No notes, books, electronic devices, or outside materials of any kind.

Read each problem carefully and simplify your answers.

Supporting work will be required on every problem worth more than 2 points.

A mistake early in your solution does not rule out your receiving full credit for later steps.

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1 (6 pts). Compute the Jacobian $\frac{\partial(x,y)}{\partial(u,v)}$ of the transformation $x = e^{u+v}$, $y = e^{2u-v}$.

2 (10 pts). Calculate the iterated integral: $\int_0^1 \int_{-1}^0 x\sqrt{x^2 - y}\, dy\, dx$

3 (16 pts). Find the maximum and minimum of $x - y^2$ along the curve $x^2 + 4y^2 = 16$.

4 (16 pts). Calculate $\iiint_S \sqrt{x^2 + y^2 + z^2}\, dV$ where S is the sphere of radius 2 centered at $(0, 0, 0)$.

5 (16 pts). Calculate $\iiint_Q y^2\, dV$, where Q is the region inside the cylinder $x^2 + y^2 = 4$ between $z = 0$ and $z = \sqrt{x^2 + y^2}$.

6 (10 pts). Let P be the solid between beneath the surface $z = xy$ and above the triangle in the xy-plane with vertices $(1, 1)$, $(2, 0)$, and $(2, 4)$. Express the volume of P as an iterated integral (or integrals), but **do not evaluate**.

7 (10 pts). Let R be the rectangle $[1, 3] \times [0, 4]$. Calculate a Riemann sum to estimate $\iint_R xy\, dA$ using $m = n = 2$ subintervals in each variable, choosing the sample points to be the upper left corners of each subrectangle (as viewed in the xy-plane).

8 (16 pts). Let D be the rectangle in the xy-plane bounded by the lines

$$x + y = 0 \qquad x + y = 1 \qquad x - 2y = 1 \qquad x - 2y = 3.$$

Write $\iint_D x\, dA$ as an iterated integral in the variables $u = x + y$ and $v = x - 2y$, but **do not evaluate**.

1.
$$\frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} x_u & x_v \\ y_u & y_v \end{vmatrix} = \begin{vmatrix} e^{u+v} & e^{u+v} \\ 2e^{2u-v} & -e^{2u-v} \end{vmatrix} = e^{u+v}e^{2u-v}\begin{vmatrix} 1 & 1 \\ 2 & -1 \end{vmatrix} = -3e^{3u}.$$

(Source: 15.9.4)

2.
$$\begin{aligned}
\int_0^1 \int_{-1}^0 x\sqrt{x^2-y}\,dy\,dx &= \int_0^1 \left(-\frac{2}{3}x(x^2-y)^{3/2}\Big|_{y=-1}^{y=0} \right) dx \\
&= -\frac{2}{3}\int_0^1 \left(x(x^2)^{3/2} - x(x^2+1)^{3/2} \right) dx \\
&= -\frac{2}{3}\int_0^1 \left(x^4 - x(x^2+1)^{3/2} \right) dx \\
&= -\frac{2}{3}\left(\frac{1}{5}x^4 - \frac{1}{5}(x^2+1)^{5/2}\right)\Big|_0^1 \\
&= -\frac{2}{3}\frac{1}{5}(2-2^{5/2}) = \frac{2}{15}(2^{5/2}-2) = \frac{4}{15}(2^{3/2}-1)
\end{aligned}$$

(Source: 15.2.12)

3. Let $f(x,y) = x - y^2$ and $g(x,y) = x^2 + 4y^2$. The max and min of f cann only occur at those points on $g(x,y) = 16$ where $\nabla f = \lambda \nabla g$ for some scalar λ or one of the gradients is zero or fails to exist.

$$\nabla f = \langle 1, -2y \rangle \qquad \nabla g = \langle 2x, 8y \rangle$$

Both of these exist at all (x,y), and the only point where either is zero is the origin $(0,0)$, which is not a point on $g(x,y) = 16$. Look for the solutions (x,y) to the system

$$\begin{aligned}
1 &= \lambda 2x \\
-2y &= \lambda 8y \\
x^2 + 4y^2 &= 16
\end{aligned}$$

From the first, observe that $x \neq 0$, so $\lambda = 1/(2x)$. Substitute this into the second to obtain $-2y = \frac{8y}{2x} = \frac{4y}{x}$, so

$$0 = \frac{4y}{x} + 2y = 2y\left(\frac{2}{x} + 1\right).$$

By setting each factor equal zero we conclude that either $y = 0$ or $x = -2$. The first would imply that $x^2 = 16$, or $x = \pm 4$, and the second would imply $4 + 4y^2 = 16$, or $y = \pm\sqrt{3}$.

Now evaluate f at the four critical points we've found.

