1 (19 pts). Find the circulation of \( \mathbf{H} = (y + e^{x^2})\mathbf{i} + (z - e^{y^2})\mathbf{j} + (x + e^{z^2})\mathbf{k} \) around the boundary of the part of the plane \( x + y + 2z = 2 \) that lies in the first octant, traveled in the positive direction when viewed from above.

2 (2 pts). Suppose \( \mathbf{E} \) is a vector field and \( \text{curl} \mathbf{E} = (1 + 2z)\mathbf{j} \). Is \( \mathbf{E} \) conservative? circle one: Yes No Maybe

3 (19 pts). Let \( \mathbf{F} = \langle x + z, -y - x, z^2 - y \rangle \). Let \( \mathbf{E} \) be the half of the sphere \( x^2 + y^2 + z^2 \leq 1 \) where \( z \geq 0 \). Note that \( \mathbf{E} \) is a solid, not a surface.

Find the flux of \( \mathbf{F} \) across the boundary of \( \mathbf{E} \) oriented outward.

4 (10 pts). Let \( \mathbf{E} \) be the solid bounded by the three surfaces \( y = x^2 \), \( y + 2z = 6 \), and \( z = 0 \) and let \( f(x, y, z) \) be some unspecified function. Express the integral \( \iiint_{\mathbf{E}} f(x, y, z) \, dV \) as an iterated integral but do not evaluate.

5 (8 pts). Evaluate the triple integral: \( \int_0^1 \int_0^z \int_0^{x+z} 6xz \, dy \, dx \, dz \)

6. This problem is about the graph of the equation \( z = 4 + x^2 \).

a (2 pts). Find the coordinates of all intercepts of this surface with the coordinate axes.

b (9 pts). Make a sketch of the surface. Show the intercepts you found in part a in your sketch. Label your \( x, y, \) and \( z \) axes and use an arrow to indicate the positive direction along each.

7. Let \( \mathbf{u} = \langle 1, 2, -1 \rangle \) and \( \mathbf{v} = \langle 3, 4, 5 \rangle \). Find the following.

a (7 pts). The vector projection of \( \mathbf{u} \) onto \( \mathbf{v} \).

b (6 pts). A nonzero vector orthogonal to both \( \mathbf{u} \) and \( \mathbf{v} \).

8. Find an equation (in any form) of the line described in each part.

a (6 pts). The line passing through the points \( (1, 0, 1) \), and \( (2, 1, -3) \).

b (9 pts). The line tangent to the curve parametrized by \( x = t - e^t, y = te^t, \) and \( z = \ln t^5 \) and the point corresponding to \( t = 1 \).

9. Find an equation of the plane described in each part.

a (8 pts). The plane passing through the points \( (0, 0, 3) \), \( (1, 0, 3) \), and \( (0, 4, 0) \).

b (9 pts). The plane tangent to the surface \( z^{-1}e^{2x-y} + 1 = 0 \) at the point \( (2, 4, -1) \).

10. Let \( \mathbf{r}(t) = 2t\mathbf{i} - \frac{1}{2}t^2\mathbf{j} - t\mathbf{k} \). Express the following vectors and scalars as functions of \( t \). Label your answers.

a (7 pts). \( \mathbf{v}, \mathbf{a}, \) and \( ds/dt \)

b (3 pts). \( \mathbf{T} \)

c (8 pts). \( a_T \) and \( a_N \)

d (6 pts). \( \kappa \)
11 (7 pts). Suppose, that along a certain curve, $\mathbf{T} = \langle \frac{3}{5} \cos(\pi t), \sin(\pi t), -\frac{4}{5} \cos(\pi t) \rangle$. Express $\mathbf{N}$ as a function of $t$.

12 (19 pts). Find the absolute maximum and minimum values of $-x + 2y + z$ along the surface $x^2 + y^2 + z^2 = 9$.

13. Suppose $T(x, y) = 2xy - x^2$ is the temperature (in °C) at the point $(x, y)$.
   a (2 pts). Find the rate of change of temperature in the $x$-direction at the point $(2, 3)$.
   b (6 pts). Find the rate of change of temperature in the direction $\langle \frac{3}{5}, -\frac{4}{5} \rangle$ at the point $(2, 3)$.
   c (9 pts). Find the direction (in the form of a unit vector) in which $T$ is increasing most rapidly at the point $(2, 3)$, and its rate of increase in that direction.

14. Let $\mathbf{G} = \langle 1 + y - e^{x-y}, x + e^{x-y} \rangle$.
   a (6 pts). Find a potential function for $\mathbf{G}$.
   b (7 pts). Evaluate the line integral $\int_C \mathbf{G} \cdot d\mathbf{r}$, where $C$ is the curve given by $\mathbf{r}(t) = \langle -\cos t, \sin t \rangle$ for $0 \leq t \leq \pi$.

