MATH 221–01 (Kunkle), Exam 3 Name: ______________________

100 pts, 75 minutes Nov 1, 2017

No notes, books, electronic devices, or outside materials of any kind.

Read each problem carefully and simplify your answers.

Unless otherwise indicated, supporting work will be required on every problem worth more than 2 points.

1(6 pts). Rewrite the iterated integral $\int_1^e \int_{\ln x}^1 f(x,y)\,dy\,dx$ by changing the order of integration, but **do not evaulate**.

2(11 pts). Find the volume of the solid enclosed by the cylinder $y = x^2$, and the planes $z = 0$ and $y + 2z = 3$. Express your answer as an iterated triple integral, but **do not evaulate**.

3(14 pts). Find the area of the surface $z = xy$ that lies inside the cyclinder $x^2 + y^2 = 4$.

4(22 pts). Evaluate $\iiint_E y^2 z\,dV$, where E is the solid that lies above the cone $\phi = \frac{\pi}{4}$ and inside the sphere $\rho = 1$.

5(14 pts). Let D be the parallelogram bounded by the lines

$$x + 2y = 1 \qquad x + 2y = 2 \qquad x - y = -1 \qquad x - y = 0$$

Write the integral $\iint_D \frac{x-y}{x+2y}\,dA$ as an iterated integral in the variables $u = x - y$ and $v = x + 2y$, but **do not evaulate**.

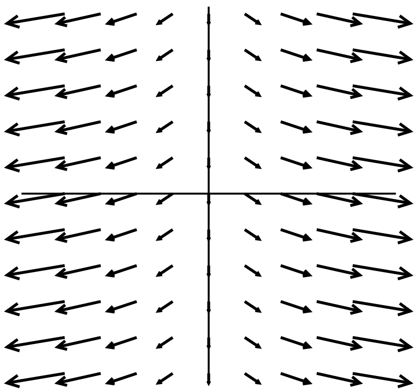
6(17 pts). Evaluate the line integral $\int_C y\,dx + (x+z)\,dy + x\,dz$, where C consists of the line segments from $(0,0,0)$ to $(0,1,1)$ and from there to $(1,0,1)$.

7(4 pts). Find cylindrical coordinates (r, θ, z) for the point whose rectangular coordinates are $(-2\sqrt{3}, 2, 3)$.

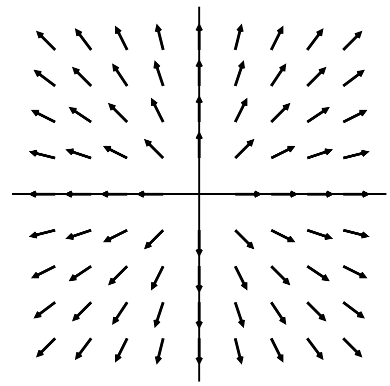
8(4 pts). Rewrite the equation $z = x^2 - y^2$ in cylindrical coordinates.

9(8 pts). Find the gradient vector field of each function from among the given graphs.

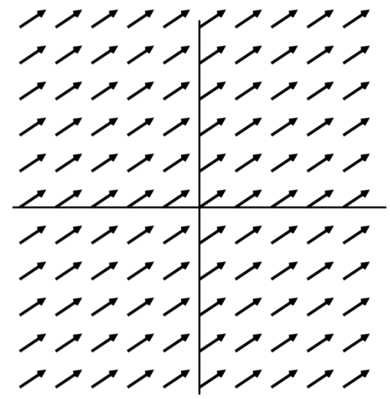
a. $f(x,y) = x^2 - y^2$ b. $g(x,y) = x^2 - y$ c. $h(x,y) = \frac{1}{2}x + \frac{1}{3}y$ d. $k(x,y) = \sqrt{x^2 + y^2}$



