No notes, books, electronic devices, or outside materials of any kind.

Read each problem carefully and simplify your answers.

Unless otherwise indicated, supporting work will be required on every problem worth more than 2 points.

1(21 pts). Let R be the region in the plane bounded by the curves $y = x^2 + 2x$ and $y = x + 6$. Express the following as definite integrals, **but do not integrate**.

a. The area of R.

b. The volume of the solid resulting by rotating R about the line $y = 8$.

c. The volume of the solid resulting by rotating R about the line $x = 3$.

2(13 pts). Find the sum of the series, or explain why it does not exist.

a. $\sum_{n=1}^{\infty} \frac{1}{1-\left(\frac{2}{3}\right)^n}$ b. $\sum_{n=0}^{\infty} \frac{4}{\pi^n}$ c. $\sum_{n=0}^{\infty} \frac{4^n}{\pi^n n!}$

3(11 pts). Determine the **radius** of convergence of the power series $\sum_{n=0}^{\infty} \frac{(x-1)^n}{2^n(n^2+1)}$. You are not required to find the interval of convergence of this series.

4(10 pts). Evaluate the improper integral, if it converges. $\int_2^{\infty} \frac{1}{x(\ln x)^2}\,dx$

5(17 pts). Determine whether the series converges absolutely, converges conditionally, or diverges.

a. $\sum_{n=2}^{\infty} \frac{(-1)^n}{n(\ln n)^2}$ b. $\sum_{n=1}^{\infty} \frac{n^3+2n}{n^5+4n+1}$

6(9 pts). Evaluate the limit, if it exists.

a. $\lim_{n\to\infty} \frac{\ln n}{\ln(n^2+1)}$ b. $\lim_{n\to\infty} \frac{3\sqrt{n}}{\sqrt{n+2}}$

7(12 pts). Find the MacLaurin series for the given function.

a. $\sin(x^3)$ b. $\frac{x^3}{(1+2x)^2}$

8(11 pts). Find the Taylor polynomial of degree 3 centered at $a = 1$ for the function e^{2x}.

9(11 pts). Find the partial fraction decomposition of $\frac{2x^2-8x+1}{(x^2+1)(x+2)}$. **You are not required to integrate this function.**

10a(8 pts). Find the general solution of the differential equation $x - 2y^2\sqrt{x^2-1}\frac{dy}{dx} = 0$.

10b(4 pts). Find the particular solution of this differential equation that passes through the point $(x, y) = (1, 2)$.

