No notes, books, electronic devices, or outside materials of any kind.

Read each problem carefully and simplify your answers.

Supporting work will be required on every problem worth more than 2 points.

A mistake early in your solution does not rule out your receiving full credit for later steps.

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1 (19 pts). Find the circulation of $\mathbf{H} = (y + e^{x^2})\mathbf{i} + (z - e^{y^2})\mathbf{j} + (x + e^{e^z})\mathbf{k}$ around the boundary of the part of the plane $x + y + 2z = 2$ that lies in the first octant, traveled in the positive direction when viewed from above.

2 (2 pts). Suppose $\mathbf{E}$ is a vector field and $\operatorname{curl} \mathbf{E} = (1 + 2z)\mathbf{j}$. Is $\mathbf{E}$ conservative?

circle one: **Yes** **No** **Maybe**

3 (19 pts). Let $\mathbf{F} = \langle x + z, -y - x, z^2 - y \rangle$.

Let E be the half of the sphere $x^2 + y^2 + z^2 \leq 1$ where $z \geq 0$. Note that E is a solid, not a surface.

Find the flux of $\mathbf{F}$ across the boundary of E oriented outward.

4 (10 pts). Let E be the solid bounded by the three surfaces $y = x^2$, $y + 2z = 6$, and $z = 0$ and let $f(x, y, z)$ be some unspecified function. Express the integral $\iiint_E f(x, y, z)\, dV$ as an iterated integral **but do not evaluate**.

5 (8 pts). Evaluate the triple integral: $\int_0^1 \int_0^z \int_0^{x+z} 6xz\, dy\, dx\, dz$

6. This problem is about the graph of the equation $z = 4 + x^2$.

a (2 pts). Find the coordinates of all intercepts of this surface with the coordinate axes.

b (9 pts). Make a sketch of the surface. Show the intercepts you found in part a in your sketch. Label your x, y, and z axes and use an arrow to indicate the positive direction along each.

7. Let $\mathbf{u} = \langle 1, 2, -1 \rangle$ and $\mathbf{v} = \langle 3, 4, 5 \rangle$. Find the following.

a (7 pts). The vector projection of $\mathbf{u}$ onto $\mathbf{v}$.

b (6 pts). A nonzero vector orthogonal to both $\mathbf{u}$ and $\mathbf{v}$.

8. Find an equation (in any form) of the line described in each part.

a (6 pts). The line passing through the points $(1, 0, 1)$, and $(2, 1, -3)$.

b (9 pts). The line tangent to the curve parametrized by $x = t - e^t$, $y = te^t$, and $z = \ln t^5$ and the point corresponding to $t = 1$.

9. Find an equation of the plane described in each part.

a (8 pts). The plane passing through the points $(0, 0, 3)$, $(1, 0, 3)$, and $(0, 4, 0)$.

b (9 pts). The plane tangent to the surface $z^{-1}e^{2x-y} + 1 = 0$ at the point $(2, 4, -1)$.

10. Let $\mathbf{r}(t) = 2t\mathbf{i} - \frac{1}{2}t^2\mathbf{j} - t\mathbf{k}$. Express the following vectors and scalars as functions of t. Label your answers.

a (7 pts). $\mathbf{v}$, $\mathbf{a}$, and ds/dt

b (3 pts). $\mathbf{T}$

c (8 pts). a_T and a_N

d (6 pts). κ

11 (7 pts). Suppose, that along a certain curve, $\mathbf{T} = \langle \frac{3}{5}\cos(\pi t), \sin(\pi t), -\frac{4}{5}\cos(\pi t)\rangle$. Express $\mathbf{N}$ as a function of t.

12 (19 pts). Find the absolute maximum and minimum values of $-x + 2y + z$ along the surface $x^2 + y^2 + z^2 = 9$.

13. Suppose $T(x, y) = 2xy - x^2$ is the temperature (in °C) at the point (x, y).

a (2 pts). Find the rate of change of temperature in the x-direction at the point $(2, 3)$.

b (6 pts). Find the rate of change of temperature in the direction $\langle \frac{3}{5}, -\frac{4}{5}\rangle$ at the point $(2, 3)$.

c (9 pts). Find the direction (in the form of a unit vector) in which T is increasing most rapidly at the point $(2, 3)$, and its rate of increase in that direction.

14. Let $\mathbf{G} = \langle 1 + y - e^{x-y},\ x + e^{x-y}\rangle$.

a (6 pts). Find a potential function for $\mathbf{G}$.

b (7 pts). Evaluate the line integral $\int_C \mathbf{G} \cdot d\mathbf{r}$, where C is the curve given by $\mathbf{r}(t) = \langle -\cos t, \sin t\rangle$ for $0 \le t \le \pi$.

