MATH 221–02 (Kunkle), Exam 2 Name: ______________________
100 pts, 75 minutes Mar. 10, 2015

No notes, books, electronic devices, or outside materials of any kind.

Read each problem carefully and simplify your answers.

Supporting work will be required on every problem worth more than 2 points.

A mistake early in your solution does not rule out your receiving full credit for later steps.

---

1 (10 pts). One of the limits below exists and the other does not. **Choose one**, and then evaluate that limit or show that it does not exist. Be sure to state clearly which limit you've chosen to work on.

a. $\displaystyle\lim_{(x,y)\to(0,0)} \frac{x^4 - 4y^4}{x^2 + 2y^2}$ b. $\displaystyle\lim_{(x,y)\to(0,0)} \frac{x^2y^2}{x^4 + 4y^4}$

2. Suppose that $\mathbf{r}(t) = \langle \sin t, \cos t, t^2 \rangle$ is the position at time t of a particle.

a (9 pts). Express the velocity, acceleration, and speed of the particle as functions of t.

b (9 pts). Find the tangential and normal components of $\mathbf{a}$ at time $t = \pi$.

3. Let $f(x, y) = xe^{y/x}$.

a (10 pts). Find f_x, f_y, f_{xx}, f_{xy}, and f_{yy}. Simplify your answers, and label them so I can tell which is which.

b (6 pts). Find the linearization of $f(x, y)$ at the point $(2, 1)$.

c (6 pts). Find $\dfrac{\partial f}{\partial s}$ if $x = e^t \tan s$ and $y = \ln(s^2 t)$. You may express your answer using all four variable x, y, s, and t.

4. Let $h(x, y, z) = xz - yx^2 + \cos(y + z)$.

a (6 pts). Find the gradient ∇h.

b (6 pts). Find the directional derivative of $h(x, y, z)$ at the point $(2, 1, -1)$ in the direction of $\mathbf{u} = \langle 1, 0, 2 \rangle$.

c (5 pts). Find the maximum rate of change of h at the point $(2, 1, -1)$ and the direction in which occurs. You can state the direction of any non-zero vector; it needn't be a unit vector.

d (5 pts). Find the equation of the plane tangent to the surface $xz - yx^3 + \cos(y + z) = -9$ at the point $(2, 1, -1)$.

5 (12 pts). Find the (x, y)-location of all local maximums and minimums and saddle points of

$$S(x, y) = x^2 - y^3 - 2x + 12y.$$

(You are not required to compute any of the local maximum or minimum values that you locate.)

6 (16 pts). Determine the absolute maximum and minimum values of

$$S(x, y) = x^2 - y^3 - 2x + 12y$$

on the rectangle $\{(x, y) : 0 \le x \le 2, 0 \le y \le 3\}$.

1a. (Source: 14.2.14) The denominator is a factor of the numerator.

$$\lim_{(x,y)\to(0,0)} \frac{x^4-4y^4}{x^2+2y^2} = \lim_{(x,y)\to(0,0)} \frac{(x^2-2y^2)(x^2+2y^2)}{x^2+2y^2} = \lim_{(x,y)\to(0,0)} (x^2+2y^2) = 0.$$

b. (Source: 14.2.12) As $(x,y)\to(0,0)$ along the x-axis, the limit is

$$\lim_{(x,0)\to(0,0)} \frac{x^2\cdot 0}{x^4} = \lim_{(x,0)\to(0,0)} \frac{0}{x^4} = \lim_{(x,0)\to(0,0)} 0 = 0.$$

But as $(x,y)\to(0,0)$ along the line $y=x$, the limit is

$$\lim_{(x,x)\to(0,0)} \frac{x^2x^2}{x^4+4x^4} = \lim_{(x,x)\to(0,0)} \frac{x^2x^2}{x^4+4x^4} = \lim_{(x,x)\to(0,0)} \frac{x^4}{5x^4} = \lim_{(x,x)\to(0,0)} \frac{1}{5} = \frac{1}{5}.$$

Since these limits disagree, the limit does not exist.
2a. (Source: 13.4.10) Velocity $= \mathbf{v} = \mathbf{r}' = \langle \cos t, -\sin t, 2t\rangle$.
Speed $= \frac{ds}{dt} = |\mathbf{v}| = \sqrt{\cos^2 t + \sin^2 t + 4t^2} = \sqrt{1+4t^2}$.
Acceleration $= \mathbf{a} = \mathbf{r}'' = \mathbf{v}' = \langle -\sin t, -\cos t, 2\rangle$.

