MATH 221–02 (Kunkle), Exam 1 Name: ______________________

100 pts, 75 minutes Feb. 3, 2015

No notes, books, electronic devices, or outside materials of any kind.

Read each problem carefully and simplify your answers.

Supporting work will be required on every problem worth more than 2 points.

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1 (4 pts). Find the equation of the sphere centered at $(2, 1, -1)$ and passing through the point $(5, 2, 0)$.

2. Let $\mathbf{m} = \langle 3, 1, 1\rangle$ and $\mathbf{n} = \langle -3, -2, 5\rangle$. Find the following.

a (4 pts). A unit vector in the same direction as $\mathbf{m}$.

b (1 pts). $\mathbf{m} \cdot \mathbf{n}$

c (6 pts). The vector projection of $\mathbf{m}$ onto $\mathbf{n}$.

3. Find an equation (in any form you like) of the line described in each part.

a (6 pts). The line passing through the points $(0, 1, 3)$ and $(5, 4, 7)$.

b (10 pts). The line of intersection between the two planes $x+y-4z = 2$ and $x+2y+z = 0$.

4 (10 pts). Find an equation of the plane passing through the points $(0, 3, -1)$, $(1, -1, 0)$, and $(2, 4, 1)$.

5. This problem is about the graph of the equation $x^2 - y^2 - 4z^2 = 0$.

a (3 pts). Find the coordinates of all intercepts of this surface with the coordinate axes.

b (3 pts). Complete the following statements by circling the correct word.

Cross-sections at $x =$ constant (when they exist) are	*parabolas*	*hyperbolas*	*ellipses*
Cross-sections at $y =$ constant (when they exist) are	*parabolas*	*hyperbolas*	*ellipses*
Cross-sections at $z =$ constant (when they exist) are	*parabolas*	*hyperbolas*	*ellipses*

c (1 pts). What is the name for such a surface?

d (8 pts). Make a sketch of the surface. Label your axes x y z and use arrows to indicate the positive direction along each.

6a (6 pts). Find an equation of the line tangent to the curve $\mathbf{r} = \langle 2\cos t, t, 2\sin t\rangle$ at the point corresponding to $t = \pi$.

6b (10 pts). Find $\mathbf{T}$ and $\mathbf{N}$ (as functions of t) along the curve in 6a.

7 (14 pts). Find the length of the curve $\mathbf{r}(t) = \langle t^2, \cos t + t\sin t, \sin t - t\cos t\rangle$ for $0 \le t \le 2$.

8 (14 pts). Find κ (as a function of t) along the curve $\mathbf{r} = \langle t, \frac{2}{3}t^{3/2}, \frac{2}{3}t^{3/2}\rangle$.

1. (Source: 12.1.13) The equation of a sphere centered at (h, k, ℓ) with radius ρ is

$$(x-h)^2 + (y-k)^2 + (z-\ell)^2 = \rho^2.$$

The center is given, and the radius is the distance between $(2, 1, -1)$ and $(5, 2, 0)$, or $\sqrt{(5-2)^2 + (2-1)^2 + (0-(-1))^2} = \sqrt{11}$. Equation of sphere is therefore

$$(x-2)^2 + (y-1)^2 + (z+1)^2 = 11.$$

2a. (Source: 12.2.21-23) $\frac{1}{\|\mathbf{m}\|}\mathbf{m} = \frac{1}{\sqrt{11}}\langle 3, 1, 1\rangle$. b. (Source: 12.3.3-9) $\langle 3, 1, 1\rangle \cdot \langle -3, -2, 5\rangle = 3(-3) + 1(-2) + 1(5) = -6$.
c. (Source: 12.3.35-37) $\frac{\mathbf{m}\cdot\mathbf{n}}{\mathbf{n}\cdot\mathbf{n}} = \frac{-6}{38} = \frac{-3}{19}$, so the projection is $\frac{\mathbf{m}\cdot\mathbf{n}}{\mathbf{n}\cdot\mathbf{n}}\mathbf{n} = \frac{-3}{19}\langle -3, -2, 5\rangle$ (or $\langle \frac{9}{19}, \frac{6}{19}, \frac{-15}{19}\rangle$).
3a. (Source: 12.5.6-9) The line is parallel the vector $\langle 5-0, 4-1, 7-3\rangle = \langle 5, 3, 4\rangle$ (as well as any nonzero multiple of this vector). In any form, your answer depends on which point you choose on the line. I'll use $(0, 1, 3)$. Then, in vector form, the line would be $\mathbf{r} = \langle 0, 1, 3\rangle + t\langle 5, 3, 4\rangle$. In parametric form, it's $x = 5t; y = 1+3t; z = 3+4t$. In symmetric form, $\frac{x}{5} = \frac{y-1}{3} = \frac{z-3}{4}$.
b. (Source: 12.5.55-58) First, you can find a point on the line by setting one variable equal zero (or any number you want) and solving for the others. Let $z = 0$. Then $x + y = 2$ and $x + 2y = 0$. Subtract the first from the second and get $y = -2$; plug this into either and find $x = 4$.
Now, to find a vector parallel the line, note that since the line lies on both planes, it must be orthogonal to both normal vectors of the planes, and is therefore parallel to their cross product.

