MATH 221–01 (Kunkle), Final Exam　　Name: ______________________

200 pts, 3 hours　　Dec 11, 2017　　

No notes, books, electronic devices, or outside materials of any kind.

Read each problem carefully and simplify your answers.

Unless otherwise indicated, supporting work will be required on every problem worth more than 2 points.

1a(4 pts). Find an equation of the line passing through the points $(-1, 2, 0)$ and $(0, -2, 3)$.

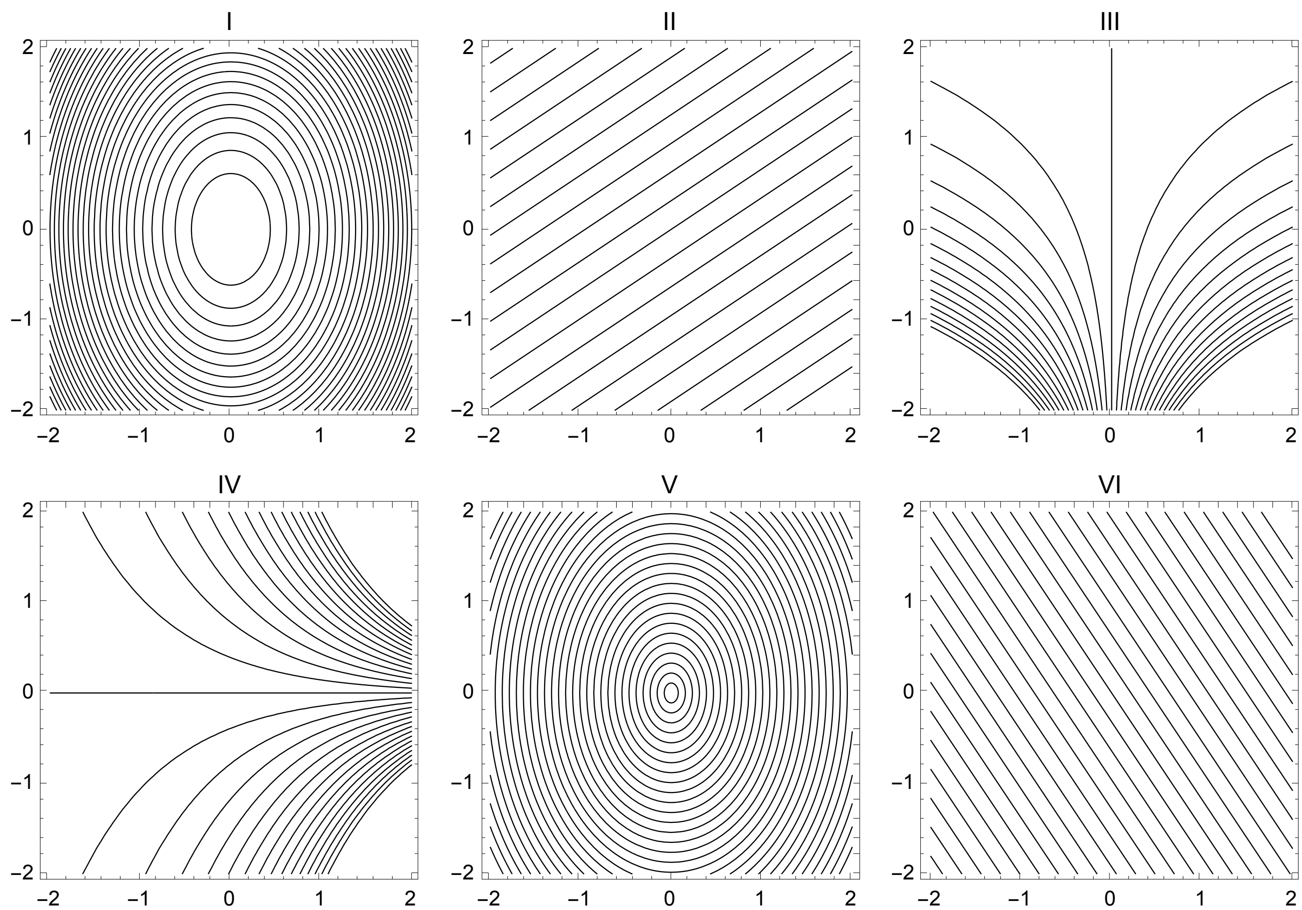
1b(4 pts). Find an equation of the line passing through the point $(-1, 2, 0)$ and perpendicular to the plane $2x - 3y + 5z = 1$.