b. (Source: 13.4.35) $\mathbf{v}(\pi) = \langle -1, 0, 2\pi\rangle$ and $\mathbf{a}(\pi) = \langle 0, 1, 2\rangle$, so

$$a_T = (\mathbf{a}\cdot\mathbf{v})/|\mathbf{v}| = \frac{4\pi}{\sqrt{1+4\pi^2}}$$

and

$$a_N = \sqrt{|\mathbf{a}|^2 - a_T^2} = \sqrt{5 - 16\pi^2/(1+4\pi^2)} = \sqrt{\frac{5+4\pi^2}{1+4\pi^2}}.$$

Could instead find a_T by differentiating speed

$$a_T = \frac{d^2s}{dt^2} = \frac{d}{dt}\sqrt{1+4t^2} = (1/2)8t(1+4t^2)^{-1/2} = \frac{4t}{\sqrt{1+4t^2}} = \frac{4\pi}{\sqrt{1+4\pi^2}}$$

and a_N by the formula

$$a_N = |\mathbf{a}\times\mathbf{v}|/|\mathbf{v}| = |\langle -2\pi, 2, -1\rangle|/\sqrt{1+4\pi^2} = \sqrt{\frac{5+4\pi^2}{1+4\pi^2}}.$$

3a. (Source: 14.3.27, 56) If we rewrite $f(x,y)$ as $xe^{yx^{-1}}$, then we won't need to use the quotient rule. Simplify the first derivatives completely before computing the second derivatives.

$$f_x = e^{yx^{-1}} + x(-yx^{-2})e^{yx^{-1}} = (1-yx^{-1})e^{yx^{-1}}$$
$$f_y = x\cdot x^{-1}e^{yx^{-1}} = e^{yx^{-1}}$$

Now take the second derivatives.

$$\begin{aligned} f_{xx} &= ((1-yx^{-1})e^{yx^{-1}})_x = yx^{-2}e^{yx^{-1}} + (1-yx^{-1})(-yx^{-2})e^{yx^{-1}} \\ &= (yx^{-2} - yx^{-2} + y^2x^{-3})e^{yx^{-1}} = y^2x^{-3}e^{yx^{-1}} \\ f_{xy} &= f_{yx} = (e^{yx^{-1}})_x = -yx^{-2}e^{yx^{-1}} \\ f_{yy} &= (e^{yx^{-1}})_y = x^{-1}e^{yx^{-1}} \end{aligned}$$

b. (Source: 14.4.11–16) Evaluate f and its two first derivatives at the point $(2, 1)$:

$$f(2,1) = 2e^{1/2} \qquad f_x(2,1) = (1-yx^{-1})e^{yx^{-1}} = \frac{1}{2}e^{1/2} \qquad f_y(2,1) = e^{yx^{-1}} = e^{1/2}$$

Then the linearization is

$$\begin{aligned} L(x,y) &= f(2,1) + f_x(2,1)(x-2) + f_y(2,1)(y-1) \\ &= 2e^{1/2} + \frac{1}{2}e^{1/2}(x-2) + e^{1/2}(y-1) \end{aligned}$$

(You could rewrite this as $(y + \frac{1}{2}x)e^{1/2}$, but that was not required.)

c. (Source: 14.5.7–12) This calls for the Chain Rule. It will help to rewrite $y = 2\ln s + \ln t$.

$$\begin{aligned} f_s &= f_x x_s + f_y y_s \\ &= (1-yx^{-1})e^{yx^{-1}} e^t \sec^2 s + e^{yx^{-1}} \cdot \frac{2}{s} \end{aligned}$$

(This doesn't simplify much at all. You could factor out the exponential but I didn't require it when grading.)