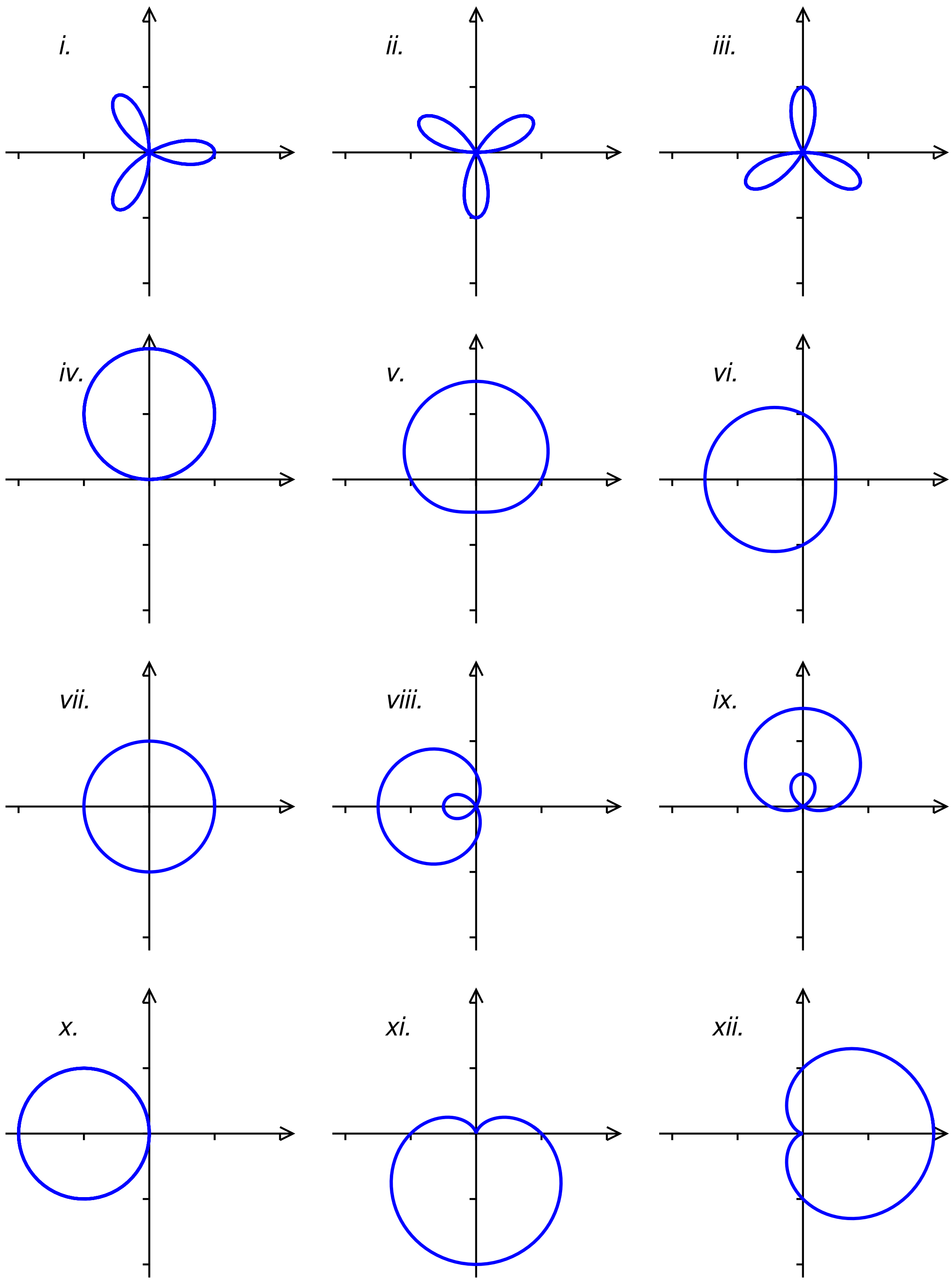
11(37 pts). Integrate.

a. $\int x \sec^2 x\,dx$ b. $\int \frac{dt}{t^2\sqrt{16-t^2}}$ c. $\int \sin^6 x\,dx$

12(6 pts). Each of the six polar equations below are graphed in the figure. Identify each equation with its graph number. (All 12 graphs displayed are drawn to the same scale.)

a. $r = 1$ b. $r = \frac{1}{2} + \sin\theta$ c. $r = -2\cos\theta$

d. $r = 1 - \frac{1}{2}\cos\theta$ e. $r = \sin(3\theta)$ f. $r = 1 - \sin\theta$



13(8 pts). Let C be the curve given by the polar equation $r = 1 - \frac{1}{2}\cos\theta$. Express the following quantities as definite integrals, **but do not integrate**.

a. The area enclosed by C.

b. The length of C.

14(4 pts). Find $\frac{dy}{dx}$ along the polar curve $r = \cos(2\theta)$. State your answer as a function of θ.

15. This problem is about approximations to the definite integral $\int_1^2 \cos(e^x)\,dx$.
Leave unsimplified, unfinished arithmetic in your answers but otherwise be completely explicit about how they are to be calculated. Your answer should not include "…" or anything else for me to fill in. On some parts, it might be helpful to know

$$|E_T| \leq \frac{K(b-a)^{?+1}}{12n^?} \qquad \text{and} \qquad |E_S| \leq \frac{K(b-a)^{?+1}}{180n^?}$$

15a(6 pts). Approximate the integral using the Trapzoid Rule with n = 5 subintervals.

15b(5 pts). I did some calculations and found that, if $f(x) = \cos(e^x)$ and $1 \leq x \leq 2$, then

$$|f'(x)| \leq 8, \quad |f''(x)| \leq 75, \quad |f^{(3)}(x)| \leq 580, \quad |f^{(4)}(x)| \leq 6000, \quad |f^{(5)}(x)| \leq 40000.$$

Assuming these are correct, how large might be the absolute error in your approximation in 15a? Report your answer in the form "Abs Error $\leq \varepsilon$" for some number ε.

15c(7 pts). If I require that the Trapezoid Rule approximate this definite integral with an absolute error less or equal 10^{-8}, how large must n be to guarantee this? Report your answer in the form "$n \geq K$" for some number K.