15 (6 pts). Evaluate the line integral $\int_C y\, ds$, where C is the curve from Problem 14b.

1. (Source: 16.8.8) Stokes's Theorem tells us that $\int_{\partial C} \mathbf{H} \cdot d\mathbf{r} = \iint_C \operatorname{curl} H \cdot \mathbf{n}\, dS$, where C is the surface in question and ∂C is its boundary. Calculate the curl:

$$\operatorname{curl} H = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y + e^{x^2} & z - e^{y^2} & x + e^{e^z} \end{vmatrix} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y & z & x \end{vmatrix} = -\mathbf{i} - \mathbf{j} - \mathbf{k}.$$

Parametrize the surface with $\mathbf{r}(x, y) = \langle x, y, 1 - \frac{1}{2}x - \frac{1}{2}y \rangle$. Then

$$\mathbf{n}\, dS = \pm(\mathbf{r}_x \times \mathbf{r}_y)\, dx\, dy = \pm \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 0 & -\frac{1}{2} \\ 0 & 1 & -\frac{1}{2} \end{vmatrix} dx\, dy = \pm(\tfrac{1}{2}\mathbf{i} + \tfrac{1}{2}\mathbf{j} + \mathbf{k})\, dx\, dy.$$

Since we want the normal to be pointing upward, use + above. The x and y limits for the surface integral must describe the triangle in the xy plane bounded by the lines $x = 0$, $y = 0$, and $x + y = 2$, and so the integral is

$$\int_0^2 \int_0^{2-x} (-\mathbf{i} - \mathbf{j} - \mathbf{k}) \cdot (\tfrac{1}{2}\mathbf{i} + \tfrac{1}{2}\mathbf{j} + \mathbf{k})\, dy\, dx = \int_0^2 \int_0^{2-x} -2\, dy\, dx,$$

and this equals -2 times the area of said triangle, or $-2 \cdot \frac{1}{2} \cdot 2 \cdot 2 = -4$.

2. (Source: 16.5.17) If $\mathbf{E}$ were conservative, its curl would be zero, so **No**.

3. (Source: 16.7.24, 15.8.23) The Divergence Theorem tells us that $\iint_{\partial E} \mathbf{F} \cdot \mathbf{n}\, dS = \iiint_E \operatorname{div} \mathbf{dV}$ where ∂E is the boundary of E. Calculate $\operatorname{div} \mathbf{F} = (x + z)_x + (-y - x)_y + (z^2 - y)_z = 2z$. The triple integral over the hemisphere is best done with spherical coordinates:

$$\int_0^{2\pi} \int_0^{\pi/2} \int_0^1 2z\rho^2 \sin\phi\, d\rho\, d\phi\, d\theta = \int_0^{2\pi} \int_0^{\pi/2} \int_0^1 2\rho \cos\phi \rho^2 \sin\phi\, d\rho\, d\phi\, d\theta$$

$$= \int_0^{2\pi} d\theta \int_0^{\pi/2} 2\cos\phi \sin\phi\, d\phi \int_0^1 \rho^3\, d\rho = 2\pi (\sin^2\phi)\Big|_0^{\pi/2} (\tfrac{1}{4}\rho^4)\Big|_0^1 = \frac{\pi}{2}$$

4. (Source: 15.6.31) It helps to picture the solid. $z = 0$ forms the floor, $z = 3 - y/2$ is the ceiling, and the parabolic cylinder $y = x^2$ is a vertical wall. There are 6 different ways to order the variables. Here's just one. $\int_{-\sqrt{6}}^{\sqrt{6}} \int_{x^2}^{6} \int_0^{3-y/2} f(x, y, z)\, dz\, dy\, dx$

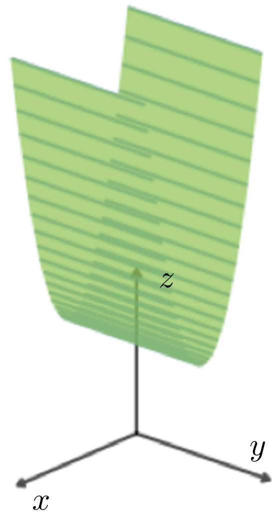
5. (Source: 15.3.3)

$$\int_0^1 \int_0^z \int_0^{x+z} 6xz\, dy\, dx\, dz = \int_0^1 \int_0^z 6xz(x + z)\, dx\, dz = \int_0^1 \int_0^z (6x^2 z + 6xz^2)\, dx\, dz$$

$$= \int_0^1 (2x^3 z + 3x^2 z^2)\Big|_0^z dz = \int_0^1 (2z^4 + 3z^4)\Big|_0^z dz = \int_0^1 5z^4\Big|_0^z dz = 1.$$

6a. (Source: 12.6.5) z cannot equal zero, so the only axis that intersects the surface is $x = y = 0$, and the intercept is $(0, 0, 4)$.