2(7 pts). Find the vector projection of $\langle 1, 2, -1\rangle$ onto $\langle 2, -1, 2\rangle$.

3(14 pts). Find an equation of the plane tangent to $z = 1 + y\ln(xy - 11)$ at the point $(3, 4, 1)$.

4(8 pts). The contour maps below show the level curves $f(x, y) = k$ for various functions f and equally spaced values of k. Identify the contour map of each of the given functions.

a. $2x^2 + y^2$　　b. $\sqrt{2x^2 + y^2}$　　c. ye^x　　d. $2x - 3y$



5. The position of a particle at time t is given by the vector $\mathbf{r}(t) = \langle e^t, e^{-t}, t\rangle$.

a(7 pts). Express the particle's velocity, speed, and acceleration as functions of t. Label your answers with those three words so I can tell which is which.

b(5 pts). Find a unit vector $\mathbf{T}$ tangent to the path of the particle at time $t = 0$.

c(7 pts). Find the curvature κ of the particle's path at time $t = 0$.

d(7 pts). Find the tangential and normal components a_T and a_N of acceleration at time $t = 0$.

e(5 pts). Express distance travelled by the particle between $t = -1$ and $t = 1$ as a definite integral, but **do not evaluate**.

6a(4 pts). Find a parametrization $r(u, v)$ for the part of the surface $z = 1 + y^2$ that lies inside the cylinder $x^2 + y^2 = 4$.

6b(12 pts). Express the area of the surface in 6a as a definite iterated integral, but **do not evaluate**.

7(8 pts). The temperature at the point (x, y) is given by a function $T(x, y)$. Here x and y are measured in cm, and T is measured in °C. A bug comes to the point $(2, 3)$ and finds it uncomfortably cold there. If $T_x(2, 3) = -4$ and $T_y(2, 3) = 1$, in which direction should the bug move in order to see the most rapid increase in temperature (in °C per cm), and at what rate is the temperature increasing in that direction? State the direction as a unit vector.

8(18 pts). Find the maximum and minimum values of $2x + 2y - z$ on the ellipsoid $4x^2 + y^2 + z^2 = 3$.

9(6 pts). Two students are working on the problem of maximizing and minimizing a function $f(x, y)$ on the disk $D = \{(x, y) \mid x^2 + y^2 \leq 1\}$.
The first student has determined that the maximum and minimum values of f on the circle $x^2 + y^2 = 1$ are 5 and -1, respectively.
The second student has determined that the only solutions to $f_x(x, y) = f_y(x, y) = 0$ are $(x, y) = (0, 2 \pm \sqrt{2})$, and that $f(0, 2 - \sqrt{2}) = -3$ and $f(0, 2 + \sqrt{2}) = 7$.
Assuming their work is correct, what, if anything, can you say about the maximum and minimum values of $f(x, y)$ on D?

10(12 pts). Find the volume of the solid bounded above by the surface $z = 1 + xy^2$ and below by $z = 0$ between $x = y^2$ and $x = 1$.

11(16 pts). Evaluate $\iiint_E z\,dV$ where E is the solid between the spheres $x^2 + y^2 + z^2 = 1$ and $x^2 + y^2 + z^2 = 2$ and above the xy plane.

12(14 pts). Let B be the solid in the first octant given by the inequalities $x^2 + z^2 \leq 1$, $x \geq 0$, $z \geq 0$, and $0 \leq y \leq 2$. Find the flux of $\mathbf{H} = xy\mathbf{i} + z\mathbf{j} + y\mathbf{k}$ out of B.

13(11 pts). Evaluate $\int_C x\,ds$ where C is the right half of the circle $x^2 + y^2 = 4$.

14(13 pts). Evaluate $\int_C (e^{x^2} + y^2)\,dx + (x^2 - \sin^2 y)\,dy$ where C is rectangle with vertices $(1, 1)$, $(3, 1)$, $(1, 2)$, $(3, 2)$ traversed in the positive direction.

15a(4 pts). Determine if $\mathbf{F} = e^{xy}\mathbf{i} - e^{xy}\mathbf{j}$ is conservative. (Any answer without supporting some work will be treated as a guess and receive no credit.)

15b(14 pts). Evaluate $\int_C e^{xy}\,dx - e^{xy}\,dy$, where the path C consists of the line segments from the point $(0, 1)$ to $(0, 2)$ and from $(0, 2)$ to $(1, 2)$.

