1a (4 pts). Find the vector of length 4 in the same direction as \( \langle 3, -1, 1 \rangle \).

1b (5 pts). Find the vector projection of \( \langle 2, 1, 0 \rangle \) onto \( \langle 3, -1, 1 \rangle \).

1c (3 pts). Find the cosine of the angle between \( \langle 2, 1, 0 \rangle \) and \( \langle 3, -1, 1 \rangle \).

2. Find an equation for the object described in each part. (You can give the equation(s) for a line in vector, parametric, or symmetric form.)

a (4 pts). The sphere centered at \( (1, -1, 2) \) with radius 5.

b (4 pts). The line passing through the points \( (1, -1, 2) \) and \( (0, 1, 5) \).

c (5 pts). The line passing through \( (1, -1, 2) \) parallel to the line \( \frac{1}{3}x = y - 1 = \frac{1}{2}(z + 2) \).

d (7 pts). The plane passing through the points \( (1, -1, 2) \), \( (2, 3, -4) \), and \( (0, -1, 5) \).

e (5 pts). The plane passing through \( (1, -1, 2) \) parallel to the plane \( 2x + 3z = 4 \).

3. The position of a particle at time \( t \) is given by \( x = \cos 4t \), \( y = 3t \), \( z = \sin 4t \).

a (5 pts). Find the speed of the particle at time \( t \).

b (8 pts). Express \( T \) and \( N \) as functions of \( t \) along this curve.

c (12 pts). Find the equations of the normal and osculating planes for the path of this particle at time \( t = 0 \). Label your answers so I can tell which is which.

4. A particle moving through space has the position function \( \mathbf{r}(t) = 2t\mathbf{i} + e^t\mathbf{j} + (t^2 - t + 3)\mathbf{k} \).

a (4 pts). Express the object’s velocity \( \mathbf{v} \) and acceleration \( \mathbf{a} \) as functions of \( t \).

b (5 pts). Find a vector tangent to the graph of \( \mathbf{r}(t) \) at \( t = 0 \), and an equation of the line tangent to the curve at the point corresponding to that time.

c (10 pts). Find \( a_N \) and \( a_T \), the normal and tangential components of acceleration, at time \( t = 0 \).

d (2 pts). Find the curvature \( \kappa \) of the graph of \( \mathbf{r} \) at \( t = 0 \).

e (3 pts). Find \( \int_0^2 \frac{d\mathbf{r}}{dt} \, dt \).

5a (8 pts). Make a rough sketch of the graph of \( -x^2 + y^2 - z^2 = 1 \). Find all intercepts and include them in your sketch. What is the name of this kind of surface?

5b (6 pts). How does the graph of \( -x^2 - 4x + y^2 - z^2 = 5 \) compare with the graph of the equation in 5a? A sketch is not required, but you can make one if you wish.
1a. (Source: 12.2.26) \[ 4 \cdot \frac{1}{\|\langle 3, -1, 1 \rangle \|} \langle 3, -1, 1 \rangle = \frac{4}{\sqrt{11}} \langle 3, -1, 1 \rangle. \]

1b. (Source: 12.3.41) \[ \frac{\langle 2, 1, 0 \rangle \cdot \langle 3, -1, 1 \rangle}{\|\langle 3, -1, 1 \rangle\|} \langle 3, -1, 1 \rangle = \frac{5}{11} \langle 3, -1, 1 \rangle, \text{ or } \frac{15}{11}, -\frac{5}{11}, \frac{5}{11}. \]

1c. (Source: 12.3.17) \[ \cos \theta = \frac{\langle 2, 1, 0 \rangle \cdot \langle 3, -1, 1 \rangle}{\|\langle 2, 1, 0 \rangle\| \|\langle 3, -1, 1 \rangle\|} = \frac{5}{\sqrt{6}\sqrt{11}}, \text{ or } \frac{5}{\sqrt{66}}. \]

2a. (Source: 12.1.13-14) \( (x - 1)^2 + (y + 1)^2 + (z - 2)^2 = 25. \)