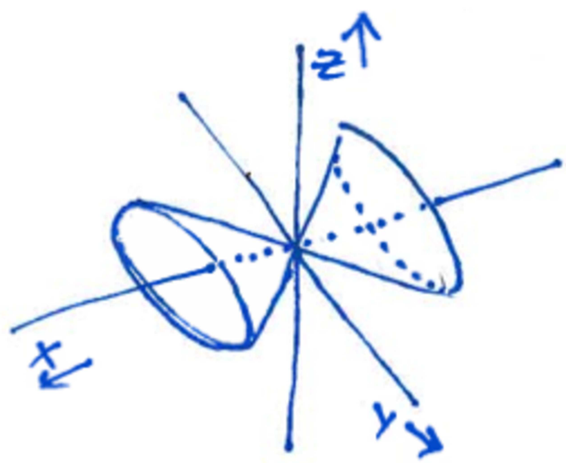
$$\begin{aligned}\langle 1, 1, -4\rangle \times \langle 1, 2, 1\rangle = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 1 & -4 \\ 1 & 2 & 1 \end{vmatrix} &= \mathbf{i}\begin{vmatrix} 1 & -4 \\ 2 & 1 \end{vmatrix} - \mathbf{j}\begin{vmatrix} 1 & -4 \\ 1 & 1 \end{vmatrix} + \mathbf{k}\begin{vmatrix} 1 & 1 \\ 1 & 2 \end{vmatrix} \\ &= 9\mathbf{i} - 5\mathbf{j} + \mathbf{k}\end{aligned}$$

Finally, the line (in parametric form) is $x = 4 + 9t$; $y = -2 - 5t$; $z = t$.
4. (Source: 12.5.31-33) To find an equation of a plane, we need a point on the plane and a normal vector. For a normal vector, we can use the cross product of two of the vectors between the three points.

$$\begin{aligned}&\langle 1-0, -1-3, 0-(-1)\rangle \times \langle 2-1, 4-(-1), 1-0\rangle \\ &= \langle 1, -4, 1\rangle \times \langle 1, 5, 1\rangle = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & -4 & 1 \\ 1 & 5 & 1 \end{vmatrix} \\ &= \mathbf{i}\begin{vmatrix} -4 & 1 \\ 5 & 1 \end{vmatrix} - \mathbf{j}\begin{vmatrix} 1 & 1 \\ 1 & 1 \end{vmatrix} + \mathbf{k}\begin{vmatrix} 1 & -4 \\ 1 & 5 \end{vmatrix} = -9\mathbf{i} + 0\mathbf{j} + 9\mathbf{k}\end{aligned}$$

which means we could also use $\langle -1, 0, 1\rangle$. With this normal, and using the point $(0, 3, -1)$ on the plane, we get the equation $-x + (z+1) = 0$ (or $x - z = 1$).

5. (Source: 12.6.12, 26, 32) a. Setting any two of the variables $= 0$ results in the third also $= 0$. Therefore, $(0,0,0)$ is the only intercept.
b. Cross-sections at $x =$ constant (when they exist) are *ellipses*.
Cross-sections at $y =$ constant (when they exist) are *hyperbolas*.
Cross-sections at $z =$ constant (when they exist) are *hyperbolas*.
c. Cone.
d. The cone is symmetric through the $x-axis$. Here's a drawing by hand.



6a. (Source: 13.2.23-26) $\mathbf{r} = \langle 2\cos t, t, 2\sin t\rangle$; $\mathbf{r}' = \langle -2\sin t, 1, 2\cos t\rangle$;
At $t = \pi$, $\mathbf{r} = \langle -2, \pi, 0\rangle$ and the tangent vector $\mathbf{r}' = \langle 0, 1, -2\rangle$. The tangent line is

$$x = -2; \qquad y = \pi + t; \qquad z = -2t.$$

6b. (Source: 13.3.17-20) Continuing, $|\mathbf{r}'| = \sqrt{4\cos^2 t + 1 + 4\sin^2 t} = \sqrt{5}$;
$\mathbf{T} = \mathbf{r}'/|\mathbf{r}'| = \frac{1}{\sqrt{5}}\langle -2\sin t, 1, 2\cos t\rangle$; $\mathbf{T}' = \frac{1}{\sqrt{5}}\langle -2\cos t, 0, -2\sin t\rangle$.
To find $\mathbf{N}$, normalize $\mathbf{T}'$ (or any positive multiple of $\mathbf{T}'$) to find $\mathbf{N} = \langle -\cos t, 0, -\sin t\rangle$.

7. (Source: 13.3.1-6) $\mathbf{r}' = \langle 2t, t\cos t, t\sin t\rangle$; $|\mathbf{r}'| = \sqrt{4t^2 + t^2\cos^2 t + t^2\sin^2 t} = t\sqrt{5}$, so the total length $= \sqrt{5}\int_0^2 t\,dt = \sqrt{5}\frac{1}{2}t^2|_0^2 = 2\sqrt{5}$.

8. (Source: 13.3.21-23) $\mathbf{r} = \langle t, \frac{2}{3}t^{3/2}, \frac{2}{3}t^{3/2}\rangle$. $\mathbf{r}' = \langle 1, t^{1/2}, t^{1/2}\rangle$. $\mathbf{r}'' = \langle 0, \frac{1}{2}t^{-1/2}, \frac{1}{2}t^{-1/2}\rangle$.
The easiest way to find κ is to use the formula $\kappa = \frac{|\mathbf{r}'\times\mathbf{r}''|}{|\mathbf{r}'|^3}$:

$$\mathbf{r}'\times\mathbf{r}'' = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & t^{1/2} & t^{1/2} \\ 0 & \frac{1}{2}t^{-1/2} & \frac{1}{2}t^{-1/2} \end{vmatrix} = 0\mathbf{i} - \frac{1}{2}t^{-1/2}\mathbf{j} + \frac{1}{2}t^{-1/2}\mathbf{k}$$

$$\kappa = \frac{|\mathbf{r}'\times\mathbf{r}''|}{|\mathbf{r}'|^3} = \frac{\sqrt{\frac{1}{4}t^{-1} + \frac{1}{4}t^{-1}}}{(1+t+t)^{3/2}} = \frac{1}{\sqrt{2t}(1+2t)^{3/2}}.$$