1a.(Source: 12.5.6) The line is parallel the vector $\langle 0,-2,3\rangle - \langle -1,2,0\rangle = \langle 1,-4,3\rangle$, so it's parametrized by $\mathbf{r}(t) = \langle -1,2,0\rangle + t\langle 1,-4,3\rangle$. This can also be expressed as

$$x = -1+t \quad y = 2-4t \quad z = 3t.$$

Solving for t gives the symmetric equations $x+1 = \frac{1}{4}(2-y) = \frac{z}{3}$.
1b.(Source: 16.5.16) The line is parallel the normal vector to the plane, $\langle 2,-3,5\rangle$, so it's parametrized by $\mathbf{r}(t) = \langle -1,2,0\rangle + t\langle 2,-3,5\rangle$.
2.(Source: 12.3.41) $\dfrac{\langle 1,2,-1\rangle \cdot \langle 2,-1,2\rangle}{\langle 2,-1,2\rangle \cdot \langle 2,-1,2\rangle}\langle 2,-1,2\rangle = \dfrac{-2}{9}\langle 2,-1,2\rangle$, or $\langle \frac{-4}{9}, \frac{2}{9}, \frac{-4}{9}\rangle$
3.(Source: 14.4.5,6) Solution 1: Let $g(x,y) = 1 + y\ln(xy-11)$. Then

$$\begin{aligned} g_x(x,y) &= y\cdot\frac{y}{xy-11} & g_y(x,y) &= \ln(xy-11) + y\cdot\frac{x}{xy-11} \\ g_x(3,4) &= 16 & g_y(3,4) &= \ln 1 + 12 = 12, \end{aligned}$$

and the tangent plane is the graph of the linearization

$$\begin{aligned} z &= g(3,4) + g_x(3,4)(x-3) + g_y(3,4)(y-4), \text{ or} \\ z &= 1 + 16(x-3) + 12(y-4). \end{aligned}$$

3. Solution 2. Let $f(x,y,z) = y\ln(xy-11) - z$. Surface in question is $f(x,y,z) = -1$. For normal vector, use the the gradient $\nabla f = \langle f_x, f_y, f_z\rangle = \langle y\cdot\frac{y}{xy-11}, \ln(xy-11) + y\cdot\frac{x}{xy-11}, -1\rangle$, which, at $(3,4,1)$, equals $\langle 16,12,-1\rangle$. The plane passing through $(3,4,1)$ normal to $\langle 16,12,-1\rangle$ has the equation $16(x-3)+12(y-4)-(z-1) = 0$.
4.(Source: 14.2.36,45-51,61-66) a: I. b: V. c: IV. d: II.
Although the level curves of both $2x^2+y^2$ and $\sqrt{2x^2+y^2}$ are ellipses, remember that $z = 2x^2+y^2$ is a paraboloid and $z = \sqrt{2x^2+y^2}$ is a cone. As z changes more rapidly on the paraboloid, the level curves come closer together. The curves $ye^x = k$ can be rewritten $y = ke^{-x}$, and $2x-3y = k$ are lines of (positive) slope $\frac{2}{3}$.
5a.(Source: 13.4.11) Velocity is the vector $\mathbf{v} = \frac{d\mathbf{r}}{dt} = \langle e^t, -e^{-t}, 1\rangle$. Speed is the magnitude of $\mathbf{v}$, $\frac{ds}{dt} = \sqrt{e^{2t}+e^{-2t}+1}$, and acceleration $= \mathbf{a} = \frac{d^2\mathbf{r}}{dt^2} = \frac{d\mathbf{v}}{dt} = \langle e^t, e^{-t}, 0\rangle$.
5b.(Source: 13.2.17-20) $\mathbf{T} = \frac{\mathbf{v}}{|\mathbf{v}|}$. At $t=0$, this equals $\frac{\langle 1,-1,1\rangle}{|\langle 1,-1,1\rangle|} = \frac{1}{\sqrt{3}}\langle 1,-1,1\rangle$.
5c.(Source: 13.3.19) At $t=0$, $\mathbf{a} = \langle 1,1,0\rangle$, and $\mathbf{v}\times\mathbf{a} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & -1 & 1 \\ 1 & 1 & 0 \end{vmatrix} = \langle -1,1,2\rangle$, and so $\kappa = |\mathbf{v}\times\mathbf{a}|/|\mathbf{v}|^3 = \frac{\sqrt{6}}{\sqrt{3}^3}$, or $\frac{\sqrt{2}}{3}$.
5d.(Source: 13.4.41) $a_T = \frac{\mathbf{a}\cdot\mathbf{v}}{|\mathbf{v}|}$, the scalar projection of $\mathbf{a}$ onto $\mathbf{v}$. Since $\mathbf{a}\cdot\mathbf{v} = 0$, this is zero. ($a_T$ is also $\frac{d^2s}{dt^2}$, the rate of change of speed, so at $t=0$, the object is neither speeding up nor slowing down.) $a_N = \sqrt{|\mathbf{a}|^2 - a_T^2} = |\mathbf{a}| = \sqrt{2}$. You could also calculate a_N by the formula $a_N = \kappa(\frac{ds}{dt})^2$.
5e.(Source: 13.3.1-6) $s = \int ds = \int_{-1}^{1} \frac{ds}{dt}\,dt = \int_{-1}^{1} \sqrt{e^{2t}+e^{-2t}+1}\,dt$