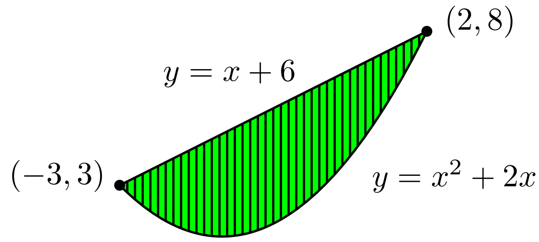
When taking limits as $n \to \infty$, it is useful to remember that, if $p(x)$ and $q(x)$ are polynomials, then the limit of $p(x)/q(x)$ as $x \to \pm\infty$ is the same as that of the lead term of $p(x)$ over the lead term of $q(x)$. For example,

$$\lim_{x\to\infty} \frac{3x^4 - 2x + 1}{2x^2 - 7} = \lim_{x\to\infty} \frac{3x^4}{2x^2} = \lim_{x\to\infty} \frac{3}{2}x^2 = \infty,$$

and

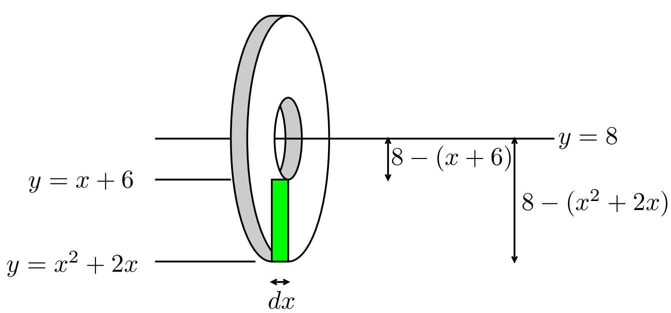
$$\lim_{x\to-\infty} \frac{7x^3 + 1}{5x^3 - 7x + 2} = \lim_{x\to-\infty} \frac{7x^3}{5x^3} = \frac{7}{5}.$$

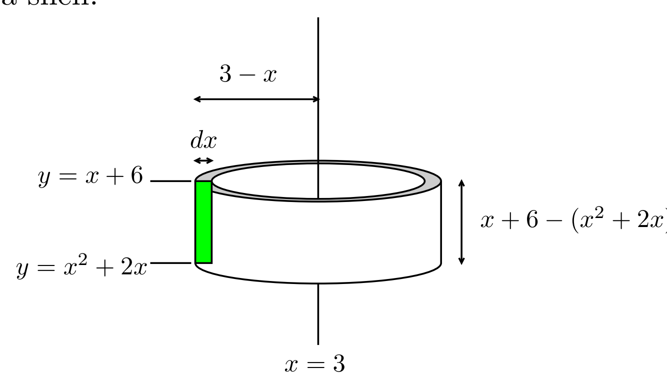
1.(Source: 6.1.8, 6.2.9, 6.3.14) $y = x^2 + 2x$ is a parabola that opens up, and that $y = x + 6$ is a line with positive slope, so R must look something like this (after slicing vertically):



a.(Source: 6.1.8) $A = \int dA = \int_{-3}^{2} (x + 6 - x^2 - 2x)\, dx = \int_{-3}^{2} (6 - x - x^2)\, dx$

Rotating each such rectangle about a horizontal line (as in part b.)results in a washer. Rotating about a vertical line (part c.) makes a shell:





b.(Source: 6.2.9) $V = \int dV = \int_{-3}^{2} \pi\big((8 - 2x - x^2)^2 - (2 - x)^2\big)\, dx$

c.(Source: 6.3.14) $V = \int dV = \int_{-3}^{2} 2\pi(3 - x)(6 - x - x^2)\, dx$

2a.(Source: 11.2.36) $\lim_{n\to\infty} \frac{1}{1-\left(\frac{2}{3}\right)^n} = \frac{1}{1-0} = 1 \neq 0$, so $\sum_{n=1}^{\infty} \frac{1}{1-\left(\frac{2}{3}\right)^n}$ diverges by the nth term test.

2b.(Source: 11.2.22) $\sum_{n=0}^{\infty} 4\left(\frac{1}{\pi}\right)^n$ is geometric with $r = \frac{1}{\pi}$. Since $|r| < 1$, the series converges to $\frac{a}{1-r} = \frac{4}{1-\frac{1}{\pi}} = \frac{4\pi}{\pi-1}$.

