No notes, books, electronic devices, or outside materials of any kind.

Read each problem carefully and simplify your answers.

Unless otherwise indicated, supporting work will be required on every problem worth more than 2 points.

1(9 pts). Let $\mathbf{G} = xy^2z\mathbf{i} + yz^3\mathbf{k}$. Find the following or state that it doesn't exist.

a. $\operatorname{div}\mathbf{G}$ b. $\operatorname{grad}\mathbf{G}$ c. $\operatorname{curl}\mathbf{G}$

2(10 pts). Let C be the triangular path consisting of line segments connecting the point $(0,0)$ to $(2,0)$ to $(2,1)$ to $(0,0)$. Use Green's Theorem to evaluate the integral

$$\int_C (2y\sin x\cos x)\,dx + (xy + \sin^2 x)\,dy$$

3(13 pts). Let $\mathbf{F} = \langle e^y, xe^y + e^z, ye^z\rangle$.

a. Find a potential function f for $\mathbf{F}$ or show that none exists.

b. Evaluate $\int_C \mathbf{F}\cdot d\mathbf{r}$, where C is the helix parametrized by $\mathbf{r} = \langle \cos t, t, \sin t\rangle$ for $0 \le t \le 2\pi$.

4. Let E be the surface parametrized by $\mathbf{r}(u,v) = \langle u, v^2, -uv\rangle$ with $-1 \le u \le 1$, $0 \le v \le 4$.

a(10 pts). Find an equation of the plane tangent to E at the point (x,y,z) corresponding to $(u,v) = (0,2)$.

b(6 pts). Express the area of E as a definite iterated integral, but **do not evaluate**.

5(9 pts). Let $\mathbf{G} = \langle x + y^2, y + z^2, x^2\rangle$, and let E be the surface of the box $0 \le x \le 1$, $0 \le y \le 2$, $-1 \le z \le 1$, oriented outward. Use the Divergence Theorem to evaluate the surface integral $\iint_E \mathbf{G}\cdot\mathbf{n}\,dS$.

6(20 pts). Find the flux of $\mathbf{H} = \langle x, y, -z\rangle$ across the cone $z = \sqrt{x^2 + y^2}$ between $z = 0$ and $z = 1$, oriented upward.

7(23 pts). Let $\mathbf{G} = \langle x + y^2, y + z^2, x^2\rangle$, and let C be the triangular path consisting of line segments connecting the point $(2,0,0)$ to $(0,2,0)$ to $(0,0,2)$ to $(2,0,0)$. Use Stokes's Theorem to evaluate the line integral $\int_C \mathbf{G}\cdot d\mathbf{r}$.

1a.(Source: 16.5.2, 16.5.12) $\operatorname{div}\mathbf{G} = \nabla\cdot\mathbf{G} = \langle \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\rangle \cdot \langle xy^2z, 0, yz^3\rangle = y^2z + 3yz^2$.

1b. $\operatorname{grad}\mathbf{G} = \nabla\mathbf{G}$ does not exist. ∇ applies to scalar-valued functions, not vector fields.

1c. $\operatorname{curl}\mathbf{G} = \nabla\times\mathbf{G} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ xy^2z & 0 & yz^3 \end{vmatrix} = \langle z^3, xy^2, -2xyz\rangle$.

2.(Source: 16.4.11) Let E denote the interior of the triangle. Since the triangular path is traversed in the positive direction, Green's Theorem says

$$\begin{aligned}\int_C (2y\sin x\cos x)\,dx + (xy+\sin^2 x)\,dy &= \iint_E \big((xy+\sin^2 x)_x - (2y\sin x\cos x)_y\big)\,dV \\ &= \iint_E (y + 2\sin x\cos x - 2\sin x\cos x)\,dV = \iint_E y\,dV \\ &= \int_0^2\int_0^{\frac{1}{2}x} y\,dy\,dx = \int_0^2 \frac{1}{2}\cdot\frac{x^2}{4}\,dx = \frac{1}{24}x^3\Big|_0^2 = \frac{1}{3}\end{aligned}$$

3a.(Source: 16.3.17) Assume that $\mathbf{F} = \nabla f$. Integrate $f_y = xe^y + e^z$ with respect to y to obtain

$$f = xe^y + ye^z + C(x,z).$$

Then, when we differentiate with respect to either x or z, we find

$$f_x = e^y + C_x = e^y \text{ and } f_z = ye^z + C_z = ye^z,$$

so $C_x = C_z = 0$, and therefore C is a constant. That is, general potential function is

$$f = xe^y + ye^z + C.$$

Since the instructions only said to find *a* potential, you can take C to be 0 (or any other constant).

