Definitions. The **Taylor series** *for* $f(x)$ *centered at* a *is the power series*

$$\sum_{k=0}^{\infty} \frac{f^{(k)}(a)}{k!}(x-a)^k.$$

The n*th* **Taylor polynomial** *for* $f(x)$ *is the partial sum*

$$T_n(x) = \sum_{k=0}^{n} \frac{f^{(k)}(a)}{k!}(x-a)^k. \tag{1}$$

The difference between $f(x)$ *and its Taylor Polynomial* $T_n(x)$ *is called the* n*th* **Remainder** *of the Taylor series:*

$$R_n(x) = f(x) - T_n(x).$$

Example 1: The first few Taylor polynomials centered at 0 for $\sin x$

We know that the MacLaurin series for $\sin x$ is

$$\sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!} = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots$$

so the first few Taylor polynomials centered at 0 for $\sin x$ are

$$T_1(x) = 1$$
$$T_3(x) = x - \frac{x^3}{3!}$$
$$T_5(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!}$$
$$T_7(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!}$$

In fact, because the Taylor coefficients of even powers of x are 0,

$$T_0(x) = 0, \quad T_1(x) = T_2(x), \quad T_3(x) = T_4(x), \quad T_5(x) = T_6(x), \quad T_7(x) = T_8(x), \ldots$$

end Example 1

Note that

$$\sum_{k=0}^{\infty} \frac{f^{(k)}(a)}{k!}(x-a)^k = f(x)$$

iff

$$T_n(x) \to f(x) \text{ as } n \to \infty$$

iff

$$R_n(x) \to 0 \text{ as } n \to \infty.$$

So, to determine whether $f(x)$ is the sum of its Taylor's series, we have to determine whether $\lim_{n\to\infty} R_n(x) = 0$.

Generally, it's not possible to take this limit without somehow rewriting R_n. (That's similar to something we saw in section 11.2: the sum of a series is the limit of its partial sums, but we can't find this limit without first getting a workable expression for s_n.) That's where the next theorem comes in handy.

Taylor's Theorem. *If f and all of its derivatives exist in an interval containing a and x, then*

$$R_n(x) = \frac{f^{(n+1)}(c)}{(n+1)!}(x-a)^{n+1}$$

for some number c between a and x.

(In the special case $n = 0$, the nth degree Taylor polynomial is just the constant $f(a)$, and Taylor's theorem (combined with the definition of the $R_n(x)$) says that

$$f(x) - f(a) = f'(c)(x-a)$$

for some number c between a and x, which is the Mean Value Theorem from Calculus I. That is, Taylor's theorem is a generalization of the MVT.)

The remarkable thing about Taylor's theorem is that it says that to obtain $f(x)$ from $T_n(x)$, we add what looks almost like the next term in the Taylor series. That is,

$$f(x) = T_n(x) + R_n(x),$$

and $R_n(x)$ is almost the next term in the series, $\frac{1}{(n+1)!}f^{(n+1)}(a)(x-a)^{n+1}$.

An immediate consequence of Taylor's Theorem is that, if $f(x)$ is a polynomial of degree n or less, then $R_n(x)$ is identically zero. Consequently, T_n and f are the same function. That is, every polynomial is the sum of its own Taylor series, whatever the center.

To see how $T_n(x)$ and $f(x)$ are related in general, first note that, as a polynomial of degree n, $T_n(x)$ is itself a Taylor series with finitely many terms:

$$T_n(x) = \sum_{k=0}^{n} \frac{T_n^{(k)}(a)}{k!}(x-a)^k.$$

When we compare this to (1), we see that T_n and its first n derivatives match f and *its* first n derivatives at $x = a$:

$$\begin{aligned} T_n(a) &= f(a) \\ T_n'(a) &= f'(a) \\ T_n^{(2)}(a) &= f^{(2)}(a) \\ &\vdots \\ T_n^{(n)}(a) &= f^{(n)}(a) \end{aligned}$$

For instance, when $n = 1$,

$$T_1(x) = f(a) + f'(a)(x-a)$$

is the linearization of f at $x = a$, and, since it matches f and f' at $x = a$, its graph is the line tangent to f at a. When $n > 1$, the graph of T_n is "super-tangent" to that of f.

(See the nice figures.)

Instead of Taylor's theorem, our text includes the following fact.

Taylor's Inequality. *If $|f^{(n+1)}(x)| \leq M$ for all x (or for all x within some fixed distance from a) then for all such x,*

$$|R_n(x)| \leq \frac{M}{(n+1)!}|x-a|^{n+1}.$$

Although it doesn't tell the whole story that Taylor's theorem does, Taylor's inequality includes all we actually need to solve convergence and approximation problems in Calc II.

