1. This question is about the surface \( x^2 - y^2 + \frac{1}{4}z^2 = 1 \).

a (2 pts). When they exist, the traces of this surface in planes \( y = \) constant are

(check one) □ parabolas □ ellipses □ hyperbolas □ lines

b (2 pts). When they exist, the traces of this surface in planes \( z = \) constant are

(check one) □ parabolas □ ellipses □ hyperbolas □ lines

c (2 pts). When they exist, the traces of this surface in planes \( x = \) constant are

(check one) □ parabolas □ ellipses □ hyperbolas □ lines

d (3 pts). Find all intersections of this graph with the coordinate axes.

e (6 pts). Make a rough sketch of this surface on the axes below. Label the axes and the intercepts you found in part d.

2. Suppose a particle has position \( \mathbf{r} = (\cos t + t \sin t, -\sin t + t \cos t, \frac{1}{2}t^2) \) at time \( t > 0 \).

a (11 pts). Express the particle’s velocity, acceleration, and speed as functions of \( t \). (Label your answers so I can tell which is which.)

b (12 pts). Express the vectors \( \mathbf{T}, \mathbf{N}, \) and \( \mathbf{B} \) along the particle’s path as functions of \( t \).

c (8 pts). Find the path’s curvature at time \( t = 2\pi \).

d (10 pts). Find the tangential and normal components of the particle’s acceleration at \( t = 2\pi \). (Label your answers so I can tell which is which.)

e (5 pts). Find the particle’s osculating plane at time \( t = 2\pi \).

3 (10 pts). Find the length of the curve \( x(t) = t - 1 \quad y(t) = \frac{2}{3}t^3 \quad z(t) = -\frac{2}{5}t^5 + 1 \) from \( t = 0 \) to \( t = 1 \).

4 (8 pts). Find an equation of the line tangent to the curve \( \mathbf{r} = (t^2, e^t, \ln t) \) at the point corresponding to \( t = 1 \). Express your answer in parametric form.

5 (3 pts). Suppose \( \mathbf{a} \cdot \mathbf{b} = 0 \). What can you conclude about the vectors \( \mathbf{a} \) and \( \mathbf{b} \)?

6 (3 pts). Suppose \( \mathbf{u} \times \mathbf{v} = 0 \). What can you conclude about the vectors \( \mathbf{u} \) and \( \mathbf{v} \)?

7 (5 pts). Let \( \mathbf{u} = \mathbf{i} - 2\mathbf{j} + \mathbf{k} \) and \( \mathbf{w} = 2\mathbf{i} + \mathbf{j} - \mathbf{k} \). Find the vector projection of \( \mathbf{w} \) onto \( \mathbf{u} \).

8 (5 pts). Find an equation of the plane parallel to \( 2x - 2y + \pi z = 8 \) that passes through \( (1, -2, 1) \).

9 (5 pts). Find a vector parallel the line of intersection of the planes \( x - 2y - 8 = 0 \) and \( 3y + z = 12 + x \).
1. \( x^2 - y^2 + \frac{1}{4} z^2 = 1 \)

- \( x = \text{const} \), \( -y^2 + \frac{1}{4} z^2 = \text{const} \), a hyperbola
- \( y = \text{const} \), \( x^2 + \frac{1}{4} z^2 = \text{const} \), an ellipse
- \( z = \text{const} \), \( x^2 - y^2 = \text{const} \), a hyperbola

\( \frac{d}{dt} \) order differed on different tests.

\( x \)-intercept \( (y = z = 0) \) \( x^2 = 1 \), \( x = \pm 1 \).

\( y \)-intercept \( (x = z = 0) \) \(-y^2 = 1 \). No real solutions, so no \( y \)-intercept.

\( z \)-intercept \( (x = y = 0) \) \( \frac{1}{4} z^2 = 1 \) \( \Rightarrow z = \pm 2 \). (Alt: \( \frac{1}{4} z^2 = 1 \) \( \Rightarrow z = \pm 2 \))

\[ A = (0, 0, 2) \quad (\text{alt} \ (0, 0, 3)) \]
\[ B = (1, 0, 0) \]
\[ C = (0, 0, -2) \quad (\text{alt} \ (0, 0, -3)) \]
\[ D = (-1, 0, 0) \]

