No notes, books, electronic devices, or outside materials of any kind.

Read each problem carefully and simplify your answers.

Supporting work will be required on every problem worth more than 2 points.

A mistake early in your solution does not rule out your receiving full credit for later steps.

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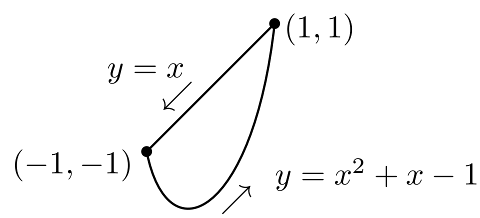
1 (16 pts). Let C be the path parametrized by $x = t$, $y = 1 - t$, $z = \frac{1}{4}t^4$ on $0 \le t \le 1$. Evaluate the following line integrals on C.

a. $\int_C y\,dx + z\,dy + x\,dz$ b. $\int_C xz\,ds$

2 (12 pts). Let $\mathbf{G} = \langle y^2, 1 + 2xy \rangle$.

a. Find $\int_C \mathbf{G} \cdot d\mathbf{r}$ along the curve C given by $\mathbf{r}(t) = \langle t - \sin t, 1 - \cos t \rangle$, $0 \le t \le \pi/2$.

b. Find $\int_B \mathbf{G} \cdot d\mathbf{r}$ along the path B graphed at right. (B begins and ends at the point $(1, 1)$.)



3 (20 pts). Let $\mathbf{F} = \langle \ln(xy), yz, x \rangle$.
Find each of the following, or explain why it doesn't exist. (Label your answers.)

a. div $\mathbf{F}$ b. curl $\mathbf{F}$ c. grad $\mathbf{F}$ d. A potential function for $\mathbf{F}$.

4 (10 pts). Rewrite the line integral $\int_C (e^x - xy)\,dx + (e^y + 2x)\,dy$, where C is the rectangle with vertices $(0, 0)$, $(2, 0)$, $(0, 3)$, and $(2, 3)$ traveled in the positive direction, as a double integral over the interior of the rectangle.
Express your answer as an iterated definite integral, but **do not evaluate**.

5 (10 pts). Find an equation of the plane tangent to the surface given by $\mathbf{r}(u, v) = \langle u + v^2, 3v + u^2, u^3 - v \rangle$ at the point corresponding to $u = 0$ and $v = 1$.

6 (16 pts). Compute the surface integral $\iint_E y\,dS$ where E is the part of the plane $z = 1 + 2x - y$ that lies above the rectangle $[0, 2] \times [0, 1]$ in the xy-plane.

7 (16 pts). Find the flux of $\langle e^x, z, -y \rangle$ across the surface in Problem 6, oriented upward.

1a. (Source: 16.2.14) Because $P_y = 1 \neq Q_x = 0$, the integral is not path independent and we have no choice but to use these

$$x = t \qquad y = 1 - t \qquad z = \frac{1}{4}t^4$$
$$dx = dt \qquad dy = -dt \qquad dz = t^3\,dt$$

to rewrite the integral in terms of t:

$$\int_C y\,dx + z\,dy + x\,dz = \int_0^1 (1-t)\,dt - \frac{1}{4}t^4\,dt + t\cdot t^3\,dt = \int_0^1 (1 - t + \frac{3}{4}t^4)\,dt$$
$$= (t - \frac{1}{2}t^2 + \frac{3}{20}t^5)\Big|_0^1 = \frac{13}{20}$$

1b. (Source: 16.2.12) To rewrite the integral in terms of t, we need

$$ds = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2}\,dt = \sqrt{1 + 1 + (t^3)^2}\,dt = \sqrt{2 + t^6}\,dt.$$

Now
$$\int_C xz\,ds = \int_0^1 t\cdot\frac{1}{4}t^4\sqrt{2+t^6}\,dt = \frac{1}{4}\int_0^1 t^5\sqrt{2+t^6}\,dt.$$

You could use the substitution $u = 2 + t^6$ to integrate. If you did it without, it would look like this:

$$= \frac{1}{24}\int_0^1 6t^5\sqrt{2+t^6}\,dt = \frac{1}{24}\cdot\frac{2}{3}(2+t^6)^{3/2}\Big|_0^1 = \frac{1}{36}\big(3^{3/2} - 2^{3/2}\big).$$

2a. (Source: 16.2.39, 16.3.21) $\int \mathbf{G}\cdot d\mathbf{r}$ is independent of path, since $\mathbf{G} = \nabla(y + xy^2)$, so all we need is to note that C begins at $(0,0)$ at time $t = 0$ and ends at $(-1 + \pi/2, 1)$ at time $t = \pi/2$ and we can use the Fundamental Theorem for line integrals.

$$\int_C \nabla(xy^2)\cdot d\mathbf{r} = (y + xy^2)\Big|_{(0,0)}^{(-1+\pi/2,1)} = \frac{\pi}{2}.$$

2b. Since $\mathbf{G}$ is conservative and C is a closed path, $\int_C \mathbf{G}\cdot \mathbf{r} = 0$.

