1 (6 pts). Compute the Jacobian \( \frac{\partial(x,y)}{\partial(u,v)} \) of the transformation \( x = e^{u+v}, \ y = e^{2u-v} \).

2 (10 pts). Calculate the iterated integral: \( \int_{0}^{1} \int_{-1}^{0} x \sqrt{x^2 - y} \, dy \, dx \)

3 (16 pts). Find the maximum and minimum of \( x - y^2 \) along the curve \( x^2 + 4y^2 = 16 \).

4 (16 pts). Calculate \( \iiint_{S} \sqrt{x^2 + y^2 + z^2} \, dV \) where \( S \) is the sphere of radius 2 centered at \( (0,0,0) \).

5 (16 pts). Calculate \( \iiint_{Q} y^2 \, dV \), where \( Q \) is the region inside the cylinder \( x^2 + y^2 = 4 \) between \( z = 0 \) and \( z = \sqrt{x^2 + y^2} \).

6 (10 pts). Let \( P \) be the solid between beneath the surface \( z = xy \) and above the triangle in the \( xy \)-plane with vertices \( (1,1), (2,0), \) and \( (2,4) \). Express the volume of \( P \) as an iterated integral (or integrals), but do not evaluate.

7 (10 pts). Let \( R \) be the rectangle \([1, 3] \times [0, 4] \). Calculate a Riemann sum to estimate \( \iint_{R} xy \, dA \) using \( m = n = 2 \) subintervals in each variable, choosing the sample points to be the upper left corners of each subrectangle (as viewed in the \( xy \)-plane).

8 (16 pts). Let \( D \) be the rectangle in the \( xy \)-plane bounded by the lines

\[
x + y = 0 \quad x + y = 1 \quad x - 2y = 1 \quad x - 2y = 3.
\]

Write \( \iint_{D} x \, dA \) as an iterated integral in the variables \( u = x + y \) and \( v = x - 2y \), but do not evaluate.
1. \[
\frac{\partial (x, y)}{\partial (u, v)} = \begin{vmatrix} x_u & x_v \\ y_u & y_v \end{vmatrix} = \begin{vmatrix} e^{u+v} & e^{u+v} \\ 2e^{2u-v} & -e^{2u-v} \end{vmatrix} = e^{u+v}e^{2u-v} \begin{vmatrix} 1 & 1 \\ 2 & -1 \end{vmatrix} = -3e^{3u}.
\]
(Source: 15.9.4)

2. \[
\int_0^1 \int_{-1}^0 x\sqrt{x^2 - y} \, dy \, dx = \int_0^1 \left( -\frac{2}{3} x(x^2 - y)^{3/2} \right)_{y=0}^{y=-1} \, dx = -\frac{2}{3} \int_0^1 (x(x^2)^{3/2} - x(x^2 + 1)^{3/2}) \, dx = -\frac{2}{3} \int_0^1 (x^4 - x(x^2 + 1)^{3/2}) \, dx = -\frac{2}{3} \left( \frac{1}{5} x^4 - \frac{1}{5} (x^2 + 1)^{5/2} \right) \bigg|_0^1 = -\frac{2}{3} \left( 2 - 2^{5/2} \right) = \frac{2}{15} (2^{3/2} - 2) = \frac{4}{15} (2^{3/2} - 1)
\]
(Source: 15.2.12)

3. Let \( f(x, y) = x - y^2 \) and \( g(x, y) = x^2 + 4y^2 \). The max and min of \( f \) can only occur at those points on \( g(x, y) = 16 \) where \( \nabla f = \lambda \nabla g \) for some scalar \( \lambda \) or one of the gradients is zero or fails to exist.
\[ \nabla f = \langle 1, -2y \rangle \quad \nabla g = \langle 2x, 8y \rangle \]
Both of these exist at all \((x, y)\), and the only point where either is zero is the origin \((0, 0)\), which is not a point on \( g(x, y) = 16 \). Look for the solutions \((x, y)\) to the system
\[
\begin{align*}
1 &= \lambda 2x \\
-2y &= \lambda 8y \\
x^2 + 4y^2 &= 16
\end{align*}
\]
From the first, observe that \( x \neq 0 \), so \( \lambda = 1/(2x) \). Substitute this into the second to obtain
\[
-2y = \frac{8y}{2x} = \frac{4y}{x}, \text{ so } 0 = \frac{4y}{x} + 2y = 2y \left( \frac{2}{x} + 1 \right).
\]
By setting each factor equal zero we conclude that either \( y = 0 \) or \( x = -2 \). The first would imply that \( x^2 = 16 \), or \( x = \pm 4 \), and the second would imply \( 4 + 4y^2 = 16 \), or \( y = \pm \sqrt{3} \).

