No notes, books, electronic devices, or outside materials of any kind.
Read each problem carefully and simplify your answers.
Unless otherwise indicated, supporting work will be required on every problem worth more than 2 points.

1a. Find an equation of the line passing through the points \((-1, 2, 0)\) and \((0, -2, 3)\).

1b. Find an equation of the line passing through the point \((-1, 2, 0)\) and perpendicular to the plane \(2x - 3y + 5z = 1\).

2. Find the vector projection of \(\langle 1, 2, -1 \rangle\) onto \(\langle 2, -1, 2 \rangle\).

3. Find an equation of the plane tangent to \(z = 1 + y \ln(xy - 11)\) at the point \((3, 4, 1)\).

4. The contour maps below show the level curves \(f(x, y) = k\) for various functions \(f\) and equally spaced values of \(k\). Identify the contour map of each of the given functions.
   a. \(2x^2 + y^2\)
   b. \(\sqrt{2x^2 + y^2}\)
   c. \(ye^x\)
   d. \(2x - 3y\)

5. The position of a particle at time \(t\) is given by the vector \(\mathbf{r}(t) = \langle e^t, e^{-t}, t \rangle\).

a. Express the particle’s velocity, speed, and acceleration as functions of \(t\). Label your answers with those three words so I can tell which is which.
b(5 pts). Find a unit vector $\mathbf{T}$ tangent to the path of the particle at time $t = 0$.

c(7 pts). Find the curvature $\kappa$ of the particle’s path at time $t = 0$.

d(7 pts). Find the tangential and normal components $a_T$ and $a_N$ of acceleration at time $t = 0$.

e(5 pts). Express distance travelled by the particle between $t = -1$ and $t = 1$ as a definite integral, but do not evaluate.

6a(4 pts). Find a parametrization $r(u, v)$ for the part of the surface $z = 1 + y^2$ that lies inside the cylinder $x^2 + y^2 = 4$.

6b(12 pts). Express the area of the surface in 6a as a definite iterated integral, but do not evaluate.

7(8 pts). The temperature at the point $(x, y)$ is given by a function $T(x, y)$. Here $x$ and $y$ are measured in cm, and $T$ is measured in °C. A bug comes to the point $(2, 3)$ and finds it uncomfortably cold there. If $T_x(2, 3) = -4$ and $T_y(2, 3) = 1$, in which direction should the bug move in order to see the most rapid increase in temperature (in °C per cm), and at what rate is the temperature increasing in that direction? State the direction as a unit vector.

8(18 pts). Find the maximum and minimum values of $2x + 2y - z$ on the ellipsoid $4x^2 + y^2 + z^2 = 3$.

9(6 pts). Two students are working on the problem of maximizing and minimizing a function $f(x, y)$ on the disk $D = \{(x, y) \mid x^2 + y^2 \leq 1\}$.

The first student has determined that the maximum and minimum values of $f$ on the circle $x^2 + y^2 = 1$ are $5$ and $-1$, respectively.

The second student has determined that the only solutions to $f_x(x, y) = f_y(x, y) = 0$ are $(x, y) = (0, 2 \pm \sqrt{2})$, and that $f(0, 2 - \sqrt{2}) = -3$ and $f(0, 2 + \sqrt{2}) = 7$.

Assuming their work is correct, what, if anything, can you say about the maximum and minimum values of $f(x, y)$ on $D$?

10(12 pts). Find the volume of the solid bounded above by the surface $z = 1 + xy^2$ and below by $z = 0$ between $x = y^2$ and $x = 1$.

11(16 pts). Evaluate $\iiint_E z \, dV$ where $E$ is the solid between the spheres $x^2 + y^2 + z^2 = 1$ and $x^2 + y^2 + z^2 = 2$ and above the $xy$ plane.

12(14 pts). Let $B$ be the solid in the first octant given by the inequalities $x^2 + z^2 \leq 1$, $x \geq 0$, $z \geq 0$, and $0 \leq y \leq 2$. Find the flux of $\mathbf{H} = x\mathbf{i} + z\mathbf{j} + y\mathbf{k}$ out of $B$.

13(11 pts). Evaluate $\int_C x \, ds$ where $C$ is the right half of the circle $x^2 + y^2 = 4$.

14(13 pts). Evaluate $\int_C (e^{x^2} + y^2) \, dx + (x^2 - \sin^2 y) \, dy$ where $C$ is rectangle with vertices $(1, 1), (3, 1), (1, 2), (3, 2)$ traversed in the positive direction.

15a(4 pts). Determine if $\mathbf{F} = e^{xy} \mathbf{i} - e^{xy} \mathbf{j}$ is conservative. (Any answer without supporting some work will be treated as a guess and receive no credit.)