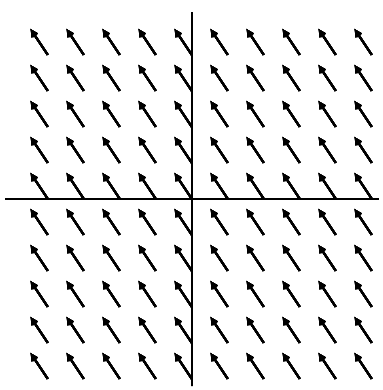
i



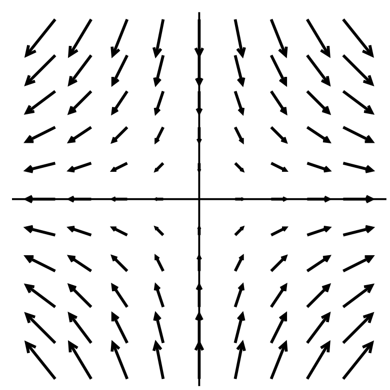
ii



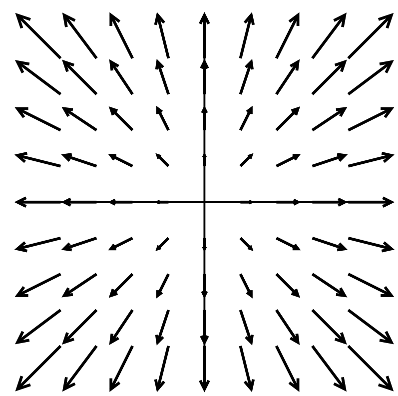
iii



iv



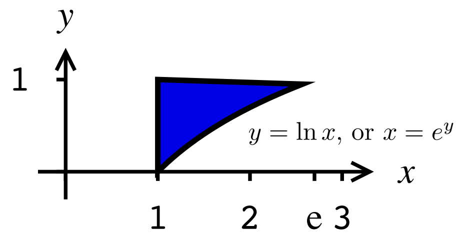
v



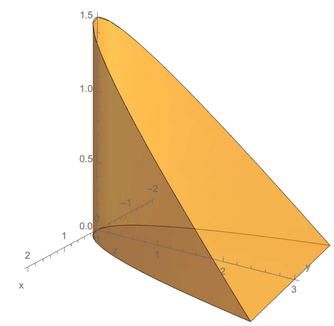
vi

1.(Source: 15.2.49) The figure shows the region of integration in the xy-plane.

$$\int_0^1 \int_1^{e^y} f(x,y)\,dx\,dy$$



2.(Source: 15.6.21,31) Here are all the ways to write its volume an iterated triple integral in xyz (any one of which received full credit).



$$\int_0^3 \int_0^{\frac{1}{2}(3-y)} \int_{-\sqrt{y}}^{\sqrt{y}} dx\,dz\,dy \qquad \int_0^{3/2} \int_0^{3-2z} \int_{-\sqrt{y}}^{\sqrt{y}} dx\,dy\,dz$$

$$\int_0^3 \int_{-\sqrt{y}}^{\sqrt{y}} \int_0^{\frac{1}{2}(3-y)} dz\,dx\,dy \qquad \int_{-\sqrt{3}}^{\sqrt{3}} \int_{x^2}^{3} \int_0^{\frac{1}{2}(3-y)} dz\,dy\,dx$$

$$\int_0^{3/2} \int_{-\sqrt{3-2z}}^{\sqrt{3-2z}} \int_{x^2}^{3-2z} dy\,dx\,dz \qquad \int_{-\sqrt{3}}^{\sqrt{3}} \int_0^{\frac{1}{2}(3-x^2)} \int_{x^2}^{3-2z} dy\,dz\,dx$$

3.(Source: 15.5.9) The area $S = \iint dS = \iint \sqrt{1+z_x^2+z_y^2}\,dA$ over the circle $x^2+y^2 \le 4$ in the xy-plane. This works very nicely in polar coordinates:

$$\begin{aligned}\iint \sqrt{1+y^2+x^2}\,dA &= \int_0^{2\pi}\int_0^2 r\sqrt{1+r^2}\,dr\,d\theta \\ &= \left(\int_0^{2\pi} d\theta\right)\left(\frac{1}{3}(1+r^2)^{3/2}\Big|_0^2\right) = \frac{2\pi}{3}(5^{3/2}-1)\end{aligned}$$

4.(Source: 15.8.22) In cylindrical coordinates, $z = \rho\cos\phi$ and $y = \rho\sin\phi\sin\theta$:

$$\begin{aligned}&\int_0^{2\pi}\int_0^{\pi/4}\int_0^1 \rho^5\sin^3\phi\cos\phi\sin^2\theta\,d\rho\,d\phi\,d\theta \\ &= \int_0^{2\pi}\sin^2\theta\,d\theta\cdot\int_0^{\pi/4}\sin^3\phi\cos\phi\,d\phi\cdot\int_0^1\rho^5\,d\rho \\ &= \int_0^{2\pi}\frac{1}{2}\big(1-\cos(2\theta)\big)\,d\theta\cdot\int_0^{\pi/4}\sin^3\phi\cos\phi\,d\phi\cdot\int_0^1\rho^5\,d\rho \\ &= \left(\frac{1}{2}\big(\theta-\frac{1}{2}\sin(2\theta)\big)\Big|_0^{2\pi}\right)\left(\frac{1}{4}\sin^4\phi\Big|_0^{\pi/4}\right)\left(\frac{1}{6}\rho^6\Big|_0^1\right) = \pi\cdot\frac{1}{16}\cdot\frac{1}{6} = \frac{\pi}{96}\end{aligned}$$