2. \( \vec{r} = \langle \cos t + tsin t, -sin t + tcos t, \frac{1}{2} t^2 \rangle \)

a. \( \vec{r}' = \langle t \cos t, -t \sin t, t \rangle = \text{velocity} \)

\[ \vec{r}'' = \langle -t \sin t - \cos t, -t \cos t - \sin t, 1 \rangle = \text{acceleration} \]

\[ |\vec{r}''| = \sqrt{t^2 \cos^2 t + t^2 \sin^2 t + 1} = \sqrt{2t^2} = t \sqrt{2} \quad (\text{since } t > 0) = \text{speed}. \]

b. \( \vec{T} = \frac{\vec{r}'}{|\vec{r}''|} = \frac{\langle t \cos t, -t \sin t, t \rangle}{t \sqrt{2}} = \frac{1}{\sqrt{2}} \langle \cos t, -\sin t, 1 \rangle \)

\[ \vec{N} = \frac{d\vec{r}'}{dt} = \frac{1}{\sqrt{2}} \langle -\sin t, -\cos t, 0 \rangle = \langle -\sin t, -\cos t, 0 \rangle \]

\[ \vec{B} = \vec{T} \times \vec{N} = \begin{vmatrix} i & j & k \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} & 0 \end{vmatrix} = \frac{1}{\sqrt{2}} \cos t \langle i \rangle - \frac{1}{\sqrt{2}} \sin t \langle j \rangle - \frac{1}{2} \langle k \rangle \]
2. alt.  \( \vec{r} = \langle \cos t + t \sin t, \sin t - t \cos t, \frac{1}{2} t^2 \rangle \)

a.  \( \vec{v} = \langle -t \cos t, t \sin t, t \rangle \),  \( \vec{a} = \langle -t \sin t, \cos t - t \cos t, 1 \rangle \)
\[ \frac{ds}{dt} = t \sqrt{2} \text{ (again)} \]

b.  \( \vec{r} = \langle \cos t, \sin t, 1 \rangle / \sqrt{2} \)
\( \vec{n} = \langle -\sin t, \cos t, 0 \rangle \)
\( \vec{B} = \langle \frac{1}{\sqrt{2}} \cos t, \frac{1}{\sqrt{2}} \sin t, \frac{1}{2} \rangle \)

2. continued.

c.  \( @ t = 2 \pi \),  \( \vec{r}' = \langle 2 \pi, 0, 2 \pi \rangle \)
\( \vec{r}'' = \langle 1, -2 \pi, 1 \rangle \)
\[ \kappa = \frac{1}{\sqrt{\left(\frac{\vec{r}' \times \vec{r}''}{|\vec{r}'| |\vec{r}''|}\right)^4}} = \frac{1}{4 \pi} \]
\[ \kappa = \frac{\sqrt{16 \pi^4 + 16 \pi^4}}{(2 \pi \cdot \sqrt{2})^3} = \frac{1}{4 \pi} \]

\( d.* a_T = \frac{d^2 s}{dt^2} = \sqrt{2} \);  \( a_N = \left( \frac{ds}{dt} \right)^2 \kappa = (2 \pi \sqrt{2})^2 \cdot \frac{1}{4 \pi} = 2 \pi \)

(\text{check:}  \ a_T \vec{r} + a_N \vec{n} = \frac{\sqrt{2}}{\sqrt{6}} \langle 1, 0, 1 \rangle + 2 \pi \langle 0, -1, 0 \rangle = \langle 1, -2 \pi, 1 \rangle = \vec{r} \)
\( @ t = 2 \pi \)

d.  \( @ t = 2 \pi \),  position = \( \langle 1, 2 \pi, 2 \pi^2 \rangle \)
\( \vec{B} = \langle \frac{1}{\sqrt{2}}, 0, \frac{1}{\sqrt{2}} \rangle \).
plane is \( \frac{1}{2 \sqrt{2}} (x - 1) + \frac{1}{\sqrt{2}} (z - 2 \pi^2) = 0 \).

2. alt.  c.d.

a.  \( @ t = 2 \pi \),  \( \vec{r}' = \langle 2 \pi, 0, 2 \pi \rangle \)
\( \vec{r}'' = \langle 1, 2 \pi, 1 \rangle \).
\[ \vec{r}' \times \vec{r}'' = \begin{vmatrix}
1 & \frac{1}{2} & \frac{1}{2} \\
1 & 2 \pi & 2 \pi \\
2 \pi & 0 & 2 \pi \\
\end{vmatrix} = -4 \pi^2 \vec{n} + 4 \pi^2 \vec{k}.  \kappa = \frac{1}{4 \pi} \text{ (again)} \]

d.  \( a_T = \frac{d^2 s}{dt^2} = \sqrt{2} \);  \( a_N = \left( \frac{ds}{dt} \right)^2 \kappa = 2 \pi \text{ (again)} \)

e.  \( \vec{r}(2 \pi) = \langle 1, -2 \pi, 2 \pi^2 \rangle \)
\( \vec{B} = \langle \frac{1}{\sqrt{2}}, 0, \frac{1}{\sqrt{2}} \rangle \).  plane is \( \frac{1}{\sqrt{2}} (x - 1) + \frac{1}{\sqrt{2}} (z - 2 \pi^2) = 0 \).