15b(14 pts). Evaluate $\int_C e^{xy} \, dx - e^{xy} \, dy$, where the path $C$ consists of the line segments from the point $(0, 1)$ to $(0, 2)$ and from $(0, 2)$ to $(1, 2)$.
1a. (Source: 12.5.6) The line is parallel to the vector \( \langle 0, -2, 3 \rangle \) \(-\langle 1, 2, 0 \rangle = \langle 1, -4, 3 \rangle \), so it’s parametrized by \( r(t) = \langle -1, 2, 0 \rangle + t\langle 1, -4, 3 \rangle \). This can also be expressed as 
\[
x = -1 + t \quad y = 2 - 4t \quad z = 3t.
\]
Solving for \( t \) gives the symmetric equations \( x + 1 = \frac{1}{4}(2 - y) = \frac{z}{3} \).

1b. (Source: 16.5.16) The line is parallel to the normal vector to the plane, \( \langle 2, -3, 5 \rangle \), so it’s parametrized by \( r(t) = \langle -1, 2, 0 \rangle + t\langle 2, -3, 5 \rangle \).

2. (Source: 12.3.41) \( \langle 1, 2, -1 \rangle \cdot \langle 2, -1, 2 \rangle = \frac{-2}{9} \langle 2, -1, 2 \rangle = \langle -\frac{4}{9}, \frac{2}{9}, -\frac{4}{9} \rangle \)

3. (Source: 14.4.5, 6) Solution 1: Let \( g(x, y) = 1 + y\ln(xy - 11) \). Then
\[
g_x(x, y) = y \cdot \frac{y}{xy - 11} \quad g_y(x, y) = \ln(xy - 11) + y \cdot \frac{x}{xy - 11}
\]
\[
g_x(3, 4) = 16 \quad g_y(3, 4) = \ln 1 + 12 = 12,
\]
and the tangent plane is the graph of the linearization
\[
z = g(3, 4) + g_x(3, 4)(x - 3) + g_y(3, 4)(y - 4), \text{ or } z = 1 + 16(x - 3) + 12(y - 4).
\]

3. Solution 2. Let \( f(x, y, z) = y\ln(xy - 11) - z \). Surface in question is \( f(x, y, z) = -1 \). For normal vector, use the the gradient \( \nabla f = \langle f_x, f_y, f_z \rangle = \langle y \cdot \frac{x}{xy - 11}, \ln(xy - 11) + y \cdot \frac{x}{xy - 11}, -1 \rangle \), which, at \( (3, 4, 1) \), equals \( \langle 16, 12, -1 \rangle \). The plane passing through \( (3, 4, 1) \) normal to \( \langle 16, 12, -1 \rangle \) has the equation \( 16(x - 3) + 12(y - 4) - (z - 1) = 0 \).


Although the level curves of both \( 2x^2 + y^2 \) and \( \sqrt{2x^2 + y^2} \) are ellipses, remember that \( z = 2x^2 + y^2 \) is a paraboloid and \( z = \sqrt{2x^2 + y^2} \) is a cone. As \( z \) changes more rapidly on the paraboloid, the level curves come closer together. The curves \( ye^x = k \) can be rewritten \( y = ke^{-x} \), and \( 2x - 3y = k \) are lines of (positive) slope \( \frac{2}{3} \).

5a. (Source: 13.4.11) Velocity is the vector \( \mathbf{v} = \frac{dr}{dt} = \langle e^t, -e^{-t}, 1 \rangle \). Speed is the magnitude of \( \mathbf{v} \), \( \frac{ds}{dt} = \sqrt{e^{2t} + e^{-2t} + 1} \), and acceleration = \( \mathbf{a} = \frac{d^2r}{dt^2} = \frac{ds}{dt} = \langle e^t, e^{-t}, 0 \rangle \).

5b. (Source: 13.2.17-20) \( \mathbf{T} = \frac{\mathbf{v}}{||\mathbf{v}||} \). At \( t = 0 \), this equals \( \frac{\langle 1, -1, 1 \rangle}{||\langle 1, -1, 1 \rangle||} = \frac{1}{\sqrt{3}}\langle 1, -1, 1 \rangle \).

