1 (10 pts). Find the flux of the curl of \( \mathbf{F} = yz \mathbf{i} - x \mathbf{j} + xy \mathbf{k} \),

\[
\int \int_E (\nabla \times \mathbf{F}) \cdot \mathbf{n} \, dS,
\]

where \( E \) is the part of the paraboloid \( z = x^2 + y^2 \) that lies below \( z = 4 \), oriented downward.

**Solution:**

1. There are two ways to find this flux. The easy way is to use Stokes’s Theorem, which says that the flux of the curl of \( \mathbf{F} \) is the circulation of \( \mathbf{F} \) around the boundary of \( E \).

   The boundary is the circle \( 4 = x^2 + y^2, z = 4 \). For the downward-bound normal to travel around the circle with \( E \) at its left means traveling around \( x^2 + y^2 = 4 \) in the negative direction (from the point of view of the positive \( z \)-axis). Let’s parametrize this circle in the positive direction with

\[
\begin{align*}
x &= 2 \cos \theta \\
y &= 2 \sin \theta \\
z &= 4 \\
dx &= -2 \sin \theta \, d\theta \\
dy &= 2 \cos \theta \, d\theta \\
dz &= 0
\end{align*}
\]

and then multiply the resulting line integral by \(-1\).

(Tip: \( \int_0^{2\pi} \cos^2 \theta \, d\theta \) and \( \int_0^{2\pi} \sin^2 \theta \, d\theta \) both equal \( \pi \).)

\[
- \int_{\partial E} yz \, dx - x \, dy + xy \, dz = - \int_0^{2\pi} 2 \sin \theta \cdot 4 \cdot (-2 \sin \theta) \, d\theta - 2 \cos \theta \cdot 2 \cos \theta \, d\theta + 0
\]

\[
= 4 \int_0^{2\pi} (4 \sin^2 \theta + \cos^2 \theta) \, d\theta
= 4 \int_0^{2\pi} (3 \sin^2 \theta + 1) \, d\theta
= 4 \int_0^{2\pi} \left( \frac{3}{2} (1 - \cos 2\theta) + 1 \right) \, d\theta
\]

\[
= 4 \left( \int_0^{2\pi} \frac{5}{2} \, d\theta - \int_0^{2\pi} \cos 2\theta \, d\theta \right)
= 4(5\pi + 0) = 20\pi
\]

(done)

Here’s a sketch of the calculation of the flux of the curl directly.

curl \( \mathbf{F} = (x, 0, -1 - z) \). Parametrize the surface with \( \mathbf{r} = (x, y, x^2 + y^2) \). Then

\[
\mathbf{r}_x \times \mathbf{r}_y = \langle 1, 0, 2x \rangle \times \langle 0, 1, 2y \rangle = \langle -2x, -2y, 1 \rangle
\]

which points upward, not downward, as required. Instead, use \( \mathbf{n} \, dS = \langle 2x, 2y, -1 \rangle \, dx \, dy \)

Then \( (\nabla \times \mathbf{F}) \cdot \mathbf{n} \, dS = (2x^2 + z + 1) \, dx \, dy = (3x^2 + y^2 + 1) \, dx \, dy \)

Compute the flux using polar coordinates:

\[
\int_0^{2\pi} \int_0^2 (2r^2 \cos^2 \theta + r^2 + 1) r \, dr \, d\theta
= \int_0^{2\pi} \int_0^2 (2r^3 \cos^2 \theta + r^3 + r) \, dr \, d\theta
= \int_0^{2\pi} (8 \cos^2 \theta + 6) \, d\theta
= 8 \cdot \pi + 6 \cdot 2\pi = 20\pi.
\]

(done)