2b. (Source: 12.5.7) The desired line is parallel to the vector \( \langle 1, -1, 2 \rangle - \langle 0, 1, 5 \rangle = \langle 1, -2, -3 \rangle \) and can be written in the parametric form

\[ x = 1 + t \quad y = -1 - 2t \quad z = 2 - 3t. \]

2c. (Source: 12.5.11) The given line is parallel to the vector \( \langle 3, 1, 4 \rangle \), so the line in question can be written in the symmetric form \( \frac{x - 1}{3} = y + 1 = \frac{z - 2}{4}. \)

2d. (Source: 12.5.31) Subtract points to find two vectors parallel to the plane in question:

\[ \langle 1, -1, 2 \rangle - \langle 2, 3, -4 \rangle = \langle -1, -4, 6 \rangle, \]
\[ \langle 1, -1, 2 \rangle - \langle 0, -1, 5 \rangle = \langle 1, 0, -3 \rangle. \]

Cross these to find a vector normal to the plane:

\[
\mathbf{n} = \begin{vmatrix}
i & j & k \\
-1 & -4 & 6 \\
1 & 0 & -3
\end{vmatrix} = \begin{vmatrix}i & -1 & 6 \\
0 & -3 & 1 \\
-4 & 1 & 0
\end{vmatrix} = \mathbf{k} = \langle 12, 3, 4 \rangle.
\]

The desired plane is \( 12(x - 1) + 3(y + 1) + 4(z - 2) = 0. \)

2e. (Source: 12.5.27) \( 2x + 3z = 4 \) has the normal vector \( \langle 2, 0, 3 \rangle \), so our plane is \( 2(x - 1) + 3(z - 2) = 0, \) or \( 2x + 3z = 8. \)

3a. (Source: 13.4.10) \( \mathbf{r} = \langle \cos 4t, 3t, \sin 4t \rangle, \quad \mathbf{v} = \frac{d\mathbf{r}}{dt} = \langle -4 \sin 4t, 3, 4 \cos 4t \rangle, \) and speed \( \frac{ds}{dt} = |\mathbf{v}| = \sqrt{16 \sin^2 4t + 9 + 16 \cos^2 4t} = \sqrt{16 \sin^2 4t + 25} = \sqrt{16 + 16} = 5. \) (That is, the particle moves at constant speed.)

3b. (Source: 13.3.17) \( \mathbf{T} = \frac{1}{|\mathbf{v}|} \mathbf{v} = \frac{1}{5} \langle -4 \sin 4t, 3, 4 \cos 4t \rangle, \text{ or } \langle -\frac{4}{5} \sin 4t, \frac{3}{5}, \frac{4}{5} \cos 4t \rangle. \)

\( \mathbf{N} = \frac{d\mathbf{T}}{dt} + \frac{d\mathbf{T}}{dt} = \frac{16}{5} \langle -\cos 4t, 0, -\sin 4t \rangle = \langle \cos 4t, 0, -\sin 4t \rangle. \)

3c. (Source: 13.3.49) Both planes pass through the point \( \mathbf{r}(0) = \langle 1, 0, 0 \rangle. \) The normal plane is perpendicular to the tangent vector \( \mathbf{T}(0) = \langle 0, \frac{3}{5}, \frac{4}{5} \rangle, \) so its equation is \( \frac{3}{5} y + \frac{4}{5} z = 0. \)

The osculating plane is parallel both \( \mathbf{T}(0) \) and \( \mathbf{N}(0) = \langle -1, 0, 0 \rangle, \) so its normal vector is

\[
\langle 0, \frac{3}{5}, \frac{4}{5} \rangle \times \langle -1, 0, 0 \rangle = \begin{vmatrix}i & j & k \\
0 & \frac{3}{5} & \frac{4}{5} \\
-1 & 0 & 0
\end{vmatrix} = \langle 0, -1, 0 \rangle.
\]

and its equation is \( \frac{4}{5} y + \frac{3}{5} z = 0. \)

(The osculating plane also contains \( \mathbf{v} \) and \( \mathbf{a}, \) so we could cross these two vectors to find a normal vector.)
4a. (Source: 13.4.9, 11) \( \mathbf{r}(t) = 2t \mathbf{i} + e^t \mathbf{j} + (t^2 - t + 3) \mathbf{k} \), so \( \mathbf{v} = \frac{d\mathbf{r}}{dt} = 2 \mathbf{i} + e^t \mathbf{j} + (2t - 1) \mathbf{k} \), and \( \mathbf{a} = \frac{d\mathbf{v}}{dt} = e^t \mathbf{j} + 2 \mathbf{k} \).