(x,y)	$x - y^2$
$(4,0)$	4
$(-4,0)$	-4
$(-2,\sqrt{3})$	-5
$(-2,-\sqrt{3})$	-5

Therefore, the maximum value of f is 4, and the minimum is -5. (Source: 14.8.5)

4. In spherical coordinates, $dV = \rho^2 \sin\phi \, d\rho \, d\phi \, d\theta$, and the integral is

$$\int_0^{2\pi} \int_0^{\pi} \int_0^{2} \rho^3 \sin\phi \, d\rho \, d\phi \, d\theta = \int_0^{2\pi} d\theta \int_0^{\pi} \sin\phi \, d\phi \int_0^{2} \rho^3 \, d\rho$$
$$= 2\pi \left(-\cos\phi \big|_0^{\pi} \right) \left(\frac{1}{4}\rho^4 \big|_0^2 \right)$$
$$= 2\pi \cdot 2 \cdot 4 = 16\pi$$

(Source: 15.8.21)

5. In cylindrical coordinates, $dV = r \, dz \, dr \, d\theta$, $\sqrt{x^2 + y^2} = r$, and $y = r \sin\theta$, so the integral is

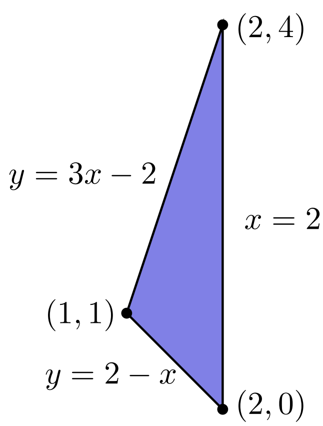
$$\int_0^{2\pi} \int_0^{2} \int_0^{r} r^3 \sin^2\theta dz \, dr \, d\theta = \int_0^{2\pi} \sin^2\theta \, d\theta \int_0^{2} r^4 \, dr$$
$$= \int_0^{2\pi} \sin^2\theta \, d\theta \left(\frac{1}{5} r^5 \big|_0^2 \right) = \frac{32}{5} \int_0^{2\pi} \sin^2\theta \, d\theta.$$

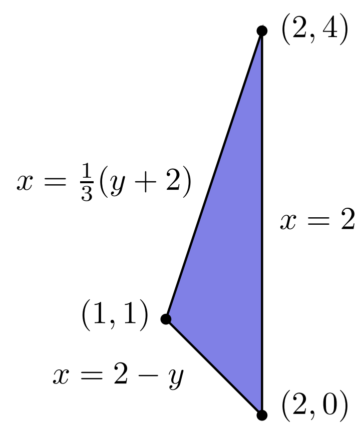
To integrate $\sin^{\theta}$, rewrite it with the half-angle identity.

$$= \frac{32}{5} \int_0^{2\pi} \frac{1}{2}(1 - \cos(2\theta)) \, d\theta = \frac{32}{5} \cdot \frac{1}{2} \left(\theta - \frac{1}{2}\sin(2\theta)\right)\big|_0^{2\pi} = \frac{32}{5}\pi.$$

(Source: 15.7.21)

6. Here's the triangle in question, as well as equations of the three sides, written twice.





It's easiest to put the x integral on the outside. See the figure on the left. You could satisfy the instructions by expressing the volume as either a double or a triple integral.

$$\int_1^{2} \int_{2-x}^{3x-2} \int_0^{xy} dz \, dy \, dx = \int_1^{2} \int_{2-x}^{3x-2} xy \, dy \, dx$$

If you chose to put the y integral on the outside, then, with the equations in the figure on the right, the volume is

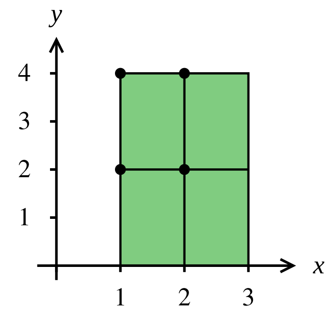
$$\int_0^1 \int_{2-y}^2 \int_0^{xy} dz\,dx\,dy + \int_1^4 \int_{\frac{1}{3}(y+2)}^2 \int_0^{xy} dz\,dx\,dy$$

or

$$\int_0^1 \int_{2-y}^2 xy\,dx\,dy + \int_1^4 \int_{\frac{1}{3}(y+2)}^2 xy\,dx\,dy$$

(Source: 15.3.21)

7. It's helpful to make a drawing of the domain. Here it is, with divided into four subrectangles. The sample point of each rectangle is marked with a •. $\Delta x = 1$ and $\Delta y = 2$ (so that the area of each subrectangle is 2). Letting $f(x, y)$ stand for xy, the Riemann sum is



$$\begin{aligned} &\Delta x \Delta y(f(1,2) + f(1,4) + f(2,2) + f(2,4)) \\ &= 2(2+4+4+8) \\ &= 36. \end{aligned}$$

(Source: 15.1.1)

8. To write x and y in terms of u and v, solve for them in the system

$$\begin{aligned} x + y &= u \\ x - 2y &= v \end{aligned}$$

Subtract the second from the first to obtain $3y = u - v$, so $y = \frac{1}{3}(u - v)$. Substituting this into the first yields $x = u - y = u - \frac{1}{3}(u - v) = \frac{2}{3}u + \frac{1}{3}v$. Compute the Jacobian:

$$\frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} x_u & x_v \\ y_u & y_v \end{vmatrix} = \begin{vmatrix} \frac{2}{3} & \frac{1}{3} \\ \frac{1}{3} & -\frac{1}{3} \end{vmatrix} = \frac{-3}{9} = \frac{-1}{3}.$$

Therefore $dV = |\frac{-1}{3}|\,du\,dv$.

Noting that the rectangle in question is described by $0 \le u \le 1$ and $1 \le v \le 3$, the integral is

$$\int_0^1 \int_1^3 \left(\frac{2}{3}u + \frac{1}{3}v\right) \frac{1}{3}\,dv\,du$$

(Source: 15.9.11,19)