MATH 221–01 (Kunkle), Exam 1 | Name: ______________________
100 pts, 75 minutes | Sept 21, 2017 |

No notes, books, electronic devices, or outside materials of any kind.

Read each problem carefully and simplify your answers.

Unless otherwise indicated, supporting work will be required on every problem worth more than 2 points.

1a(4 pts). Find the vector of length 4 in the same direction as $\langle 3, -1, 1\rangle$.

1b(5 pts). Find the vector projection of $\langle 2, 1, 0\rangle$ onto $\langle 3, -1, 1\rangle$.

1c(3 pts). Find the cosine of the angle between $\langle 2, 1, 0\rangle$ and $\langle 3, -1, 1\rangle$.

2. Find an equation for the object described in each part. (You can give the equation(s) for a line in vector, parametric, or symmetric form.)

a(4 pts). The sphere centered at $(1, -1, 2)$ with radius 5.
b(4 pts). The line passing through the points $(1, -1, 2)$, and $(0, 1, 5)$.
c(5 pts). The line passing through $(1, -1, 2)$ parallel to the line $\frac{1}{3}x = y - 1 = \frac{1}{4}(z + 2)$.
d(7 pts). The plane passing through the points $(1, -1, 2)$, $(2, 3, -4)$, and $(0, -1, 5)$.
e(5 pts). The plane passing through $(1, -1, 2)$ parallel to the plane $2x + 3z = 4$.

3. The position of a particle at time t is given by $x = \cos 4t$, $y = 3t$, $z = \sin 4t$.

a(5 pts). Find the speed of the particle at time t.

b(8 pts). Express $\mathbf{T}$ and $\mathbf{N}$ as functions of t along this curve.

c(12 pts). Find the equations of the normal and osculating planes for the path of this particle at time $t = 0$. Label your answers so I can tell which is which.

4. A particle moving through space has the position function $\mathbf{r}(t) = 2t\mathbf{i} + e^t\mathbf{j} + (t^2 - t + 3)\mathbf{k}$.

a(4 pts). Express the object's velocity $\mathbf{v}$ and acceleration $\mathbf{a}$ as functions of t.
b(5 pts). Find a vector tangent to the graph of $\mathbf{r}(t)$ at $t = 0$, and an equation of the line tangent to the curve at the point corresponding to that time.
c(10 pts). Find a_N and a_T, the normal and tangential components of acceleration, at time $t = 0$.
d(2 pts). Find the curvature κ of the graph of $\mathbf{r}$ at $t = 0$.
e(3 pts). Find $\displaystyle\int_0^2 \frac{d\mathbf{r}}{dt}\,dt$.

5a(8 pts). Make a rough sketch of the graph of $-x^2 + y^2 - z^2 = 1$. Find all intercepts and include them in your sketch. What is the name of this kind of surface?

5b(6 pts). How does the graph of $-x^2 - 4x + y^2 - z^2 = 5$ compare with the graph of the equation in 5a? A sketch is not required, but you can make one if you wish.

1a.(Source: 12.2.26) $4 \cdot \frac{1}{|\langle 3,-1,1\rangle|}\langle 3,-1,1\rangle = \frac{4}{\sqrt{11}}\langle 3,-1,1\rangle$.

1b.(Source: 12.3.41) $\dfrac{\langle 2,1,0\rangle \cdot \langle 3,-1,1\rangle}{\langle 3,-1,1\rangle \cdot \langle 3,-1,1\rangle}\langle 3,-1,1\rangle = \dfrac{5}{11}\langle 3,-1,1\rangle$, or $\langle \frac{15}{11}, -\frac{5}{11}, \frac{5}{11}\rangle$.

1c.(Source: 12.3.17) $\cos\theta = \dfrac{\langle 2,1,0\rangle \cdot \langle 3,-1,1\rangle}{|\langle 2,1,0\rangle||\langle 3,-1,1\rangle|} = \dfrac{5}{\sqrt{6}\sqrt{11}}$, or $\frac{5}{\sqrt{66}}$.

2a.(Source: 12.1.13-14) $(x-1)^2 + (y+1)^2 + (z-2)^2 = 25$.