5c. (Source: 13.3.19) At \( t = 0 \), \( \mathbf{a} = \langle 1, 1, 0 \rangle \), and \( \mathbf{v} \times \mathbf{a} = \begin{vmatrix} i & j & k \\ 1 & -1 & 1 \\ 1 & 1 & 0 \end{vmatrix} = \langle -1, 1, 2 \rangle \), and so
\[
\kappa = ||\mathbf{v} \times \mathbf{a}||/||\mathbf{v}||^3 = \frac{\sqrt{2}}{\sqrt{3}}, \text{ or } \frac{\sqrt{2}}{3}.
\]

5d. (Source: 13.4.41) \( a_T = \frac{\mathbf{a} \cdot \mathbf{v}}{||\mathbf{v}||} \), the scalar projection of \( \mathbf{a} \) onto \( \mathbf{v} \). Since \( \mathbf{a} \cdot \mathbf{v} = 0 \), this is zero. \( a_T \) is also \( \frac{ds}{dt} \), the rate of change of speed, so at \( t = 0 \), the object is neither speeding up nor slowing down.) \( a_N = \sqrt{||\mathbf{a}||^2 - a_T^2} = ||\mathbf{a}|| = \sqrt{2} \). You could also calculate \( a_N \) by the formula \( a_N = \kappa (\frac{ds}{dt})^2 \).

5e. (Source: 13.3.1-6) \( s = \int ds = \int_{-1}^{1} \frac{ds}{dt} dt = \int_{-1}^{1} \sqrt{e^{2t} + e^{-2t} + 1} dt \)
6a. (Source: 16.6.26) We could use polar coordinates to write \( \mathbf{r} = \langle r \cos \theta, r \sin \theta, 1 + r^2 \sin^2 \theta \rangle \), but this and 6b below are both easier if we choose \( x \) and \( y \) for the parameters, so that \( \mathbf{r}(x, y) = \langle x, y, 1 + y^2 \rangle \).

6b. (Source: 15.5.9, 16.6.45) Since \( z \) is a function of \( x \) and \( y \) along this surface, \( dS = \sqrt{1 + z_x^2 + z_y^2} \, dx \, dy \), and so the area equals \( \int_2^3 \int_{\sqrt{4-y^2}}^{\sqrt{4-y^2}} \sqrt{1 + 4y^2} \, dx \, dy \).

7. (Source: 14.6.21-26) \( T \) increases most rapidly (in \( ^\circ \text{C} \) per cm) when the bug moves in the direction of the gradient \( \nabla T(2,3) = \langle -4, 1 \rangle \). Normalize this to obtain the unit vector \( \frac{\nabla T}{|\nabla T|} = \frac{\langle -4, 1 \rangle}{\sqrt{17}} \).

8. (Source: 14.8.7) Use Lagrange multipliers. The max and min of \( f = 2x + 2y - z \) can only occur at critical points on \( 4x^2 + y^2 + z^2 = 3 \), i.e., those points at which \( \nabla(4x^2 + y^2 + z^2) = \lambda \nabla(2x + 2y - z) \):

\[
8x = 2\lambda \\
2y = 2\lambda \\
2z = -\lambda \\
x = \frac{1}{4}\lambda \\
y = \lambda \\
z = -\frac{1}{2}\lambda
\]

Substitute these for \( x, y, z \) into the equation for the ellipsoid and solve for \( \lambda \):

\[
3 = 4x^2 + y^2 + z^2 = \frac{1}{4}\lambda^2 + \lambda^2 + \frac{1}{4}\lambda^2 = \frac{6}{4}\lambda^2
\]

which implies \( \lambda = \pm \sqrt{2} \). Therefore, the critical points are \( (x, y, z) = \pm (\sqrt{2}/2, \sqrt{2}/2, -\sqrt{2}/2) \).

Since \( f(\sqrt{2}/2, \sqrt{2}/2, -\sqrt{2}/2) = 3\sqrt{2} \) and \( f(-\sqrt{2}/2, -\sqrt{2}/2, \sqrt{2}/2) = -3\sqrt{2} \), the maximum is \( 3\sqrt{2} \) and the minimum is \( -3\sqrt{2} \).

9. (Source: 14.8.31-42) The maximum and minimum values of \( f \) on \( D \) must occur either on the boundary or at an interior critical point. The only critical point inside \( D \) is \( (0,2-\sqrt{2}) \), where \( f = -3 \). Therefore \( f \)'s maximum on \( D \) is 5, and its minimum is \( -3 \).

10. (Source: 15.2.24) \( V = \int_{-1}^{1} \int_{y_0}^{1} \int_{y_0}^{1+xy^2} dz \, dx \, dy = \int_{-1}^{1} \int_{y_0}^{1} (1 + xy^2) \, dx \, dy \)

\[
= \int_{-1}^{1} (x + \frac{1}{2}x^2y^2) \bigg|_{x=y^2} dx \, dy = \int_{y_0}^{1} (1 + \frac{1}{2}y^2 - y^2 - \frac{1}{2}y^6) \, dy = \int_{y_0}^{1} (1 - \frac{1}{6}y^2 - \frac{1}{2}y^6) \, dy
\]

\[
= (y - \frac{1}{6}y^3 - \frac{1}{14}y^7) \bigg|_{y_0}^{1} = 2(1 - \frac{1}{6} - \frac{1}{14}) = \frac{32}{21}.
\]