6a.(Source: 16.6.26) We could use polar coordinates to write $\mathbf{r} = \langle r\cos\theta, r\sin\theta, 1+r^2\sin^2\theta\rangle$, but this and 6b below are both easier if we choose x and y for the parameters, so that $\mathbf{r}(x,y) = \langle x, y, 1+y^2\rangle$.

6b.(Source: 15.5.9, 16.6.45) Since z is a function of x and y along this surface, $dS = \sqrt{1+z_x^2+z_y^2}\,dx\,dy$, and so the area equals $\int_{-2}^{2}\int_{-\sqrt{4-y^2}}^{\sqrt{4-y^2}} \sqrt{1+4y^2}\,dx\,dy$

7.(Source: 14.6.21-26) T increases most rapidly (in °C per cm) when the bug moves in the direction of the gradient $\nabla T(2,3) = \langle -4, 1\rangle$. Normalize this to obtain the unit vector $\frac{1}{\sqrt{17}}\langle -4,1\rangle$. In this direction, the derivative is $|\nabla T| = \sqrt{17}$.

8.(Source: 14.8.7) Use Lagrange multipliers. The max and min of $f = 2x+2y-z$ can only occur at critical points on $4x^2+y^2+z^2 = 3$, i.e., those points at which $\nabla(4x^2+y^2+z^2) = \lambda\nabla(2x+2y-z)$:

$$8x = 2\lambda \qquad 2y = 2\lambda \qquad 2z = -\lambda$$

$$x = \frac{1}{4}\lambda \qquad y = \lambda \qquad z = -\frac{1}{2}\lambda$$

Substitute these for x, y, z into the equation for the ellipsoid and solve for λ:

$$3 = 4x^2+y^2+z^2 = \frac{1}{4}\lambda^2 + \lambda^2 + \frac{1}{4}\lambda^2 = \frac{6}{4}\lambda^2$$

which implies $\lambda = \pm\sqrt{2}$. Therefore, the critical points are $(x,y,z) = \pm(\frac{\sqrt{2}}{4}, \sqrt{2}, -\frac{\sqrt{2}}{2})$. Since $f(\frac{\sqrt{2}}{4}, \sqrt{2}, -\frac{\sqrt{2}}{2}) = 3\sqrt{2}$ and $f(-\frac{\sqrt{2}}{4}, -\sqrt{2}, \frac{\sqrt{2}}{2}) = -3\sqrt{2}$, the maximum is $3\sqrt{2}$ and the minimum is $-3\sqrt{2}$.

9.(Source: 14.8.31-42) The maximum and minimum values of f on D must occur either on the boundary or at an interior critical point. The only critical point inside D is $(0, 2-\sqrt{2})$, where $f = -3$. Therefore f's maximum on D is 5, and its minimum is -3.

10.(Source: 15.2.24) $V = \int_{-1}^{1}\int_{y^2}^{1}\int_{0}^{1+xy^2} dz\,dx\,dy = \int_{-1}^{1}\int_{y^2}^{1}(1+xy^2)\,dx\,dy$
$= \int_{-1}^{1}(x+\frac{1}{2}x^2y^2)\Big|_{x=y^2}^{x=1} dy = \int_{-1}^{1}(1+\frac{1}{2}y^2-y^2-\frac{1}{2}y^6)\,dy = \int_{-1}^{1}(1-\frac{1}{2}y^2-\frac{1}{2}y^6)\,dy$
$= (y-\frac{1}{6}y^3-\frac{1}{14}y^7)\Big|_{-1}^{1} = 2(1-\frac{1}{6}-\frac{1}{14})$, or $\frac{32}{21}$.