4b. (Source: 13.2.17, 18, 23) \( \mathbf{v}(0) = 2 \mathbf{i} + \mathbf{j} - \mathbf{k} = \langle 2, 1, -1 \rangle \) is a tangent vector. At time 0, the curve passes through the point \( \mathbf{r}(0) = \mathbf{j} + 3 \mathbf{k} = \langle 0, 1, 3 \rangle \), so the tangent line is given parametrically by

\[
x = 0 + 2t \quad y = 1 + t \quad z = 3 - t.
\]

4c. (Source: 13.4.38) Calculate

\[
\mathbf{a}(0) = \mathbf{j} + 2 \mathbf{k} = \langle 0, 1, 2 \rangle \quad \text{and} \quad \mathbf{v}(0) \times \mathbf{a}(0) = \langle 3, -4, 2 \rangle.
\]

\[
\mathbf{a}_T = \frac{\mathbf{v} \cdot \mathbf{a}}{|\mathbf{v}|} = \frac{-1}{\sqrt{6}}, \quad \text{and} \quad \mathbf{a}_N = \frac{\mathbf{v} \times \mathbf{a}}{|\mathbf{v}|} = \sqrt{\frac{29}{6}}.
\]

Note that \( \mathbf{a}_T = \frac{d^2 s}{dt^2} < 0 \), so the particle is slowing down at time 0. We could also have calculated \( \mathbf{a}_N = \sqrt{|\mathbf{a}|^2 - \mathbf{a}_T^2} \).

4d. (Source: 13.2.22) \( \kappa = \frac{|\mathbf{v} \times \mathbf{a}|}{|\mathbf{v}|^3} = \frac{29}{6\sqrt{6}} \).

4e. (Source: 13.2.35) By the Fundamental Theorem of Calculus,

\[
\int_0^2 \left( \frac{d\mathbf{r}}{dt} \right)^2 dt = \mathbf{r}(2) - \mathbf{r}(0) = \langle 4, e^2, 5 \rangle - \langle 0, 1, 3 \rangle = \langle 4, e^2 - 1, 2 \rangle.
\]

5a. (Source: 12.6.24) To find the intercepts, set two variables equal zero and solve for the third.

\[
x = y = 0 \quad \Rightarrow \quad \text{no real solutions}
\]

\[
y = z = 0 \quad \Rightarrow \quad \text{no real solutions}
\]

\[
x = z = 0 \quad \Rightarrow \quad y = \pm 1
\]

So the surface doesn’t intersect the \( x \)- or \( z \)-axis, and intersects the \( y \)-axis at \((0, 1, 0)\) and \((0, -1, 0)\).

Its cross-sections with the \( yz \)-plane \( x = 0 \) or with the \( xy \)-plane \( z = 0 \) are hyperbolas. If \( y^2 > 1 \), its cross-sections parallel the \( xz \)-plane are circles. The surface is question is a hyperboloid of two sheets.

Here’s a graph I got online. The \( z \) axis is vertical, but I’ve rotated the \( xy \) plane a bit for a better view. As a result, the \( y \)-axis runs pretty much left-to-right, and the \( x \)-axis runs front to back, mostly.

5b. (Source: 12.1.17-20, 12.6.35-37) Complete the square:

\[
-x^2 - 4x + y^2 - z^2 = 5
\]

\[
-(x^2 + 4x + 4) + y^2 - z^2 = 5 - 4
\]

\[
-(x + 2)^2 + y^2 - z^2 = 1
\]

This surface is obtained by shifting the surface in 5a \(-2\) units in the \( x \)-direction. This means, for instance, that the two “vertices” are at \((-2, \pm 1, 0)\).