Now evaluate \( f \) at the four critical points we’ve found.
<table>
<thead>
<tr>
<th>((x, y))</th>
<th>(x - y^2)</th>
</tr>
</thead>
<tbody>
<tr>
<td>((4, 0))</td>
<td>4</td>
</tr>
<tr>
<td>((-4, 0))</td>
<td>-4</td>
</tr>
<tr>
<td>((-2, \sqrt{3}))</td>
<td>-5</td>
</tr>
<tr>
<td>((-2, -\sqrt{3}))</td>
<td>-5</td>
</tr>
</tbody>
</table>

Therefore, the maximum value of \( f \) is 4, and the minimum is -5. (Source: 14.8.5)
4. In spherical coordinates, \( dV = \rho^2 \sin \phi \, d\rho \, d\phi \, d\theta \), and the integral is

\[
\int_0^{2\pi} \int_0^\pi \int_0^2 \rho^3 \sin \phi \, d\rho \, d\phi \, d\theta = \int_0^{2\pi} \theta \int_0^\pi \sin \phi \, d\phi \int_0^2 \rho^3 \, d\rho \\
= 2\pi \left( -\cos \phi \right|_0^\pi \left( \frac{1}{4} \rho^4 \right|_0^2 \right) \\
= 2\pi \cdot 2 \cdot 4 = 16\pi
\]

(Source: 15.8.21)

5. In cylindrical coordinates, \( dV = r \, dz \, dr \, d\theta \), \( \sqrt{x^2 + y^2} = r \), and \( y = r \sin \theta \), so the integral is

\[
\int_0^{2\pi} \int_0^2 \int_0^r r^3 \sin^2 \theta \, dz \, dr \, d\theta = \int_0^{2\pi} \sin^2 \theta \, d\theta \int_0^2 r^4 \, dr \\
= \int_0^{2\pi} \sin^2 \theta \, d\theta \left( \frac{1}{5} r^5 \right|_0^2 = \frac{32}{5} \int_0^{2\pi} \sin^2 \theta \, d\theta.
\]

To integrate \( \sin^\theta \), rewrite it with the half-angle identity.

\[
= \frac{32}{5} \int_0^{2\pi} \frac{1}{2} (1 - \cos(2\theta)) \, d\theta = \frac{32}{5} \cdot \frac{1}{2} (\theta - \frac{1}{2} \sin(2\theta)) \bigg|_0^{2\pi} = \frac{32}{5} \pi.
\]

(Source: 15.7.21)

6. Here’s the triangle in question, as well as equations of the three sides, written twice.

\[
y = 3x - 2 \\
(1,1) \\
x = 2 \\
(2,0) \\
y = 2 - x \\
(2,4)
\]

\[
x = \frac{1}{3} (y + 2) \\
(1,1) \\
x = 2 - y \\
(2,0) \\
x = 2 \\
(2,4)
\]

It’s easiest to put the \( x \) integral on the outside. See the figure on the left. You could satisfy the instructions by expressing the volume as either a double or a triple integral.

\[
\int_1^2 \int_{2-x}^{3x-2} \int_0^{xy} dz \, dy \, dx = \int_1^2 \int_{2-x}^{3x-2} xy \, dy \, dx
\]
If you chose to put the \( y \) integral on the outside, then, with the equations in the figure on the right, the volume is

\[
\int_0^1 \int_{2-y}^2 \int_0^{xy} dz \, dx \, dy + \int_1^4 \int_{\frac{1}{3}(y+2)}^2 \int_0^{xy} dz \, dx \, dy
\]

or

\[
\int_0^1 \int_{2-y}^2 xy \, dx \, dy + \int_1^4 \int_{\frac{1}{3}(y+2)}^2 xy \, dx \, dy
\]

(Source: 15.3.21)

7. It’s helpful to make a drawing of the domain. Here it is, with divided into four subrectangles. The sample point of each rectangle is marked with a •. \( \Delta x = 1 \) and \( \Delta y = 2 \) (so that the area of each subrectangle is 2). Letting \( f(x, y) \) stand for \( xy \), the Riemann sum is

\[
\Delta x \Delta y (f(1, 2) + f(1, 4) + f(2, 2) + f(2, 4))
\]

\[
= 2(2 + 4 + 4 + 8)
\]

\[
= 36.
\]

(Source: 15.1.1)

8. To write \( x \) and \( y \) in terms of \( u \) and \( v \), solve for them in the system

\[
x + y = u
\]
\[
x - 2y = v
\]

Subtract the second from the first to obtain \( 3y = u - v \), so \( y = \frac{1}{3}(u - v) \). Substituting this into the first yields \( x = u - y = u - \frac{1}{3}(u - v) = \frac{2}{3}u + \frac{1}{3}v \). Compute the Jacobian:

\[
\frac{\partial(x, y)}{\partial(u, v)} = \begin{vmatrix} x_u & x_v \\ y_u & y_v \end{vmatrix} = \begin{vmatrix} \frac{2}{3} & \frac{1}{3} \\ \frac{1}{3} & -\frac{1}{3} \end{vmatrix} = \frac{-3}{9} = \frac{-1}{3}.
\]

Therefore \( dV = \left| \frac{-1}{3} \right| \, du \, dv \).

Noting that the rectangle in question is described by \( 0 \leq u \leq 1 \) and \( 1 \leq v \leq 3 \), the integral is

\[
\int_0^1 \int_1^3 \left( \frac{2}{3}u + \frac{1}{3}v \right) \frac{1}{3} \, dv \, du
\]

(Source: 15.9.11,19)