2b.(Source: 12.5.7) The desired line is parallel the vector $\langle 1,-1,2\rangle - \langle 0,1,5\rangle = \langle 1,-2,-3\rangle$ and can be written in the parametric form

$$x = 1+t \qquad y = -1-2t \qquad z = 2-3t.$$

2c.(Source: 12.5.11) The given line is parallel the vector $\langle 3,1,4\rangle$, so the line in question can be written in the symmetric form $\frac{x-1}{3} = y+1 = \frac{z-2}{4}$.

2d.(Source: 12.5.31) Subtract points to find two vectors parallel the plane in question:

$$\langle 1,-1,2\rangle - \langle 2,3,-4\rangle = \langle -1,-4,6\rangle,$$
$$\langle 1,-1,2\rangle - \langle 0,-1,5\rangle = \langle 1,0,-3\rangle.$$

Cross these to find a vector normal to the plane:

$$\mathbf{n} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -1 & -4 & 6 \\ 1 & 0 & -3 \end{vmatrix} = \begin{vmatrix} -4 & 6 \\ 0 & -3 \end{vmatrix}\mathbf{i} - \begin{vmatrix} -1 & 6 \\ 1 & -3 \end{vmatrix}\mathbf{j} + \begin{vmatrix} -1 & -4 \\ 1 & 0 \end{vmatrix}\mathbf{k} = \langle 12,3,4\rangle.$$

The desired plane is $12(x-1) + 3(y+1) + 4(z-2) = 0$.

2e.(Source: 12.5.27) $2x+3z=4$ has the normal vector $\langle 2,0,3\rangle$, so our plane is $2(x-1) + 3(z-2) = 0$, or $2x+3z=8$.

3a.(Source: 13.4.10) $\mathbf{r} = \langle \cos 4t, 3t, \sin 4t\rangle$, $\mathbf{v} = \frac{d\mathbf{r}}{dt} = \langle -4\sin 4t, 3, 4\cos 4t\rangle$, and speed $= \frac{ds}{dt} = |\mathbf{v}| = \sqrt{16\sin^2 4t + 9 + 16\cos^2 4t} = \sqrt{16(\sin^2 4t + \cos^2 4t) + 9} = \sqrt{16+9} = 5$. (That is, the particle moves at constant speed.)

3b.(Source: 13.3.17) $\mathbf{T} = \frac{1}{|\mathbf{v}|}\mathbf{v} = \frac{1}{5}\langle -4\sin 4t, 3, 4\cos 4t\rangle$, or $\langle \frac{-4}{5}\sin 4t, \frac{3}{5}, \frac{4}{5}\cos 4t\rangle$.

$\mathbf{N} = \frac{d\mathbf{T}}{dt} \div \left|\frac{d\mathbf{T}}{dt}\right| = \frac{16}{5}\langle -\cos 4t, 0, -\sin 4t\rangle \div \frac{16}{5} = \langle -\cos 4t, 0, -\sin 4t\rangle$.

3c.(Source: 13.3.49) Both planes pass through the point $\mathbf{r}(0) = \langle 1,0,0\rangle$. The normal plane is perpendicular to the tangent vector $\mathbf{T}(0) = \langle 0, \frac{3}{5}, \frac{4}{5}\rangle$, so its equation is $\frac{3}{5}y + \frac{4}{5}z = 0$. The osculating plane is parallel both $\mathbf{T}(0)$ and $\mathbf{N}(0) = \langle -1,0,0\rangle$, so its normal vector is

$$\langle 0, \frac{3}{5}, \frac{4}{5}\rangle \times \langle -1,0,0\rangle = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & \frac{3}{5} & \frac{4}{5} \\ -1 & 0 & 0 \end{vmatrix} = \langle 0, \frac{4}{5}, \frac{3}{5}\rangle$$

and its equation is $\frac{4}{5}y + \frac{3}{5}z = 0$.

(The osculating plane also contains $\mathbf{v}$ and $\mathbf{a}$, so we could cross these two vectors to find a normal vector.)

4a.(Source: 13.4.9,11) $\mathbf{r}(t) = 2t\mathbf{i} + e^t\mathbf{j} + (t^2 - t + 3)\mathbf{k}$, so $\mathbf{v} = \frac{d\mathbf{r}}{dt} = 2\mathbf{i} + e^t\mathbf{j} + (2t-1)\mathbf{k}$, and $\mathbf{a} = \frac{d\mathbf{v}}{dt} = e^t\mathbf{j} + 2\mathbf{k}$.