15 (6 pts). Evaluate the line integral $\int_C y \, ds$, where $C$ is the curve from Problem 14b.
1. (Source: 16.8.8) Stokes’s Theorem tells us that \( \int_{\partial C} \mathbf{H} \cdot d\mathbf{r} = \iint_{C} \text{curl} \mathbf{H} \cdot \mathbf{n} \, dS \), where \( C \) is the surface in question and \( \partial C \) is its boundary. Calculate the curl:

\[
\text{curl} \mathbf{H} = \begin{vmatrix}
\frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\
\frac{\partial}{\partial y} & \frac{\partial}{\partial z} & \frac{\partial}{\partial x} \\
y + e^{x^2} & z - e^{y^2} & x + e^{z^2}
\end{vmatrix}
= -\mathbf{i} - \mathbf{j} - \mathbf{k}.
\]

Parametrize the surface with \( \mathbf{r}(x, y) = (x, y, 1 - \frac{1}{2}x - \frac{1}{2}y) \). Then

\[
\mathbf{n} \, dS = \pm (\mathbf{r}_x \times \mathbf{r}_y) \, dx \, dy = \pm \begin{vmatrix}
\mathbf{i} & \mathbf{j} & \mathbf{k} \\
1 & 0 & -\frac{1}{2} \\
0 & 1 & -\frac{1}{2}
\end{vmatrix} \, dx \, dy = \pm (\frac{1}{2}\mathbf{i} + \frac{1}{2}\mathbf{j} + \mathbf{k}) \, dx \, dy.
\]

Since we want the normal to be pointing upward, use + above. The \( x \) and \( y \) limits for the surface integral must describe the triangle in the \( xy \) plane bounded by the lines \( x = 0 \), \( y = 0 \), and \( x + y = 2 \), and so the integral is

\[
\int_{0}^{2} \int_{0}^{2-x} (-\mathbf{i} - \mathbf{j} - \mathbf{k}) \cdot (\frac{1}{2}\mathbf{i} + \frac{1}{2}\mathbf{j} + \mathbf{k}) \, dy \, dx = \int_{0}^{2} \int_{0}^{2-x} -2 \, dy \, dx,
\]
and this equals \(-2\) times the area of said triangle, or \(-2 \cdot \frac{1}{2} \cdot 2 = -4\).

2. (Source: 16.5.17) If \( \mathbf{E} \) were conservative, its curl would be zero, so \( \mathbf{N} \).

3. (Source: 16.7.24, 15.8.23) The Divergence Theorem tells us that \( \iint_{\partial E} \mathbf{F} \cdot \mathbf{n} \, dS = \iiint_{E} \text{div} \mathbf{F} \, dV \) where \( \partial E \) is the boundary of \( E \). Calculate \( \text{div} \mathbf{F} = (x + z)x + (-y - x)y + (z^2 - y)z = 2z \).

The triple integral over the hemisphere is best done with spherical coordinates:

\[
\int_{0}^{2\pi} \int_{0}^{\pi/2} \int_{0}^{1} 2z\rho^2 \sin \phi \, d\rho \, d\phi \, d\theta = \int_{0}^{2\pi} \int_{0}^{\pi/2} \int_{0}^{1} 2\rho \cos \phi \rho^2 \sin \phi \, d\rho \, d\phi \, d\theta
= \int_{0}^{2\pi} d\theta \int_{0}^{\pi/2} 2 \cos \phi \sin \phi \, d\phi \int_{0}^{1} \rho^3 \, d\rho = 2\pi (\sin^2 \phi) \int_{0}^{\pi/2} (\frac{1}{4} \rho^4) \bigg|_{0}^{1} = \frac{\pi}{2}
\]

4. (Source: 15.6.31) It helps to picture the solid. \( z = 0 \) forms the floor, \( z = 3 - y/2 \) is the ceiling, and the parabolic cylinder \( y = x^2 \) is a vertical wall. There are 6 different ways to order the variables. Here’s just one. \( \int_{-\sqrt{6}}^{\sqrt{6}} \int_{x^2}^{6 - y/2} f(x, y, z) \, dz \, dy \, dx \)

5. (Source: 15.3.3)

\[
\int_{0}^{1} \int_{0}^{z} \int_{0}^{z + z} 6xz \, dy \, dx \, dz = \int_{0}^{1} \int_{0}^{z} 6xz(x + z) \, dx \, dz = \int_{0}^{1} \int_{0}^{z} (6x^2z + 6xz^2) \, dx \, dz
= \int_{0}^{1} \left(2x^3z + 3x^2z^2\right) \bigg|_{0}^{z} \, dz = \int_{0}^{1} \left(2z^4 + 3z^4\right) \bigg|_{0}^{z} \, dz = \int_{0}^{1} 5z^4 \bigg|_{0}^{z} \, dz = 1.
\]

6a. (Source: 12.6.5) \( z \) cannot equal zero, so the only axis that intersects the surface is \( x = y = 0 \), and the intercept is \((0, 0, 4)\).
6b. The surface is a parabolic cylinder. Draw the parabola \( z = 4 + x^2 \) in the \( xz \)-plane, and then drag that curve in the \( y \) direction. Here’s a graph.