5.(Source: 15.9.23) Subtract $u = x-y$ from $v = x+2y$ to obtain $v-u = 3y$, so $y = \frac{1}{3}v - \frac{1}{3}u$ and $x = y+u = \frac{1}{3}v + \frac{2}{3}u$. Calculate the Jacobian:

$$\frac{\partial(x,y)}{\partial(u,v)} = \left\| \begin{matrix} x_u & x_v \\ y_u & y_v \end{matrix} \right\| = \left\| \begin{matrix} 2/3 & 1/3 \\ -1/3 & 1/3 \end{matrix} \right\| = |2/9 - (-1/9)| = 1/3$$

The integral is

$$\iint_D \frac{x-y}{x+2y}\,dx\,dy = \int_1^2 \int_{-1}^0 \frac{u}{v}\frac{\partial(x,y)}{\partial(u,v)}\,du\,dv = \frac{1}{3}\int_1^2\int_{-1}^0 \frac{u}{v}\,du\,dv$$

Note: since $\frac{\partial(x,y)}{\partial(u,v)}$ is a constant, it could also be calculated as the reciprocal of $\frac{\partial(u,v)}{\partial(x,y)}$, which avoids the work of solving for x and y in terms of u and v.

6.(Source: 16.2.15) Along the first line segment, take $x = 0$ and $y = z = t$, so $dx = 0$ and $dy = dz = dt$, and the integral becomes $\int_0^1 \big(t \cdot 0 + (0+t)\,dt + 0\,dt\big) = \int_0^1 t\,dt = \frac{1}{2}$.
Along the second segment, $x = t$, $y = 1-t$, and $z = 1$, so $dx = dt$, $dy = -dt$, and $dz = 0$, and the integral becomes $\int_0^1 \big((1-t)\,dt + (t+1)\,(-dt) + t\cdot 0\big) = \int_0^1 -2t\,dt = -1$.
The total line integral is the sum of the two parts: $\frac{1}{2} - 1 = -\frac{1}{2}$.

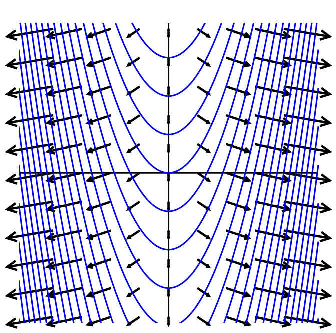
7.(Source: 15.7.3b) Can use $r = \sqrt{x^2+y^2} = \sqrt{4\cdot 3 + 4} = \sqrt{16} = 4$. The vector $\langle -2\sqrt{3}, 2\rangle$ points into Quadrant II an makes an angle $\pi/6$ with the negative x-axis, so its angle with the positive x-axis is $\theta = 5\pi/6$. Therefore $(r,\theta,z) = (4, 5\pi/6, 3)$.

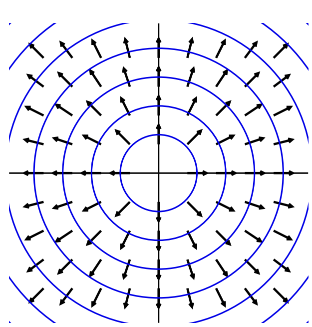
8.(Source: 15.7.9b) $z = (r\cos\theta)^2 - (r\sin\theta)^2$, or $z = r^2(\cos^\theta - \sin^2\theta)$ (which can be simplified to $r^2\cos(2\theta)$ but was not required)

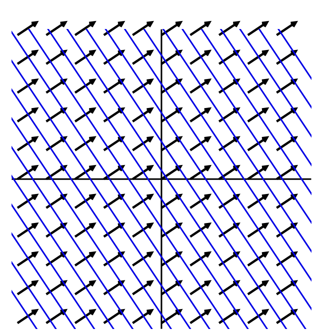
9.(Source: 16.1.29-32)

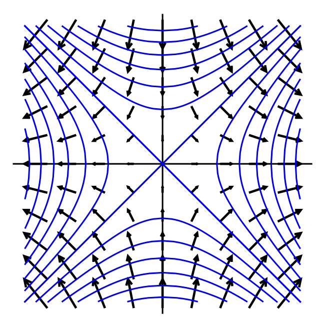
part	graph	gradient
a.	v	$\langle 2x, -2y\rangle$
b.	i	$\langle 2x, -1\rangle$
c.	iii	$\langle \frac{1}{2}, \frac{1}{3}\rangle$
d.	ii	$\left\langle \frac{x}{\sqrt{x^2+y^2}}, \frac{y}{\sqrt{x^2+y^2}} \right\rangle = \langle x,y\rangle / \lvert\langle x,y\rangle\rvert$

It might be useful to picture the level curves of each of the four functions, since the gradient must always be perpendicular to these.


i
$x^2 - y = C$


ii
$\sqrt{x^2+y^2} = C$


iii
$\frac{1}{2}x + \frac{1}{3}y = C$


v
$x^2 - y^2 = C$