No notes, books, electronic devices, or outside materials of any kind.

Read each problem carefully and simplify your answers.

Unless otherwise indicated, supporting work will be required on every problem worth more than 2 points.

1a(8 pts). Sketch the graph of the equation $x - y^2 + z^2 = 0$. Label your axes x y z and use arrows to indicate the positive direction along each.

1b(3 pts). Complete the following statements about the surface in 1a by circling the correct word.

Cross-sections at $x =$ constant (when they exist) are *parabolas* *hyperbolas* *ellipses* .
Cross-sections at $y =$ constant (when they exist) are *parabolas* *hyperbolas* *ellipses* .
Cross-sections at $z =$ constant (when they exist) are *parabolas* *hyperbolas* *ellipses* .

2(10 pts). Find and sketch the domain of the function $l(x, y) = \arccos(y - 2) + \sqrt{1 - x}$.

3(19 pts). Let $f(x, y) = \frac{2x-3y}{x+4y}$ and $g(x, y) = \sin(2 - xy^2)$. Find the following derivatives.

a. f_x b. f_y c. g_{xx} d. g_{xy}

4(12 pts). Evaluate the limit or explain why it does not exist. Hint: one of these exists and the other does not.

a. $\lim_{(x,y)\to(0,0)} \frac{5y^2 \cos x^2}{x^2-y^2}$ b. $\lim_{(x,y)\to(0,0)} \frac{x^2-3xy-10y^2}{x-5y}$

5(5 pts). Find the vector projection of $\langle 4, 1, 1\rangle$ onto $\langle -2, 2, 1\rangle$.

6(4 pts). Find $\mathbf{u} \cdot \mathbf{v}$ if the angle between the vectors $\mathbf{u}$ and $\mathbf{v}$ is $\frac{\pi}{4}$ and $|\mathbf{u}| = |\mathbf{v}| = 2$.

7(12 pts). Find $\mathbf{T}(t)$ and $\mathbf{N}(t)$ along the curve $\mathbf{r}(t) = \langle \frac{1}{2}t^2, t\cos t - \sin t, t \sin t + \cos t\rangle$. $(t \geq 0.)$

8(27 pts). The problem is about the motion of a particle having position $\mathbf{r}(t) = \langle t^{-1}, t^2, 2t\rangle$.

a. Express the particle's velocity, speed, and acceleration as functions of t.

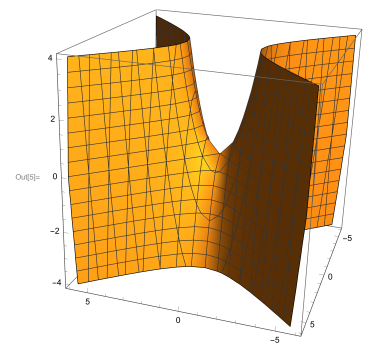
b. Find the distance traveled by the particle between time $t = 1$ and $t = 2$. Express your answer as a definite integral (that is, one with limits), but **do not evaluate**.

c. Find an equation of the line tangent to the particle's path at the point in space corresponding to $t = 1$.

d. Find a vector perpendicular to both $\mathbf{v}$ and $\mathbf{a}$ at $t = 1$.

e. Find equations of the normal and osculating planes to the particle's path at the point corresponding to $t = 1$.

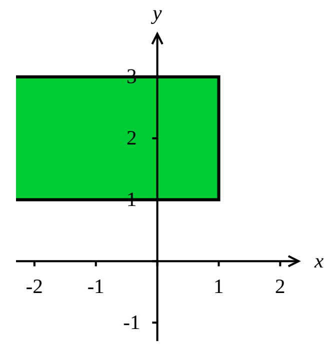
f. Find the curvature of the particle's path at the point corresponding to $t = 1$.



1a.(Source: 12.6.20) Here's a nice drawing from Mathematica, rotated to show the way I usually draw this surface. In the picture, the positive x-direction is up, the positive y-direction is to the left, and positive z-direction is back-to-front.

1b. Cross-sections at x = constant are **hyperbolas**. Cross-sections at y = constant are **parabolas**. Cross-sections at z = constant are **parabolas**.

2.(Source: 14.1.13,20) For the arccosine to be defined, we need $-1 \leq y - 2 \leq 1$, and for the squareroot to be defined, we need $1 - x \geq 0$. So, the domain consists of all points (x, y) for which $1 \leq y \leq 3$ and $x \leq 1$. Here's a graph.



3.(Source: 14.3.23,57,58) Use the quotient rule to find f_x and f_y:

$$f_x = \frac{2(x+4y) - (2x-3y)1}{(x+4y)^2} = \frac{11y}{(x+4y)^2}$$
$$f_y = \frac{-3(x+4y) - (2x-3y)4}{(x+4y)^2} = \frac{-11x}{(x+4y)^2}$$

First find g_x with the chain rule. Then differentiate g_x with respect to x and y. Finding g_{xy} will require the product rule.

$$\begin{aligned} g_x &= \big(\cos(2 - xy^2)\big)(-xy^2)_x = -y^2\cos(2 - xy^2) \\ g_{xx} &= (g_x)_x = (-y^2)\big(-\sin(2 - xy^2)\big)(-y^2) = -y^4\sin(2 - xy^2) \\ g_{xy} &= (g_x)_y = -2y\cos(2 - xy^2) + (-y^2)\big(-\sin(2 - xy^2)\big)(-xy^2)_y \\ &= -2y\cos(2 - xy^2) - 2xy^3\sin(2 - xy^2) \end{aligned}$$

4a.(Source: 14.2.10) The limit taken along the y-axis ($x = 0$) is the limit of $\frac{5y^2\cos x^2}{x^2 - y^2} = \frac{5y^2}{-y^2} = -5$, but the limit taken along the x-axis ($y = 0$) is of $\frac{5y^2\cos x^2}{x^2-y^2} = \frac{0}{x^2} = 0$, so the overall limit as $(x, y) \to (0, 0)$ does not exist.