7a. (Source: 12.3.37) \( \text{proj}_v \mathbf{u} = \frac{\mathbf{u} \cdot \mathbf{v}}{\mathbf{v} \cdot \mathbf{v}} \mathbf{v} = \frac{6}{50} \mathbf{v} = \frac{3}{25} (3, 4, 5) \).

7b. (Source: 12.4.19) \( \mathbf{u} \times \mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 2 & -1 \\ 3 & 4 & 5 \end{vmatrix} = (14, -8, -2) \).

8a. (Source: 12.5.7) The vector \( \langle 2, 1, -3 \rangle - \langle 1, 0, 1 \rangle = \langle 1, 1, -4 \rangle \) is parallel the line. Using the point \( (1, 0, 1) \), the line is parametrized by \( \mathbf{r}(t) = \langle 1, 0, 1 \rangle + t(1, 1, -4) \).

8b. (Source: 13.2.25) Evaluate \( (t - e^t, te^t, 5 \ln t) \) at \( t = 1 \) to find the point of tangency on the line, and the derivative \( \frac{d}{dt}(t - e^t, te^t, 5 \ln t) = (1 - e^t, te^t + e^t, 5/t) \) at \( t = 1 \) to obtain a vector parallel the line. The parametrization is

\[
\mathbf{r}(t) = \langle 1 - e, e, 0 \rangle + t(1 - e, 2e, 5) .
\]

9a. (Source: 12.5.31) The two vectors \( \langle 1, 0, 3 \rangle - \langle 0, 0, 3 \rangle = \langle 1, 0, 0 \rangle \) and \( \langle 0, 4, 0 \rangle - \langle 0, 0, 3 \rangle = \langle 0, 4, -3 \rangle \) are parallel the plane, so their cross product is a normal:

\[
\mathbf{n} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 0 & 0 \\ 0 & 4 & -3 \end{vmatrix} = (0, 3, 4) .
\]

Using the point \( 1, 0, 3 \) results in the equation \( 3(y - 0) + 4(z - 3) = 0 \).

9b. (Source: 14.6.43) The gradient \( \nabla (z^{-1}e^{2x-y}) = \langle 2z^{-1}e^{2x-y}, -z^{-1}e^{2x-y}, -1z^{-2}e^{2x-y} \rangle \) is normal to the level surfaces of \( g \). Evaluate this at the point \( (2, 4, -1) \) to obtain \( \mathbf{n} = \langle -2, 1, -1 \rangle \). The plane is \(-2(x - 2) + (y - 4) - 1(z + 1) = 0\).

10. (Source: 13.3.20&22, 13.4.9&26)

\[
\mathbf{r}(t) = \langle 2t, -\frac{1}{2}t^2, -t \rangle
\]

\[
\mathbf{v} = \mathbf{r}' = \langle 2, -t, -1 \rangle
\]

\[
\mathbf{a} = \mathbf{r}'' = \langle 0, -1, 0 \rangle
\]

\[
ds/dt = |\mathbf{v}| = \sqrt{5 + t^2}
\]

\[
\mathbf{T} = \frac{1}{|\mathbf{v}|} \mathbf{v} = \frac{1}{\sqrt{5 + t^2}} \langle 2, -t, -1 \rangle
\]

\[
\mathbf{a}_N = \sqrt{\mathbf{a}^2 - \mathbf{a}_T^2} = \sqrt{1 - \frac{t^2}{5 + t^2}} = \sqrt{\frac{5}{5 + t^2}}
\]

\[
\kappa = \frac{|\mathbf{v} \times \mathbf{a}|}{|\mathbf{v}|^3} = \frac{1}{(5 + t^2)^{3/2}} \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & -t & -1 \\ 0 & -1 & 0 \end{vmatrix} = \frac{\sqrt{5}}{(5 + t^2)^{3/2}}
\]

