Here are some limits that come in handy in this chapter.

Famous Limits Everybody Should Know (FLESK)

1. If r is a constant, then $\lim_{n\to\infty} r^n = \begin{cases} \infty & \text{if } r > 1, \\ 1 & \text{if } r = 1, \text{ and} \\ 0 & \text{if } -1 < r < 1, \end{cases}$ but does not exist if $r \le -1$.

2. If c is a positive constant, then $\lim_{n\to\infty} c^{1/n} = 1$.

3. If p is a real constant, then $\lim_{x\to\infty} x^p = \begin{cases} \infty & \text{if } p > 0, \\ 1 & \text{if } p = 0, \text{ and} \\ 0 & \text{if } p < 0. \end{cases}$

4. If k is a real constant, then $\lim_{x\to\infty} \left(1 + \frac{k}{x}\right)^x = e^k$.

5. If $p(x)$ and $q(x)$ are polynomials, then

$$\lim_{x\to\infty} \frac{p(x)}{q(x)} = \lim_{x\to\infty} \frac{\text{lead term of } p(x)}{\text{lead term of } q(x)}$$

Consequently, $\lim_{x\to\infty} p(x) = \lim_{x\to\infty} (\text{lead term of } p(x))$

and $$\lim_{x\to\infty} \frac{p(x)}{q(x)} = \begin{cases} \pm\infty & \text{if } \deg p > \deg q, \\ \dfrac{\text{lead coefficient of } p}{\text{lead coefficient of } q} & \text{if } \deg p = \deg q, \text{ and} \\ 0 & \text{if } \deg p < \deg q. \end{cases}$$

6. If $p(x)$ is a polynomial, then $\lim_{x\to\infty} \frac{p(x+1)}{p(x)} = 1$.

7. If $p(x)$ is a nonzero polynomial, then $\lim_{x\to\infty} |p(x)|^{1/x} = 1$.

Example 1: It's important to note that Limit 1 applies only when r is a constant, as in these examples:

$$\lim_{n\to\infty} 3^n = \infty \qquad \lim_{n\to\infty} (-\tfrac{1}{2})^n = 0 \qquad \lim_{n\to\infty} (-1)^n \text{ DNE}$$
$$\lim_{n\to\infty} (1.01)^n = \infty \qquad \lim_{n\to\infty} (0.99)^n = 0 \qquad \lim_{n\to\infty} (-2)^n \text{ DNE}$$

Limit 1 does not apply to $\lim_{n\to\infty} \left(\frac{n+3}{n}\right)^n$. (For that limit, see Example 4 below.)

Example 2: $\lim_{n\to\infty} 2^{1/n} = \lim_{n\to\infty} (1/3)^{1/n} = 1$, but Limit 2 does not apply to $\lim_{n\to\infty} (n^2+1)^{1/n}$. (For that limit, see Example 7.)

Example 3: $\lim_{n\to\infty} n^\pi = \infty = \lim_{x\to\infty} x^{0.001}$, and $\lim_{n\to\infty} n^{-0.001} = 0$.

Example 4: $\lim_{n\to\infty} \left(\frac{n+3}{n}\right)^n = \lim_{n\to\infty} \left(1 + \frac{3}{n}\right)^n = e^3$.

Limit 4 is the classic example of a limit of the form 1^∞ which doesn't equal 1.

Proof: Let $y = \left(1+\frac{k}{x}\right)^x$. Then $\ln y = x\ln\left(1+\frac{k}{x}\right) = \frac{\ln(1+kx^{-1})}{x^{-1}}$ As $x \to \infty$, $\ln y$ is of the indeterminate form $\frac{0}{0}$, so we can try l'Hospital's Rule:

$$\lim_{x\to\infty} \frac{\frac{1}{1+kx^{-1}}(-kx^{-2})}{-x^{-2}} = \lim_{x\to\infty} \frac{\frac{1}{1+kx^{-1}}(k)}{1} = k$$

Therefore $\lim\limits_{x\to\infty} \ln y$ must also equal k, and so $\lim\limits_{x\to\infty} y = \lim\limits_{x\to\infty} e^{\ln y} = e^k$.

Example 5: $\lim\limits_{x\to\infty} \dfrac{4x^3+2x-1}{3x^4-x^3+9x} = \lim\limits_{x\to\infty} \dfrac{4x^3}{3x^4} = \lim\limits_{x\to\infty} \dfrac{4}{3x} = 0.$

$\lim\limits_{x\to\infty} \dfrac{4x^4-2x^3-x}{3x^4-x^3+9x} = \lim\limits_{x\to\infty} \dfrac{4x^4}{3x^4} = \lim\limits_{x\to\infty} \dfrac{4}{3} = \dfrac{4}{3}.$

Proof: To see why Limit 5 is true, suppose that $p(x) = b_m x^m + b_{m-1}x^{m-1} + \cdots b_1 x + b_0$, and $q(x) = c_n x^n + c_{n-1}x^{n-1} + \cdots c_1 x + c_0$ where neither b_m nor c_n equals zero. Now factor out the leading terms of p and q:

$$\begin{aligned}\lim_{x\to\infty}\frac{p(x)}{q(x)} &= \lim_{x\to\infty}\frac{b_m x^m + b_{m-1}x^{m-1} + \cdots b_1 x + b_0}{cx^n + c_{n-1}x^{n-1} + \cdots c_1 x + c_0}\\ &= \lim_{x\to\infty}\frac{b_m x^m(1 + \frac{b_{m-1}}{b_m}x^{-1} + \cdots \frac{b_1}{b_m}x^{1-m} + \frac{b_0}{b_m}x^{-m})}{c_n x^n(1 + \frac{c_{n-1}}{c_n}x^{-1} + \cdots \frac{c_1}{c_n}x^{1-n} + \frac{c_0}{c_n}x^{-n})}\end{aligned}$$

The negative powers of x go to zero, and therefore this limit and

$$\lim_{x\to\infty}\frac{b_m x^m}{c_n x^n}$$

must be the same.

Example 6: $\lim\limits_{x\to\infty} \dfrac{4(x+1)^3+2(x+1)-1}{4x^3+2x-1} = 1$

Proof: Limit 6 is true for reasons that could be seen by expanding the numerator in the last example: the two polynomials $p(x)$ and $p(x+1)$ have the same lead term, so the limit of their quotient is 1 by Limit 5.

Example 7: Limit 7 applies to all polynomials other than 0. For example both $\lim\limits_{n\to\infty}(n^2+1)^{1/n} = 1$ and $\lim\limits_{n\to\infty}|1+2n-4n^3|^{1/n} = 1$, but $\lim\limits_{n\to\infty} 0^{1/n} = \lim\limits_{n\to\infty} 0 = 0$.

Proof: If p is a constant, then $\lim_{x\to\infty}|p(x)|^{1/x} = 1$ by Limit 2. Otherwise, $|p(x)| \to \infty$ as $x \to \infty$, and so when we let $y = |p(x)|^{1/x}$, its logarithm $\ln y = \dfrac{\ln|p(x)|}{x}$ has the indeterminate form $\frac{\infty}{\infty}$ as $x \to \infty$. Applying l'Hospital's Rule, we obtain $\lim\limits_{x\to\infty}\dfrac{p'(x)}{p(x)}$, which equals 0 by Limit 5, since the degree of $p'(x)$ is less than that of $p(x)$. Therefore, $\lim\limits_{x\to\infty} y = \lim\limits_{x\to\infty} e^{\ln y} = e^0 = 1$.