11. (Source: 15.8.23) In spherical coordinates, \( z = \rho \cos \phi \) and \( dV = \rho^2 \sin \phi \, d\rho \, d\phi \, d\theta \), and the integral is \( \int_{\phi_0}^{2\pi} \int_{\pi/2}^{\pi/2} \int_{1}^{\sqrt{2}} \rho^3 \cos \phi \sin \phi \, d\rho \, d\phi \, d\theta = \)

\[
\left( \int_{0}^{2\pi} d\theta \right) \left( \int_{1}^{\sqrt{2}} \rho^3 \, d\rho \right) \left( \int_{\pi/2}^{\pi/2} \cos \phi \sin \phi \, d\phi \right) = 2\pi \left( \frac{1}{4} \rho^4 \bigg|_{1}^{\sqrt{2}} \right) \left( \frac{1}{2} \sin^2 \phi \bigg|_{0}^{\pi/2} \right)
\]

\[
= 2\pi \cdot \frac{1}{4} \cdot (4 - 1) \cdot \frac{1}{2} (1 - 0) = \frac{3\pi}{4}
\]

12. (Source: 16.9.7) The boundary of \( B \) consists of 5 difference surfaces, so it’s best to use the Divergence Theorem. \( \text{div} \mathbf{H} = \langle \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \rangle \cdot \langle xy, z, y \rangle = y \). The flux out of \( B \) is \( \iiint E \text{div} \mathbf{H} \, dV = \iiint_E y \, dy \, dx \, dz \), where \( E \) is the quarter circle in the \( xz \) plane given by
\[ x^2 + z^2 \leq 1, \quad x \geq 0, \quad z \geq 0. \] First calculate \( \int_0^2 y \, dy = 2 \), and then \( \iiint_E 2 \, dx \, dz = 2 \) times the area of \( E \). Therefore the flux is \( 2 \cdot \frac{\pi}{4} = \frac{\pi}{2} \).

13. (Source: 16.2.3) Parametrize the right half of the circle as \( x = 2 \cos \theta \) and \( y = 2 \sin \theta \), where \(-\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2}\). (The order we travel \( C \) doesn’t matter since neither \( x \) nor \( ds \) depends on the direction.) Then differential arc length is

\[
 ds = \frac{ds}{d\theta} \, d\theta = \left| \frac{dr}{d\theta} \right| \, d\theta = \sqrt{\left( \frac{dx}{d\theta} \right)^2 + \left( \frac{dy}{d\theta} \right)^2} \, d\theta = \sqrt{4 \sin^2 \theta + 4 \cos^2 \theta} \, d\theta = 2 \, d\theta
\]

and the integral is \( \int_{-\pi/2}^{\pi/2} 2 \cos \theta \cdot 2 \, d\theta = 4 \sin \theta \bigg|_{-\pi/2}^{\pi/2} = 4(1 - (-1)) = 8 \).

14. (Source: 16.4.6, 15.2.15) Since this is a closed path, we can use Green’s Theorem to evaluate the line integral.

\[
 \int_C P \, dx + Q \, dy = \int_1^3 \int_1^2 (Q_x - P_y) \, dy \, dx = \int_1^3 \int_1^2 (2x - 2y) \, dy \, dx \\
= \int_1^3 (2xy - y^2) \bigg|_1^2 \, dx = \int_1^3 (4x - 4 - 2x + 1) \, dx = \int_1^3 (2x - 3) \, dx = (x^2 - 3x) \bigg|_1^3 = 2
\]

15a. (Source: 16.3.7) \( \text{curl} \mathbf{F} = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ e^{xy} & -e^{xy} & 0 \end{vmatrix} = (-y e^{xy} - x e^{xy}) \mathbf{k} \neq \mathbf{0} \). Consequently, \( \mathbf{F} \) is not conservative.

15b. (Source: 16.2.7) Since \( \mathbf{F} \) is not conservative, the path integral cannot be counted on to be path independent and we must calculate the integral along \( C \) directly. On the first segment of \( C \), we can let \( x = 0 \) (so \( dx = 0 \)) and \( y \) will serve as the path parameter. The integral along this line segment is

\[
 \int_1^2 (-e^0) \, dy = -1.
\]

On the second line segment, we can let \( y = 2 \) (so \( dy = 0 \)) and \( x \) will serve as the path parameter. The integral along this line segment is

\[
 \int_0^1 e^{2x} \, dx = \frac{1}{2} \left. e^{2x} \right|_0^1 = \frac{1}{2} (e^2 - 1).
\]

Altogether, the line integral along \( C \) equals the sum

\[-1 + \frac{1}{2} (e^2 - 1) = \frac{1}{2} (e^2 - 3).\]