4.a. (Source: 14.6.9–10) $\nabla h = \langle h_x, h_y, h_z\rangle = \langle z - 2yx, -x^2 - \sin(y+z), x - \sin(y+z)\rangle$.

b. (Source: 14.6.15–17) Evaluate ∇h at $(2, 1, -1)$:

$$\nabla h(2,1,-1) = \langle -5, -4, 2\rangle.$$

Normalize $\mathbf{u}$ to get the unit vector $\frac{1}{|\mathbf{u}|}\mathbf{u} = \frac{1}{\sqrt{5}}\langle 1, 0, 2\rangle$. The directional derivative is the dot product of this with the gradient:

$$\frac{1}{\sqrt{5}}\langle 1,0,2\rangle \cdot \langle -5,-4,2\rangle = \frac{1}{\sqrt{5}}(-5-4) = -\frac{1}{\sqrt{5}}.$$

c. (Source: 14.6.24–26) The greatest rate of change of h is $|\nabla h(2,1,-1)| = \sqrt{25+16+4} = \sqrt{45} (= 3\sqrt{5})$, and this occurs in the direction of the gradient $\langle -5, -4, 2\rangle$.

d. (Source: 14.6.39–44) To find the equation of a plane, we need a point on the plane and a vector normal to the plane. The surface in question is a level surface of the function on the left side, so the gradient is normal to the surface.
The gradient of the function of the left side

$$\nabla(xz - yx^3 + \cos(y+z)) = \langle z - 3yx^2, -x^3 - \sin(y+z), x - \sin(y+z)\rangle,$$

evaluated at $(2, 1, -1)$ is

$$= \langle -13, -8, 2\rangle.$$

The plane passing through the point $(2, 1, -1)$ with this normal vector is

$$-13(x-2) - 8(y-1) + 2(z+1) = 0.$$

Note. I originally intended part d. to use the same function as in parts a-c. If you solved part d. using the same function as in a-c., you got full credit. If you solved the problem as it was actually written, you got full credit plus 4 bonus points for the extra work this required.

5. (Source: 14.7.5–7, 10, 11) Find all critical points by setting S_x and S_y equal zero and solving:

$$S_x = 2x - 2 = 0 \Longrightarrow x = 1. \qquad S_y = 12 - 3y^2 = 0 \Longrightarrow y^2 = 4 \Longrightarrow y = \pm 2.$$

so there are two critical points, $(1, 2)$ and $(1, -2)$. Perform the second derivative test at each of these. Start by finding D.

$$\begin{aligned} D(x,y) &= \begin{vmatrix} S_{xx} & S_{xy} \\ S_{yx} & S_{yy} \end{vmatrix} \\ &= S_{xx}S_{yy} - S_{xy}^2 = (2)(-6y) - 0 = -12y. \\ D(1,2) &= -24 \\ D(1,-2) &= 24 \end{aligned}$$

Therefore S has a saddle point at $(1, 2)$ and *either* a local max or a local min at $(1, -2)$. Since $S_{yy} = -6y = 12 > 0$ there, S has a local min at $(1, -2)$.

6 (1 pts). (Source: 14.7.36) The only critical point inside the rectangle is $(1, 2)$, and

$$S(1, 2) = 15.$$

Now find the max and min of S along the boundary of the rectangle. Remember from calculus 1 that the max and min of a function of one variable along a closed interval can only occur at the endpoints and at interior critical points.
Segment 1: $y = 0$, $0 \le x \le 2$. Here, $S(x, 0) = x^2 - 2x$. Setting the derivative $2x - 2$ equal 0, we find (as in Problem 5) that the only critical number is $x = 1$:

$$S(0,0) = 0 \qquad S(1,0) = -1 \qquad S(2,0) = 0.$$

Segment 2: $y = 3$, $0 \le x \le 2$. Here, $S(x, 3) = x^2 - 2x + 9$. Again, the only critical number is $x = 1$:

$$S(0,3) = 9 \qquad S(1,3) = 8 \qquad S(2,3) = 9.$$

Segment 4: $x = 0$, $0 \le y \le 3$: Here $S(0, y) = 12y - y^3$. Again, setting $12 - 3y^2 = 0$ yields only $y = 2$. We've already computed S at the endpoints of this segment above, so we need only compute

$$S(0,2) = 16.$$

Segment 3: $x = 2$, $0 \le y \le 3$: Here $S(2, y)$ also equals $12y - y^3$, so the only critical number is $y = 2$. We've already computed S at the endpoints of this segment, so we need only compute

$$S(2,2) = 16.$$

Since the absolute max and min must be at an interior critical point or on the boundary, $\max S = 16$ and $\min S = -1$.