6b. The surface is a parabolic cylinder. Draw the parabola $z = 4 + x^2$ in the xz-plane, and then drag that curve in the y direction. Here's a graph.



7a. (Source: 12.3.37) $\text{proj}_{\mathbf{v}}\, \mathbf{u} = \dfrac{\mathbf{u}\cdot\mathbf{v}}{\mathbf{v}\cdot\mathbf{v}}\mathbf{v} = \dfrac{6}{50}\mathbf{v} = \dfrac{3}{25}\langle 3, 4, 5\rangle.$

7b. (Source: 12.4.19) $\mathbf{u}\times\mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 2 & -1 \\ 3 & 4 & 5 \end{vmatrix} = \langle 14, -8, -2\rangle.$

8a. (Source: 12.5.7) The vector $\langle 2, 1, -3\rangle - \langle 1, 0, 1\rangle = \langle 1, 1, -4\rangle$ is parallel the line. Using the point $(1, 0, 1)$, the line is parametrized by $\mathbf{r}(t) = \langle 1, 0, 1\rangle + t\langle 1, 1, -4\rangle$.

8b. (Source: 13.2.25) Evaluate $\langle t - e^t, te^t, 5\ln t\rangle$ at $t = 1$ to find the point of tangency on the line, and the derivative $\frac{d}{dt}\langle t-e^t, te^t, 5\ln t\rangle = \langle 1-e^t, te^t+e^t, 5/t\rangle$ at $t = 1$ to obtain a vector parallel the line. The parametrization is

$$\mathbf{r}(t) = \langle 1 - e, e, 0\rangle + t\langle 1 - e, 2e, 5\rangle.$$

9a. (Source: 12.5.31) The two vectors $\langle 1, 0, 3\rangle - \langle 0, 0, 3\rangle = \langle 1, 0, 0\rangle$ and $\langle 0, 4, 0\rangle - \langle 0, 0, 3\rangle = \langle 0, 4, -3\rangle$ are parallel the plane, so their cross product is a normal:

$$\mathbf{n} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 0 & 0 \\ 0 & 4 & -3 \end{vmatrix} = \langle 0, 3, 4\rangle.$$

Using the point $91, 0, 3)$ results in the equation $3(y - 0) + 4(z - 3) = 0$.

9b. (Source: 14.6.43) The gradient $\nabla(z^{-1}e^{2x-y}) = \langle 2z^{-1}e^{2x-y}, -z^{-1}e^{2x-y}, -1z^{-2}e^{2x-y}\rangle$ is normal to the level surfaces of g. Evaluate this at the point $(2, 4, -1)$ to obtain $\mathbf{n} = \langle -2, 1, -1\rangle$. The plane is $-2(x - 2) + (y - 4) - 1(z + 1) = 0$.

10. (Source: 13.3.20&22, 13.4.9&26)

$$\begin{aligned} \mathbf{r}(t) &= \langle 2t, -\tfrac{1}{2}t^2, -t\rangle \\ \mathbf{v} = \mathbf{r}' &= \langle 2, -t, -1\rangle \\ \mathbf{a} = \mathbf{r}'' &= \langle 0, -1, 0\rangle \\ \frac{ds}{dt} &= |\mathbf{v}| = \sqrt{5 + t^2} \\ \mathbf{T} &= \frac{1}{|\mathbf{v}|}\mathbf{v} = \frac{1}{\sqrt{5+t^2}}\langle 2, -t, -1\rangle \end{aligned}$$

$$\begin{aligned} a_T &= = \frac{\mathbf{a}\cdot\mathbf{v}}{|\mathbf{v}|} = \frac{t}{\sqrt{5+t^2}} \\ a_N &= \sqrt{|\mathbf{a}|^2 - a_T^2} = \sqrt{1 - \frac{t^2}{5+t^2}} = \sqrt{\frac{5}{5+t^2}} \\ \kappa &= \frac{|\mathbf{v}\times\mathbf{a}|}{|\mathbf{v}|^3} = \frac{1}{(5+t^2)^{3/2}}\left\|\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & -t & -1 \\ 0 & -1 & 0 \end{vmatrix}\right\| \\ &= \frac{1}{(5+t^2)^{3/2}}|-\mathbf{i} - 2\mathbf{k}| = \frac{\sqrt{5}}{(5+t^2)^{3/2}} \end{aligned}$$

Could also use the formulas

$$a_T = d^2s/dt^2 = \frac{d}{dt}\sqrt{5+t^2} \qquad a_N = |\mathbf{a}\times\mathbf{v}|/|\mathbf{v}| \qquad \kappa = a_N/|\mathbf{v}|^2.$$