Example 2: *Show that $\sin x$ is the sum of its MacLaurin series.*

A MacLaurin series is just a Taylor series that's centered at zero, so $a = 0$ in this example. Contrary to what you might expect, writing out the MacLaurin series for $\sin x$ isn't helpful in this problem. Instead, let's use Taylor's inequality to show that $R_n(x) \to 0$.

For any value of n, the nth derivative of $f(x) = \sin x$ can only be $\pm \sin x$ or $\pm \cos x$, and so $|f^{(n)}(x)| \leq 1$ for all x, and Taylor's inequality says

$$0 \leq |R_n(x)| \leq \frac{|x|^{n+1}}{(n+1)!}.$$

We know that $\lim_{n\to\infty} \frac{|x|^{n+1}}{(n+1)!} = 0$ for all x by the nth term test, since the Ratio Test tells us that the series $\sum_{n=0}^{\infty} \frac{|x|^{n+1}}{(n+1)!}$ converges. Therefore, since the left side and the right side above both go to zero, $\lim_{n\to\infty} R_n(x) = 0$ by the Squeeze Theorem.

end Example 2

Taylor's Theorem is useful when we want to bound the absolute error between some given f and one of its Taylor polynomials T_n on an interval about the center a:

$$|f(x) - T_n(x)| \leq B \qquad \text{on } [a-b, a+b]$$

Typical problems give us two of $\{n, B, b\}$ and ask for the third.

Example 3: *Bound the absolute error between $f(x) = \ln(1-x)$ and its third Taylor polynomial centered at 1 on the interval $[-0.9, 0.9]$*

The error depends on size of the fourth derivative, $f^{(4)}(x) = -6(1-x)^{-4}$. Note that $|-6(1-x)^{-4}|$ is largest when x is closest to 1, so we can take M to be the upper bound

$$|-6(1-x)^{-4}| \leq 6(1-0.9)^{-4} = M.$$

and therefore

$$|f(x) - T_3(x)| \leq 6 \cdot 10^{-4} \frac{|x-1|^4}{4!},$$

which, on the interval $[-0.9, 0.9]$, is $\leq 6 \cdot 10^{-4} \frac{1.9^4}{4!}$

end Example 3

Example 4: *Let $f(x) = \cos x$ and $a = 0$. Find b so that the absolute error between f and T_2 is at most 10^{-2} on $[-b, b]$.*

We'll get a better answer if we make use of the fact that $T_2(x) = T_3(x)$ for this function, so take $n = 3$. Since the 4th derivative is $-\sin x$, we can take $M = 1$ regardless of b. Taylor's Inequality then implies

$$|\cos x - T_3(x)| \leq \frac{1}{4!}b^4$$

so to make the error less than 10^{-2}, choose b so as to make

$$\frac{1}{4!}b^4 \leq 10^{-2}, \text{ or } b \leq (4!10^{-2})^{1/4}.$$

end Example 4

Example 5: *What degree MacLaurin polynomial approximates e^x on the interval $[-1, 1]$ with an absolute error at most 10^{-5}?*

By error, we mean the difference between the target function e^x and the approximation $T_n(x)$. That is, error is another word for the Taylor remainder $R_n(x)$.

Note that all the derivatives of e^x are e^x, and on the interval $[-1, 1]$,

$$|f^{(n)}(x)| = e^x \leq e^1,$$

so Taylor's inequality says that

$$|R_n(x)| \leq \frac{e^1}{(n+1)!}|x|^{n+1} \leq \frac{e^1}{(n+1)!}1^{n+1} = \frac{e^1}{(n+1)!}$$

We will ensure that the absolute error $|R_n(x)|$ is less than 10^{-5} by picking n so that

$$\frac{e^1}{(n+1)!} < 10^{-5}.$$

The factorial makes it impossible to solve this inequality algebraically, but the left side is decreasing, so we can just start calculating until we find the first n that satisfies this inequality. Here's what I found:

n	$e/(n+1)!$
2	$4.53E-01$
3	$1.13E-01$
4	$2.27E-02$
5	$3.78E-03$
6	$5.39E-04$
7	$6.74E-05$
8	$7.49E-06$

The first n for which $e/(n+1)! < 10^{-5}$ is $n = 8$, so the first MacLaurin polynomial to be within $\pm 10^{-5}$ of e^x on $[-1, 1]$ is the one of degree 8, or

$$1 + x + \frac{1}{2!}x^2 + \frac{1}{3!}x^3 + \frac{1}{4!}x^4 + \frac{1}{5!}x^5 + \frac{1}{6!}x^6 + \frac{1}{7!}x^7 + \frac{1}{8!}x^8.$$

end Example 5