3a. (Source: 16.5.7)

$$\operatorname{div}\mathbf{F} = \nabla\cdot\mathbf{F} = \left\langle \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right\rangle \cdot \langle \ln(xy), yz, x\rangle$$
$$= (\ln x + \ln y)_x + (yz)_y + (x)_z = \frac{1}{x} + z.$$

3b. (Source: 16.5.7)

$$\operatorname{curl}\mathbf{F} = \nabla \times \mathbf{F} = \left\langle \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right\rangle \times \langle \ln(xy), yz, x \rangle = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \ln(xy) & yz & x \end{vmatrix}$$

$$= \mathbf{i}\begin{vmatrix} \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ yz & x \end{vmatrix} - \mathbf{j}\begin{vmatrix} \frac{\partial}{\partial x} & \frac{\partial}{\partial z} \\ \ln(xy) & x \end{vmatrix} + \mathbf{k}\begin{vmatrix} \frac{\partial}{\partial x} & \frac{\partial}{\partial y} \\ \ln(xy) & yz \end{vmatrix} = -y\mathbf{i} - \mathbf{j} + \frac{1}{y}\mathbf{k}$$

3c. (Source: 16.5.12) grad $\mathbf{F}$ doesn't exist, since $\mathbf{F}$ is not a scalar field.

3d. (Source: 16.5.13-18) Does not exist. $\mathbf{F}$ is not conservative, since curl $\mathbf{F} \neq \mathbf{0}$.

4. (Source: 16.4.13) By Green's Theorem, the curve integral is the same as the double integral of $Q_x - P_y = 2 + x$ over the interior of the rectangle: $\int_0^2 \int_0^3 (2 + x)\, dy\, dx$.

5. (Source: 16.6.33) To find an equation of a plane, we need a point on the plane and a vector normal to the plane. For the point, use the point of tangency, $\mathbf{r}(0, 1) = \langle 1, 3, -1 \rangle$, and for the normal vector, use

$$\mathbf{r}_u \times \mathbf{r}_v = \langle 1, 2u, 3u^2 \rangle \times \langle 2v, 3, -1 \rangle = \langle 1, 0, 0 \rangle \times \langle 2, 3, -1 \rangle = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 0 & 0 \\ 2 & 3 & -1 \end{vmatrix} = \mathbf{j} + 3\mathbf{k}.$$

The equation of the plane is $0(x - 1) + 1(y - 3) + 3(z - (-1)) = 0$, or $y + 3z = 0$.

6. (Source: 16.7.5) Letting $f(x, y) = 1 + 2x - y$, we can parametrize E by $\mathbf{r}(x, y) = \langle x, y, f(x, y) \rangle$. Then $dS = \sqrt{1 + f_x^2 + f_y^2}\, dA$ and the surface integral is

$$\int_0^2 \int_0^1 y\sqrt{1 + 2^2 + (-1)^2}\, dy\, dx = \sqrt{6} \int_0^2 \int_0^1 y\, dy\, dx = \sqrt{6} \cdot 2 \cdot \frac{1}{2} = \sqrt{6}.$$

7. (Source: 16.7.21) The flux of $\mathbf{F}$ is the surface integral $\iint_E \mathbf{F} \cdot \mathbf{n}\, dS$. Parametrize the surface by $\mathbf{r}(x, y) = \langle x, y, 1 + 2x - y \rangle$. Then

$$\mathbf{n}\, dS = \mathbf{r}_x \times \mathbf{r}_y\, dx\, dy = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 0 & 2 \\ 0 & 1 & -1 \end{vmatrix}\, dx\, dy = \langle -2, 1, 1 \rangle\, dx\, dy.$$

Note that positive $\mathbf{k}$ coefficient means that $\langle -2, 1, 1 \rangle$ is the upward-bound normal vector, as required in the problem. The surface integral is

$$\begin{aligned} \int_0^2 \int_0^1 \langle e^x, z, -y \rangle \cdot \langle -2, 1, 1 \rangle\, dy\, dx &= \int_0^2 \int_0^1 (-2e^x + z - y)\, dy\, dx \\ &= \int_0^2 \int_0^1 (-2e^x + 1 + 2x - 2y)\, dy\, dx. \\ &= \int_0^2 (-2e^x + 1 + 2x - 1)\, dx \\ &= (-2e^x + 2x)\, dx = -2e^2 + 2e^0 + 4 = 6 - 2e^2 \end{aligned}$$