Could also use the formulas

\[
a_T = \frac{d^2 s}{dt^2} = \frac{d}{dt} \sqrt{5 + t^2}
\]

\[
a_N = |\mathbf{a} \times \mathbf{v}|/|\mathbf{v}|
\]

\[
\kappa = a_N/|\mathbf{v}|^2.
\]
11. (Source: 13.3.44)
\[
\frac{dT}{dt} = \left\langle \frac{-3\pi}{5} \sin(\pi t), \pi \cos(\pi t), \frac{4\pi}{5} \sin(\pi t) \right\rangle
\]
\[
= \pi \left\langle \frac{-3}{5} \sin(\pi t), \cos(\pi t), \frac{4}{5} \sin(\pi t) \right\rangle
\]
\[
\frac{|dT|}{dt} = \pi \sqrt{\frac{9}{25} \sin^2(\pi t) + \cos^2(\pi t) + \frac{16}{25} \sin^2(\pi t)} = \pi
\]
\[
N = \frac{dT}{dt} / |dT/dt| = \left\langle \frac{-3}{5} \sin(\pi t), \cos(\pi t), \frac{4}{5} \sin(\pi t) \right\rangle
\]

12. (Source: 14.8.7) Use the method of Lagrange multipliers. Let \( f = -x + 2y + z \) and \( g = x^2 + y^2 + z^2 \). The absolute extrema can only occur at points along \( g = 9 \) at which \( \lambda \nabla f = \lambda (-1, 2, 1) = (2x, 2y, 2z) = \nabla g \):
\[-\lambda = 2x \quad 2\lambda = 2y \quad \lambda = 2z \quad x^2 + y^2 + z^2 = 9\]

Solve for \( x, y, z \) in terms of \( \lambda \) and substitute into the last equation.
\[
x = -\frac{\lambda}{2} \quad y = \lambda \quad z = \frac{\lambda}{2}
\]
\[
\lambda^2 + 4 \lambda^2 + \frac{\lambda^2}{4} = 9 \quad \Rightarrow \quad \frac{3\lambda^2}{2} = 9 \quad \Rightarrow \quad \lambda = \pm \sqrt{6}
\]
The two “critical points” are \((x, y, z) = (-\frac{\sqrt{6}}{2}, \sqrt{6}, \frac{\sqrt{6}}{2})\) and \((\frac{\sqrt{6}}{2}, -\sqrt{6}, -\frac{\sqrt{6}}{2})\). Evaluate and compare \( f \) at both: \( f \left( -\frac{\sqrt{6}}{2}, \sqrt{6}, \frac{\sqrt{6}}{2} \right) = 3\sqrt{6} \) is the max and \( f \left( \frac{\sqrt{6}}{2}, -\sqrt{6}, -\frac{\sqrt{6}}{2} \right) = -3\sqrt{6} \) is the min.

(Had we used \( \nabla f = \lambda \nabla g \), we’d have found \( \lambda = \pm 1/\sqrt{6} \) but the same critical points.)

13a. (Source: 14.3.15) At \((x, y) = (2, 3)\), \( T_x = 2y - 2x = 2 \).

13b. (Source: 14.6.33) \( \nabla T = \langle 2y - 2x, 2x \rangle = \langle 2, 4 \rangle \) at \((x, y) = (2, 3)\). The directional derivative \( D_u T(2, 3) = u \cdot \nabla T = \langle \frac{3}{\pi}, -\frac{4}{\pi} \rangle \cdot \langle 2, 4 \rangle = \frac{6}{\pi} - \frac{16}{\pi} = -2 \).

13c. (Source: 14.6.21) \( T \) increases most rapidly in the direction of its gradient. \( \langle 1, 2 \rangle \) is in the same direction as \( \langle 2, 4 \rangle \), so normalize \( \langle 1, 2 \rangle \) to obtain \( \frac{1}{\sqrt{5}} \langle 1, 2 \rangle \). In that direction, the directional derivative is \( |\nabla T(2, 3)| = |\langle 2, 4 \rangle| = \sqrt{20} \).

14a. (Source: 16.3.17) \( g_x = 1 + y - e^{x-y} \) implies \( g = x + xy - e^{x-y} + B(y) \) for some function \( B \) of \( y \), and so \( g_y = x + e^{x-y} + B'(y) \). Compare this with the second component of \( \mathbf{G} \) to conclude \( B'(y) = 0 \) and therefore \( B \) is a constant. Taking \( B = 0 \), we find that one potential function is \( g = x + xy - e^{x-y} \).

14b. \( C \) begins at the point \((-1, 0)\) when \( t = 0 \) and ends at \((1, 0)\) when \( t = \pi \). By the fundamental theorem for line integrals, \( \int_C \mathbf{G} \cdot d\mathbf{r} = (x + xy - e^{x-y}) \bigg|_{(-1, 0)}^{(1, 0)} = 2 - e + e^{-1} \).

15. (Source: 16.2.3) Along \( C \), velocity is \( \langle \sin t, \cos t \rangle \) and \( \frac{ds}{dt} = \sqrt{\sin^2 t + \cos^2 t} = 1 \), so \( \int_C y \, ds = \int_C y \frac{ds}{dt} \, dt = \int_0^\pi \sin t \, dt = 2 \).