2c.(Source: 11.10.76) This is the MacLaurin series for e^x, so $\sum_{n=0}^{\infty} \frac{(4/\pi)^n}{n!} = e^{4/\pi}$.

3.(Source: 11.8.13) Ratio test: $\left| \frac{(x-1)^{n+1}}{2^{n+1}((n+1)^2+1)} \cdot \frac{2^n(n^2+1)}{(x-1)^n} \right| = \frac{|x-1|}{2} \cdot \frac{n^2+1}{n^2+2n+2} \to \frac{|x-1|}{2}$ as $n \to \infty$. The series converges if $\frac{|x-1|}{2} < 1$, or $|x - 1| < 2$. The radius of convergence is 2.

4.(Source: 7.8.24) Let $u = \ln x$ and $du = \frac{1}{x}\,dx$, and the integral becomes $\int_{\ln 2}^{\infty} u^{-2}\,du = \lim_{M\to\infty}\int_{\ln 2}^{M} u^{-2}\,du = \lim_{M\to\infty}(-u^{-1})\Big|_{\ln 2}^{M} = \lim_{M\to\infty}\left(-\frac{1}{M}+\frac{1}{\ln 2}\right) = \frac{1}{\ln 2}$.

5a.(Source: 11.6.38) Test for absolute convergence first. Since $f(x) = \frac{1}{x(\ln x)^2}$ is positive and decreasing, we can apply the integral test. As we saw in Problem 4, $\int_2^{\infty}\frac{1}{x(\ln x)^2}\,dx$ converges, so by the integral test, $\sum_{n=2}^{\infty}\frac{1}{n(\ln n)^2}$ converges, and therefore $\sum_{n=2}^{\infty}\frac{(-1)^n}{n(\ln n)^2}$ converges absolutely.

5b.(Source: 11.4.19) The series in question cannot converge conditionally since it is positive. Limit-compare it with $\sum_{n=1}^{\infty}\frac{n^3}{n^5} = \sum_{n=1}^{\infty}\frac{1}{n^2}$:

$$\lim_{n\to\infty}\frac{n^3+2n}{n^5+4n+1} \div \frac{1}{n^2} = \lim_{n\to\infty}\frac{n^5+2n^3}{n^5+4n+1} = 1,$$

which is positive and finite. The p-series $\sum_{n=1}^{\infty}\frac{1}{n^2}$ converges since $p = 2 > 1$. Therefore, $\sum_{n=1}^{\infty}\frac{n^3+2n}{n^5+4n+1}$ also converges, hence converges conditionally.

6a.(Source: 11.1.38,50) $\lim\limits_{n\to\infty}\dfrac{\ln n}{\ln(n^2+1)} = \dfrac{\infty}{\infty}$. Try l'Hospital's Rule: $\lim\limits_{n\to\infty}\dfrac{n^{-1}}{2n(n^2+1)^{-1}} = \lim\limits_{n\to\infty}\dfrac{n^2+1}{2n^2} = \dfrac{1}{2}$. Therefore, the original limit must also converge to $\frac{1}{2}$.

6b.(Source: 11.1.28,31-34) $\lim\limits_{n\to\infty} 3\sqrt{\dfrac{n}{n+2}} = 3\sqrt{\lim\limits_{n\to\infty}\dfrac{n}{n+2}} = 3\sqrt{1} = 3.$

7a.(Source: 11.10.45) $\sin(x^3) = \displaystyle\sum_{n=0}^{\infty}(-1)^n\frac{(x^3)^{2n+1}}{(2n+1)!} = \sum_{n=0}^{\infty}(-1)^n\frac{x^{6n+3}}{(2n+1)!}$

7b.(Source: 11.9.17,11.10.33) As in section 11.9, we can obtain the desired series from the geometric series. First, $\dfrac{1}{(1-x)^2} = \dfrac{d}{dx}\dfrac{1}{1-x} = \dfrac{d}{dx}\displaystyle\sum_{n=0}^{\infty}x^n = \sum_{n=1}^{\infty}nx^{n-1}$. Now

$$\frac{x^3}{(1+2x)^2} = x^3\left(\frac{1}{(1+2x)^2}\right) = x^3\sum_{n=1}^{\infty}n(-2x)^{n-1} = \sum_{n=1}^{\infty}(-1)^{n-1}n2^{n-1}x^{n+2}.$$

(done)

