phi \ d\theta \ = \frac{3\pi}{5} 2^7.$$