4b.(Source: 14.2.9,14) $\lim_{(x,y)\to(0,0)} \frac{x^2-3xy-10y^2}{x-5y} = \lim_{(x,y)\to(0,0)} \frac{(x-5y)(x+2y)}{x-5y} = \lim_{(x,y)\to(0,0)} x + 2y = 0$.

5.(Source: 12.3.41) $\frac{\langle 4,1,1\rangle\cdot\langle -2,2,1\rangle}{\langle -2,2,1\rangle\cdot\langle -2,2,1\rangle}\langle -2, 2, 1\rangle = \frac{-5}{9}\langle -2, 2, 1\rangle$, or $\langle \frac{10}{9}, -\frac{10}{9}, -\frac{5}{9}\rangle$.

6.(Source: 12.3.10) $\mathbf{u}\cdot\mathbf{v} = |\mathbf{u}||\mathbf{v}|\cos\frac{\pi}{4} = 2\cdot 2\cdot\frac{1}{\sqrt{2}} = 2\sqrt{2}$

7.(Source: 13.3.18) First use the product rule to calculuate $\mathbf{v} = \frac{d}{dt}\langle \frac{1}{2}t^2, t\cos t - \sin t, t\sin t + \cos t\rangle = \langle t, -t\sin t, t\cos t\rangle = t\langle 1, -\sin t, \cos t\rangle$. We could normalize this to find the unit tangent vector $\mathbf{T}$, but this will be same as normalizing $\langle 1, -\sin t, \cos t\rangle$, since $\mathbf{v}$ is a positive multiple of this vector:

$$\mathbf{T}(t) = \frac{\langle 1, -\sin t, \cos t\rangle}{|\langle 1, -\sin t, \cos t\rangle|} = \frac{\langle 1, -\sin t, \cos t\rangle}{\sqrt{1+\sin^2 t+\cos^2 t}} = \frac{1}{\sqrt{2}}\langle 1, -\sin t, \cos t\rangle.$$

Differentiate $\mathbf{T}$:

$$\frac{d\mathbf{T}}{dt} = \frac{1}{\sqrt{2}}\langle 0, -\cos t, -\sin t\rangle.$$

Normalize this (or equivalently, normalize $\langle 0, -\cos t, -\sin t\rangle$) to find

$$\mathbf{N}(t) = \frac{\langle 0, -\cos t, -\sin t\rangle}{|\langle 0, -\cos t, -\sin t\rangle|} = \langle 0, -\cos t, -\sin t\rangle.$$

8a.(Source: 13.4.12) Velocity $= \mathbf{v} = \frac{d\mathbf{r}}{dt} = \langle -t^{-2}, 2t, 2\rangle$. Speed $= \frac{ds}{dt} = |\mathbf{v}| = \sqrt{t^{-4}+4t^2+4}$. Acceleration $\mathbf{a} = \frac{d\mathbf{v}}{dt} = \langle 2t^{-3}, 2, 0\rangle$.

8b.(Source: 13.3.5) Arclength $s = \int_{t=1}^{t=2} \frac{ds}{dt}\,dt = \int_1^2 \sqrt{t^{-4}+4t^2+4}\,dt$.

8c.(Source: 13.2.23-26) To find the equation(s) of a line, we need a point on the line and vector parallel the line. For the point, use $\mathbf{r}(1) = \langle 1, 1, 2\rangle$. The velocity $\mathbf{v}(1) = \langle -1, 2, 2\rangle$ is tangent to the curve at time $t = 1$, so the line is parametrized by $\langle 1, 1, 2\rangle + t\langle -1, 2, 2\rangle$, or

$$x = 1 - t \qquad y = 1 + 2t \qquad z = 2 + 2t.$$

8d.(Source: 12.4.1) $\mathbf{a}(1) = \langle 2, 2, 0\rangle$. Can use

$$\mathbf{v}(1)\times\mathbf{a}(1) = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -1 & 2 & 2 \\ 2 & 2 & 0 \end{vmatrix} = \mathbf{i}\begin{vmatrix} 2 & 2 \\ 2 & 0 \end{vmatrix} - \mathbf{j}\begin{vmatrix} -1 & 2 \\ 2 & 0 \end{vmatrix} + \mathbf{k}\begin{vmatrix} -1 & 2 \\ 2 & 2 \end{vmatrix} = -4\mathbf{i} + 4\mathbf{j} - 6\mathbf{k}.$$

(Any scalar multiple of this this will also answer the question.)

8e.(Source: 13.3.50) Both planes in question pass through the point $\mathbf{r}(1) = \langle 1, 1, 2\rangle$. The normal plane is perpendicular to $\mathbf{v}(1) = \langle -1, 2, 2\rangle$, while the osculating plane is parallel to both $\mathbf{v}(1)$ and $\mathbf{a}(1)$, and hence perpendicular to $\mathbf{v}(1)\times\mathbf{a}(1) = \langle -4, 4, -6\rangle$ (or any scalar multiple of this).

$$\begin{aligned} \text{normal plane:} \quad & -(x-1) + 2(y-1) + 2(z-2) = 0 \\ \text{osculating plane:} \quad & -4(x-1) + 4(y-1) - 6(z-2) = 0 \end{aligned}$$

8f.(Source: 13.3.25) At time $t = 1$, $\kappa = \frac{|\mathbf{v}(1)\times\mathbf{a}(1)|}{|\mathbf{v}(1)|^3} = \frac{\sqrt{68}}{(\sqrt{9})^3} = \frac{68}{27}$.