You could also have used the Binomial Series $(1+x)^k = \sum_{n=0}^{\infty}\binom{k}{n}x^n$:

$$x^3(1+2x)^{-2} = x^3\sum_{n=0}^{\infty}\binom{-2}{n}(2x)^n = \sum_{n=0}^{\infty}\binom{-2}{n}2^nx^{n+3}$$

8.(Source: 11.11.3) We need the first three derivatives of $f(x) = e^{2x}$ to find $T_3(x)$.

n	0	1	2	3
$f^{(n)}(x)$	e^{2x}	$2e^{2x}$	$4e^{2x}$	$8e^{2x}$
$f^{(n)}(1)/n!$	e^2	$2e^2$	$\frac{4e^2}{2} = 2e^2$	$\frac{8e^2}{6} = \frac{4}{3}e^2$

Therefore $T_3(x) = e^2\big(1 + 2(x-1) + 2(x-1)^2 + \frac{4}{3}(x-1)^3\big)$ **(done)**

You could also have found $T_3(x)$ by finding the cubic part of the Taylor series for e^{2x} centered at 1:

$$e^{2x} = e^2 e^{2(x-1)} = e^2 \sum_{n=0}^{\infty} \frac{\big(2(x-1)\big)^n}{n!} = \sum_{n=0}^{\infty} \frac{1}{n!} e^2 2^n (x-1)^n$$
$$= e^2 + e^2 2(x-1) + \frac{1}{2}e^2 4(x-1)^2 + \frac{1}{6}e^2 8(x-1)^3 + \cdots$$

9.(Source: 7.4.23) The PDF has the form

$$\frac{2x^2 - 8x + 1}{(x^2+1)(x+2)} = \frac{Ax+B}{x^2+1} + \frac{C}{x+2}$$

Clear the fractions from the equation by multiplying both sides by $(x^2+1)(x+2)$:

$$2x^2 - 8x + 1 = (Ax+B)(x+2) + C(x^2+1)$$

Evaluating at $x = -2$ and 0 and then equating x^2-coefficients yields

$$\{ \quad 25 = 5C, \quad 1 = 2B + C, \quad 2 = A + C \quad \} \quad \Longrightarrow \quad C = 5, \quad B = -2 \quad A = -3,$$

so the PDF is

$$\frac{2x^2 - 8x + 1}{(x^2+1)(x+2)} = \frac{-3x + -2}{x^2+1} + \frac{5}{x+2}$$

10a.(Source: 9.3.15) Separate variables and integrate:

$$x = 2y^2\sqrt{x^2-1}\frac{dy}{dx} \quad \Longrightarrow \quad \int x(x^2-1)^{-1/2}\,dx = \int 2y^2\,dy$$

Letting $w = x^2 - 1$ and $dw = 2x\,dx$ makes $\int x(x^2-1)^{-1/2}\,dx = \int \frac{1}{2}u^{-1/2}\,du = u^{1/2} + C$, so the general solution is

$$(x^2-1)^{1/2} = \frac{2}{3}y^3 + C$$

10b. Substitute $x = 1$ and $y = 2$, and solve for the constant of integration. $0 = \frac{2}{3}8 + C$ implies that $\frac{-16}{3} = C$ and the solution is $(x^2-1)^{1/2} = \frac{2}{3}y^3 - \frac{16}{3}$.

11a.(Source: 7.1.13) Use integration by parts:

$$u = x \quad dv = \sec^2 x\, dx$$
$$du = dx \quad v = \tan x$$

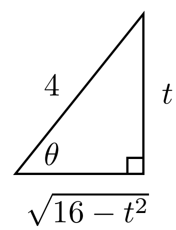
and the indefinite integral becomes

$$\int x \sec^2 x\, dx = \int u\, dv = uv - \int v\, du = x \tan x - \int \tan x\, dx$$
$$= x \tan x - \int \frac{\sin x}{\cos x}\, dx = x \tan x + \ln|\cos x| + C.$$

11b.(Source: 7.3.8) In order that $16 - t^2 = 16 - 16\sin^2\theta = 16\cos^2\theta$, we let $t = 4\sin\theta$, so that $dt = 4\cos\theta\, d\theta$, and the integral becomes

$$\int \frac{4\cos\theta\, d\theta}{16\sin^2\theta\sqrt{16\cos^2\theta}} = \int \frac{d\theta}{16\sin^2\theta} = \frac{1}{16}\int \csc^2\theta d\theta = -\frac{1}{16}\cot\theta + C$$

To rewrite this answer in terms of the original variable t, draw a right triangle with interior angle θ. Label two sides using $\sin\theta = t/4$, and then find the third side by the Pythagorean theorem. Therefore, $-\frac{1}{16}\cot\theta + C = -\frac{1}{16}\frac{\sqrt{16-t^2}}{t} + C$.