3b. The endpoints of C are $(1,0,0)$ when $t=0$ and $(1,2\pi,0)$ when $t=2\pi$. By the Fundamental Theorem of Calculus for line integrals,

$$\int_C \mathbf{F}\cdot d\mathbf{r} = \int_C \nabla f\cdot d\mathbf{r} = f(x,y,z)\Big|_{(1,0,0)}^{(1,2\pi,0)} = f(1,2\pi,0) - f(1,0,0) = e^{2\pi} + 2\pi - 1.$$

4a.(Source: 16.6.35) To find the equation of a plane, we need a point and a normal vector. Since $\mathbf{r}(0,2) = \langle 0,4,0\rangle$, the point of tangency is $(0,4,0)$. For the normal, use $\mathbf{r}_u\times\mathbf{r}_v = \langle 1,0,-v\rangle\times\langle 0,2v,-u\rangle = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 0 & -v \\ 0 & 2v & -u \end{vmatrix} = \langle 2v^2, u, 2v\rangle$, which, at $(u,v) = (0,2)$, equals $\langle 8,0,4\rangle$. The plane is $8(x-0) + 0(y-4) + 4(z-0) = 0$, or $2x+z=0$.

4b.(Source: 16.6.49) The surface area S equals

$$\iint_E dS = \int_E |\mathbf{r}_u\times\mathbf{r}_v|\,du\,dv = \int_0^4\int_{-1}^1 \sqrt{4v^4+u^2+4v^2}\,du\,dv$$

5.(Source: 16.9.1) Letting B denote the interior of the box, the Divergence Theorem says that the flux of $\mathbf{G}$ out of B, $\iint_E \mathbf{G}\cdot\mathbf{n}\,dS = \iiint_B \operatorname{div}\mathbf{G}\,dV$. The integrand is $\nabla\cdot\mathbf{G} = \langle \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\rangle \cdot \langle x+y^2, y+z^2, x^2\rangle = 2$, so the triple integral is 2 times the volume of the box, or 8.

6.(Source: 16.7.24) Since the surface in question E is not closed, so the Divergence Theorem does not apply. We'll have to calculate the surface integral $\iint_E \mathbf{H}\cdot\mathbf{n}\,dS$ directly. Parametrize E with $\mathbf{r}(r,\theta) = \langle r\cos\theta, r\sin\theta, r\rangle$ for $0 \le r \le 1$ and $0 \le \theta \le 2\pi$. Then

$$\mathbf{n}\,dS = \pm(\mathbf{r}_r \times \mathbf{r}_\theta)\,dr\,d\theta = \pm\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \cos\theta & \sin\theta & 1 \\ -r\sin\theta & r\cos\theta & 0 \end{vmatrix} dr\,d\theta = \pm\langle -r\cos\theta, -r\sin\theta, r\rangle\,dr\,d\theta$$

We want the $\mathbf{k}$ component to be positive for upward orientation, so use the + above. Then

$$\iint_E \langle x, y, -z\rangle\cdot\mathbf{n}\,dS = \int_0^{2\pi}\int_0^1 \langle r\cos\theta, r\sin\theta, -r\rangle\cdot\langle -r\cos\theta, -r\sin\theta, r\rangle\,dr\,d\theta$$
$$= \int_0^{2\pi}\int_0^1 (-r^2\cos^2\theta - r^2\sin^2\theta - r^2)\,dr\,d\theta = \int_0^{2\pi}\int_0^1 -2r^2\,dr\,d\theta = -\frac{4\pi}{3}$$

7.(Source: 16.8.7) Letting E denote the interior of the triangle, Stokes's Theorem says that the circulation of $\mathbf{G}$ around C is the flux of $\operatorname{curl}\mathbf{G}$ across E:

$$\int_C \mathbf{G}\cdot\underset{\cdot}{\mathbf{r}} = \iint_E \nabla\times\mathbf{G}\cdot\mathbf{n}\,dS$$

The surface is contained on the plane $x+y+z=2$, so we can parametrize E with $\mathbf{r}(x,y) = \langle x, y, 2-x-y\rangle$. Then $\mathbf{n}\,dS = \pm(\mathbf{r}_x\times\mathbf{r}_y)\,dr\,d\theta = \pm\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 0 & -1 \\ 0 & 1 & -1 \end{vmatrix} dx\,dy = \pm\langle 1,1,1\rangle\,dx\,dy$.
We use the + so that C is traversed in the positive direction from the point of view of $\mathbf{n}$.
Next, calculate the curl: $\nabla\times\mathbf{G} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x+y^2 & y+z^2 & x^2 \end{vmatrix} = \langle -2z, -2x, -2y\rangle$.
Then

$$\nabla\times\mathbf{G}\cdot\mathbf{n}\,dS = \langle -2(2-x-y), -2x, -2y\rangle\cdot\langle 1,1,1\rangle\,dx\,dy = -4\,dx\,dy$$

and the surface integral is -4 times the area of the region of integration in the xy-plane. That's the triangle whose vertices are $(0,0)$, $(2,0)$, $(0,2)$, so the integral equals -8.