4b.(Source: 13.2.17,18, 23) $\mathbf{v}(0) = 2\mathbf{i} + \mathbf{j} - \mathbf{k} = \langle 2, 1, -1\rangle$ is a tangent vector. At time 0, the curve passes through the point $\mathbf{r}(0) = \mathbf{j} + 3\mathbf{k} = \langle 0, 1, 3\rangle$, so the tangent line is given parametrically by

$$x = 0 + 2t \qquad y = 1 + t \qquad z = 3 - t.$$

4c.(Source: 13.4.38) Calculate $\mathbf{a}(0) = \mathbf{j} + 2\mathbf{k} = \langle 0, 1, 2\rangle$ and $\mathbf{v}(0) \times \mathbf{a}(0) = \langle 3, -4, 2\rangle$. Then

$$a_T = \frac{\mathbf{v}\cdot\mathbf{a}}{|\mathbf{v}|} = \frac{-1}{\sqrt{6}}, \text{ and}$$
$$a_N = \frac{|\mathbf{v}\times\mathbf{a}|}{|\mathbf{v}|} = \sqrt{\frac{29}{6}}.$$

Note that $a_T = \frac{d^2s}{dt^2} < 0$, so the particle is slowing down at time 0. We could also have calculated $a_N = \sqrt{|\mathbf{a}|^2 - a_T^2}$.

4d.(Source: 13.2.22) $\kappa = \frac{|\mathbf{v}\times\mathbf{a}|}{|\mathbf{v}|^3} = \frac{\sqrt{29}}{6\sqrt{6}}$.

4e.(Source: 13.2.35) By the Fundamental Theorem of Calculus,
$\int_0^2 \frac{d\mathbf{r}}{dt}\,dt = \mathbf{r}(t)\Big|_0^2 = \mathbf{r}(2) - \mathbf{r}(0) = \langle 4, e^2, 5\rangle - \langle 0, 1, 3\rangle = \langle 4, e^2 - 1, 2\rangle$.

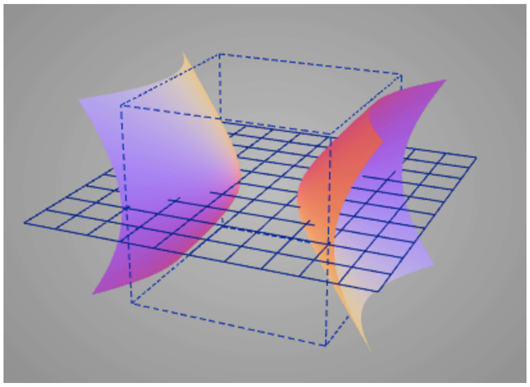
5a.(Source: 12.6.24) To find the intercepts, set two variables equal zero and solve for the third.

$$\begin{aligned} x = y = 0 &\implies \text{no real solutions} \\ y = z = 0 &\implies \text{no real solutions} \\ x = z = 0 &\implies y = \pm 1 \end{aligned}$$

So the surface doesn't intersect the x- or z-axis, and intersects the y-axis at $(0, 1, 0)$ and $(0, -1, 0)$.

Its cross-sections with the yz-plane $x = 0$ or with the xy-plane $z = 0$ are hyperbolas. If $y^2 > 1$, its cross-sections parallel the xz-plane are circles. The surface is question is a hyperboloid of two sheets.

Here's a graph I got online. The z axis is vertical, but I've rotated the xy plane a bit for a better view. As a result, the y-axis runs pretty much left-to-right, and the x-axis runs front to back, mostly.



5b.(Source: 12.1.17-20,12.6.35-37) Complete the square:

$$\begin{aligned} -x^2 - 4x + y^2 - z^2 &= 5 \\ -(x^2 + 4x + 4) + y^2 - z^2 &= 5 - 4 \\ -(x+2)^2 + y^2 - z^2 &= 1 \end{aligned}$$

This surface is obtained by shifting the surface in 5a -2 units in the x-direction. This means, for instance, that the two "vertices" are at $(-2, \pm 1, 0)$.