11c.(Source: Euler.10.e) By Euler's formula, $\sin^6 x = \left(\frac{e^{ix} - e^{-ix}}{2i}\right)^6$. The sixth row of Pascal's triangle is 1 6 15 20 15 6 1, so the integrand is

$$\frac{1}{(2i)^6}\left(e^{i6x} - 6e^{i4x} + 15e^{i2x} - 20 + 15e^{-i2x} - 6e^{-i4x} + e^{-i6x}\right)$$
$$= \frac{1}{2^6(i^2)^3}\left(e^{i6x} + e^{-i6x} - 6e^{i4x} - 6e^{-i4x} + 15e^{i2x} + 15e^{-i2x} - 20\right)$$
$$= -\frac{1}{2^5}\left(\frac{e^{i6x} + e^{-i6x}}{2} + 6\frac{e^{i4x} + e^{-i4x}}{2} + 15\frac{e^{i2x} + e^{-i2x}}{2} - 10\right)$$
$$= -\frac{1}{32}\left(\cos(6x) - 6\cos(4x) + 15\cos(2x) - 10\right).$$

Now the integration is straightforward:

$$\int \sin^6 x\, dx = -\frac{1}{32}\left(\frac{1}{6}\sin(6x) - \frac{3}{2}\sin(4x) + \frac{15}{2}\sin(2x) - 10x\right) + C.$$

12.(Source: 10.3.15, 31-40)

a. $r = 1$ *vii* b. $r = \frac{1}{2} + \sin\theta$ *ix* c. $r = -2\cos\theta$ *x*

d. $r = 1 - \frac{1}{2}\cos\theta$ *vi* e. $r = \sin(3\theta)$ *ii* f. $r = 1 - \sin\theta$ *xi*

13. As we saw in Problem 12, the graph of $r = 1 - \frac{1}{2}\cos\theta$ is a simply closed curve drawn once when $0 \le \theta \le 2\pi$.

a.(Source: 10.4.11) $A = \int dA = \int_0 2\pi \frac{1}{2} r^2 \, d\theta = \int_0 2\pi \frac{1}{2}(1 - \frac{1}{2}\cos\theta)^2 \, d\theta$.

b.(Source: 10.4.48) $s = \int ds = \int_0^{2\pi} \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2} \, d\theta = \int_0^{2\pi} \sqrt{(1 - \frac{1}{2}\cos\theta)^2 + (\frac{1}{2}\sin\theta)^2} \, d\theta$.

14.(Source: 10.3.55-60) Every polar curve of the form $r = r(\theta)$ is a parametric curve. In this case, the curve is given by $x = r\cos\theta = \cos(2\theta)\cos\theta$ and $y = r\sin\theta = \cos(2\theta)\sin\theta$. Using the Chain Rule,

$$\frac{dy}{dx} = \frac{dy/d\theta}{dx/d\theta} = \frac{-2\sin(2\theta)\sin\theta + \cos(2\theta)\cos\theta}{-2\sin(2\theta)\cos\theta - \cos(2\theta)\sin\theta}$$

15.(Source: 7.7.19) $\Delta x = (2-1)/5 = 1/5$ so subintervals have endpoints $1, 1.2, 1.4, 1.6, 1.8, 2$.
$\texttt{TRAP} = \frac{1}{10}(\cos(e^1) + 2\cos(e^{1.2}) + 2\cos(e^{1.4}) + 2\cos(e^{1.6}) + 2\cos(e^{1.8}) + \cos(e^2))$.

15a. In the Trapezoid Rule, K is an upper bound for $|f^{(2)}|$, so

$$|E_T| \le \frac{75(b-a)^3}{12n^2} = \frac{75 \cdot 1^3}{12 \cdot 5^2} = \frac{1}{4}$$

15b. To ensure that $10^{-8} \ge |E_T|$, choose n so that

$$10^{-8} \ge \frac{75}{12n^2} = \frac{25}{4n^2} \quad \text{so} \quad n \ge \sqrt{\frac{25}{4} 10^8} = \frac{5}{2} \cdot 10^4 = 25,000.$$