11. (Source: 13.3.44)

$$\frac{d\mathbf{T}}{dt} = \left\langle -\frac{3\pi}{5}\sin(\pi t), \pi\cos(\pi t), \frac{4\pi}{5}\sin(\pi t)\right\rangle$$
$$= \pi\left\langle -\frac{3}{5}\sin(\pi t), \cos(\pi t), \frac{4}{5}\sin(\pi t)\right\rangle$$
$$\left|\frac{d\mathbf{T}}{dt}\right| = \pi\sqrt{\frac{9}{25}\sin^2(\pi t) + \cos^2(\pi t) + \frac{16}{25}\sin^2(\pi t)} = \pi$$
$$\mathbf{N} = d\mathbf{T}/dt/|d\mathbf{T}/dt| = \left\langle -\frac{3}{5}\sin(\pi t), \cos(\pi t), \frac{4}{5}\sin(\pi t)\right\rangle$$

12. (Source: 14.8.7) Use the method of Lagrange multipliers. Let $f = -x + 2y + z$ and $g = x^2 + y^2 + z^2$ The absolute extrema can only occur at points along $g = 9$ at which $\lambda\nabla f = \lambda\langle -1, 2, 1\rangle = \langle 2x, 2y, 2z\rangle = \nabla g$:

$$-\lambda = 2x \qquad 2\lambda = 2y \qquad \lambda = 2z \qquad x^2 + y^2 + z^2 = 9$$

Solve for x, y, z in terms of λ and substitute into the last equation.

$$x = -\frac{\lambda}{2} \qquad y = \lambda \qquad z = \frac{\lambda}{2}$$
$$\frac{\lambda^2}{4} + \lambda^2 + \frac{\lambda^2}{4} = 9 \Longrightarrow \frac{3\lambda^2}{2} = 9 \Longrightarrow \lambda = \pm\sqrt{6}$$

The two "critical points" are $(x, y, z) = (-\frac{\sqrt{6}}{2}, \sqrt{6}, \frac{\sqrt{6}}{2})$ and $(\frac{\sqrt{6}}{2}, -\sqrt{6}, -\frac{\sqrt{6}}{2})$. Evaluate and compare f at both: $f\left(-\frac{\sqrt{6}}{2}, \sqrt{6}, \frac{\sqrt{6}}{2}\right) = 3\sqrt{6}$ is the max and $f\left(\frac{\sqrt{6}}{2}, -\sqrt{6}, -\frac{\sqrt{6}}{2}\right) = -3\sqrt{6}$ is the min.

(Had we used $\nabla f = \lambda\nabla g$, we'd have found $\lambda = \pm 1/\sqrt{6}$ but the same critical points.)

13a. (Source: 14.3.15) At $(x, y) = (2, 3)$, $T_x = 2y - 2x = 2$.

13b. (Source: 14.6.33) $\nabla T = \langle 2y - 2x, 2x\rangle = \langle 2, 4\rangle$ at $(x, y) = (2, 3)$. The directional derivative $D_{\mathbf{u}}T(2, 3) = \mathbf{u}\cdot\nabla T = \langle\frac{3}{5}, -\frac{4}{5}\rangle\cdot\langle 2, 4\rangle = \frac{6}{5} - \frac{16}{5} = -2$.

13c. (Source: 14.6.21) T increases most rapidly in the direction of its gradient. $\langle 1, 2\rangle$ is in the same direction as $\langle 2, 4\rangle$, so normalize $\langle 1, 2\rangle$ to obtain $\frac{1}{\sqrt{5}}\langle 1, 2\rangle$. In that direction, the directional derivative is $|\nabla T(2, 3)| = |\langle 2, 4\rangle| = \sqrt{20}$.

14a. (Source: 16.3.17) $g_x = 1 + y - e^{x-y}$ implies $g = x + xy - e^{x-y} + B(y)$ for some function B of y, and so $g_y = x + e^{x-y} + B'(y)$. Compare this with the second component of $\mathbf{G}$ to conclude $B'(y) = 0$ and therefore B is a constant. Taking $B = 0$, we find that one potential function is $g = x + xy - e^{x-y}$.

14b. C begins at the point $(-1, 0)$ when $t = 0$ and ends at $(1, 0)$ when $t = \pi$. By the fundamental theorem for line integrals, $\int_C \mathbf{G}\cdot d\mathbf{r} = (x + xy - e^{x-y})\Big|_{(-1,0)}^{(1,0)} = 2 - e + e^{-1}$.

15. (Source: 16.2.3) Along C, velocity is $\langle\sin t, \cos t\rangle$ and $\dfrac{ds}{dt} = \sqrt{\sin^2 t + \cos^2 t} = 1$, so $\int_C y\,ds = \int_C y\frac{ds}{dt}\,dt = \int_0^\pi \sin t\,dt = 2$.