*Note  \( a_T \) also = \( \frac{\vec{a} : \vec{v}}{|v|^2} = \vec{a} \cdot \left( \frac{\vec{v}}{|v|^2} \right) = \vec{a} \cdot \vec{T} \), the scalar projection of \( \vec{a} \) onto \( \vec{T} \).

Likewise  \( a_N = \vec{a} \cdot \vec{N} \),  \( a_N \) also = \( \frac{|\vec{a} : \vec{v}|}{|v|^2} = \sqrt{|\vec{a}|^2 - a_T^2} \).
3. \( s = \int_{t=0}^{t=1} \sqrt{1 + (2t^2)^2 + (-2t^4)^2} \, dt \)
\[
= \int_{t=0}^{t=1} \sqrt{1 + 4t^4 + 4t^8} \, dt
= \int_{1}^{2} \sqrt{(1 + 2t^4)^2} \, dt
\]
\[
= \int_{1}^{2} (1 + 2t^4) \, dt = t + \frac{2}{5}t^5 \bigg|_{1}^{2} = 1 + \frac{2}{5} = \frac{7}{5}.
\]

4. \( \vec{r} = \langle t^2, e^t, \arctan t \rangle \), \( \vec{r}' = \langle 2t, e^t, \frac{1}{1+t^2} \rangle \)

At \( t=1 \), \( \vec{p} = \langle 1, e, 0 \rangle \), \( \vec{r}' = \langle 2, e, 1 \rangle \), equation of line \( \vec{r}' \) and passing through \( (1, e, 0) \) is
\[
X = 1 + 2t, \quad Y = e + te, \quad Z = t.
\]

4a) \( X = e + te, \quad Y = t, \quad Z = 1 + 2t. \)

5. \( \vec{a} \cdot \vec{b} = 0 \Rightarrow \vec{a} \text{ and } \vec{b} \text{ are perpendicular.} \)

6. \( \vec{a} \times \vec{b} = \vec{0} \Rightarrow \vec{a} \text{ and } \vec{b} \text{ are parallel.} \)

5a) \( \vec{a} \times \vec{b} = \vec{0} \Rightarrow \vec{a}, \vec{b} \text{ parallel.} \)

5b) \( \vec{a} \cdot \vec{b} = 0 \Rightarrow \vec{a}, \vec{b} \text{ perpendicular.} \)

7. a) \( \vec{u} = \langle 1, -2, 1 \rangle, \quad \vec{w} = \langle 2, 1, -1 \rangle \)
\[
\vec{u} \cdot \vec{w} = 1 \cdot 2 + (-2) \cdot 1 + 1 \cdot (-1) = 0.
\]
\( \cos \theta = \frac{\vec{a} \cdot \vec{b}}{|\vec{a}| \cdot |\vec{b}|} = \frac{0}{1 \cdot 3} = 0 \).

b) \( \frac{\vec{w} \cdot \vec{a}}{\vec{a} \cdot \vec{a}} \vec{a} = \frac{-1}{6} \langle 1, -2, 1 \rangle = \langle -\frac{1}{6}, \frac{1}{3}, \frac{1}{6} \rangle. \)

7a) \( \vec{p} = \langle 1, -3, 1 \rangle, \quad \vec{w} = \langle 3, 1, -1 \rangle \), \( \theta = \cos^{-1} \left( \frac{-1}{11} \right) \).

b) \( \frac{\vec{w} \cdot \vec{a}}{\vec{a} \cdot \vec{a}} \vec{a} = \frac{-1}{11} \langle 1, -3, 1 \rangle = \langle -\frac{1}{11}, \frac{3}{11}, \frac{-1}{11} \rangle. \)

8. Plane must have the form \( 2x - 2y + 3z = \text{const} \), so
\( 2x - 2y + 3z = 6 + \pi \quad (act), \quad 2x - 2y + 3z = 4 + 2\pi. \)

a) \( \vec{b}_1 = \langle 1, -2, 0 \rangle, \quad \vec{b}_2 = \langle -1, 3, 1 \rangle. \quad \vec{v} = \vec{b}_1 \times \vec{b}_2 = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & -2 & 0 \\ 1 & 3 & 1 \end{vmatrix} = \langle -2, -1, 1 \rangle \)

a) \( \vec{v} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & 2 & 0 \\ -1 & -3 & 1 \end{vmatrix} = \langle -2, -1, 5 \rangle. \)