11.(Source: 15.8.23) In spherical coordinates, $z = \rho\cos\phi$ and $dV = \rho^2\sin\phi\,d\rho\,d\phi\,d\theta$, and the integral is $\int_0^{2\pi}\int_0^{\pi/2}\int_1^{\sqrt{2}} \rho^3\cos\phi\sin\phi\,d\rho\,d\phi\,d\theta =$

$$\left(\int_0^{2\pi} d\theta\right)\left(\int_1^{\sqrt{2}} \rho^3\,d\rho\right)\left(\int_0^{\pi/2}\cos\phi\sin\phi\,d\phi\right) = 2\pi\left(\frac{1}{4}\rho^4\Big|_1^{\sqrt{2}}\right)\left(\frac{1}{2}\sin^2\phi\Big|_0^{\pi/2}\right)$$
$$= 2\pi\cdot\frac{1}{4}(4-1)\cdot\frac{1}{2}(1-0) = \frac{3\pi}{4}$$

12.(Source: 16.9.7) The boundary of B consists of 5 difference surfaces, so it's best to use the Divergence Theorem. $\operatorname{div}\mathbf{H} = \langle\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\rangle\cdot\langle xy, z, y\rangle = y$. The flux out of B is $\iiint \operatorname{div}\mathbf{H}\,dV = \iint_E\int_0^2 y\,dy\,dx\,dz$, where E is the quarter circle in the xz plane given by

$x^2 + z^2 \leq 1$, $x \geq 0$, $z \geq 0$. First calculate $\int_0^2 y\,dy = 2$, and then $\iint_E 2\,dx\,dz = 2$ times the area of E. Therefore the flux is $2 \cdot \frac{\pi}{4} = \frac{\pi}{2}$.

13.(Source: 16.2.3) Parametrize the right half of the circle as $x = 2\cos\theta$ and $y = 2\sin\theta$, where $-\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2}$. (The order we travel C doesn't matter since neither x nor ds depends on the direction.) Then differential arc length is

$$ds = \frac{ds}{d\theta}\,d\theta = |\frac{d\mathbf{r}}{d\theta}|\,d\theta = \sqrt{(\frac{dx}{d\theta})^2 + (\frac{dy}{d\theta})^2}\,d\theta = \sqrt{4\sin^2\theta + 4\cos^2\theta}\,d\theta = 2\,d\theta$$

and the integral is $\int_{-\pi/2}^{\pi/2} 2\cos\theta \cdot 2\,d\theta = 4\sin\theta\Big|_{-\pi/2}^{\pi/2} = 4(1-(-1)) = 8$.

14.(Source: 16.4.6, 15.2.15) Since this is a closed path, we can use Green's Theorem to evaluate the line integral.

$$\int_C P\,dx + Q\,dy = \int_1^3 \int_1^2 (Q_x - P_y)\,dy\,dx = \int_1^3 \int_1^2 (2x - 2y)\,dy\,dx$$

$$= \int_1^3 (2xy - y^2)\Big|_1^2\,dx = \int_1^3 (4x - 4 - 2x + 1)\,dx = \int_1^3 (2x - 3)\,dx = (x^2 - 3x)\Big|_1^3 = 2$$

15a.(Source: 16.3.7) $\operatorname{curl}\mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ e^{xy} & -e^{xy} & 0 \end{vmatrix} = (-ye^{xy} - xe^{xy})\mathbf{k} \neq \mathbf{0}$. Consequently, $\mathbf{F}$ is not conservative.

15b.(Source: 16.2.7) Since $\mathbf{F}$ is not conservative, the path integral cannot be counted on to be path independent and we must calculate the integral along C directly.

On the first segment of C, we can let $x = 0$ (so $dx = 0$) and y will serve as the path parameter. The integral along this line segment is

$$\int_1^2 (-e^0)\,dy = -1.$$

On the second line segment, we can let $y = 2$ (so $dy = 0$) and x will serve as the path parameter. The integral along this line segment is

$$\int_0^1 e^{2x}\,dx = \frac{1}{2}e^{2x}\Big|_0^1 = \frac{1}{2}(e^2 - 1).$$

Altogether, the line integral along C equals the sum

$$-1 + \frac{1}{2}(e^2 - 1) = \frac{1}{2}(e^2 - 3).$$