1. Let $f(x) = x \ln x$.
   a. Find the second degree Taylor polynomial $T_2(x)$ of $f(x)$ centered at $a = 1$.
   b. Find an upper bound for the absolute value of the error when $f(x)$ is approximated by $T_2(x)$ on the interval $[0.9, 1.1]$.

2. Find the MacLaurin series for each function.
   a. $\int \cos(x^3) \, dx$
   b. $e^{2x}$

3. Find a power series representation for each function.
   a. $\sec^{-1} x$
   b. $\ln(1 + x^2)$
   c. $x^2 \sqrt{1-x}$

4. A particle is at position $x = 2 + 3 \sin t$, $y = -2 \cos t$ at time $t$. Sketch the path of the particle as $t$ increases from 0 to $\frac{3\pi}{2}$. Include the particle’s starting and stopping positions and the direction it travels as $t$ increases. Help me understand your sketch by describing the particle’s path in words.

5. Sketch the curve given by the polar equation $r = 1 + 2 \sin \theta$. Clearly indicate in your drawing the $\theta$’s at which $r$ is a maximized, minimized, or zero.

6. Find the general solution to the differential equation $\cos x - (4y + e^{3y}) \frac{dy}{dx} = 0$.

7. Find the solution of the differential equation in 6a that passes through the point $(x, y) = (\pi/2, 0)$.

8. Find an equation of the line tangent to the curve given by the parametric equations $x = 5 - t^2$, $y = t^4 - 32t$ at the point corresponding to $t = 1$.

9. Find the time(s) $t$, if any, when the curve in 7a has a horizontal tangent line.

10. Find the time(s) $t$, if any, when the curve in 7a has a vertical tangent line.

11. Find $\frac{d^2y}{dx^2}$ along the curve in 7a. Express your answer as a function of $t$. 
1a. (Source: 11.11.20) For the second degree Taylor polynomial, we need the first and second derivatives of $f$.

<table>
<thead>
<tr>
<th>$n$</th>
<th>$f^{(n)}(x)$</th>
<th>$f^{(n)}(1)/n!$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>$x \ln x$</td>
<td>0</td>
</tr>
<tr>
<td>1</td>
<td>$1 + \ln x$</td>
<td>1</td>
</tr>
<tr>
<td>2</td>
<td>$x^{-1}$</td>
<td>$1/2$</td>
</tr>
</tbody>
</table>

Therefore $T_2(x) = (x - 1) + \frac{1}{2}(x - 1)^2$.

1b. The error depends on the size of $f'''(x) = -x^{-2}$. On the interval in question, $|f'''(x)| \leq 1$.


so Taylor’s inequality implies that

$$|f(x) - T_2(x)| \leq \frac{1}{(0.9)^2} |x - 1|^3 \leq \frac{(0.1)^3}{(0.9)^2 3!}$$

2a. (Source: 11.10.54) Starting with $\cos x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!}$, replace $x$ with $x^3$.

$$\cos(x^3) = \sum_{n=0}^{\infty} (-1)^n \frac{(x^3)^{2n}}{(2n)!} = \sum_{n=0}^{\infty} (-1)^n \frac{x^{6n}}{(2n)!}$$

Now integrate:

$$\int \cos(x^3) \, dx = \int \sum_{n=0}^{\infty} (-1)^n \frac{x^{6n}}{(2n)!} \, dx = \sum_{n=0}^{\infty} (-1)^n \frac{x^{6n+1}}{(2n)!(6n+1)} + C.$$  

2b. (Source: 11.10.14) $e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$ for all real numbers $x$, so $e^{2x} = \sum_{n=0}^{\infty} \frac{(2x)^n}{n!} = \sum_{n=0}^{\infty} \frac{2^n x^n}{n!}$.

3. These can be obtained from series derived in class and appearing in Table 1, p. 768.

3a. (Source: 11.9.16) $\tan^{-1} x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{2n+1}$.

3b. (Source: 11.9.22) $\ln(1 + x) = \sum_{n=1}^{\infty} (-1)^{n-1} \frac{x^n}{n}$, so $\ln(1 + x^2) = \sum_{n=1}^{\infty} (-1)^{n-1} \frac{(x^2)^n}{n} = \sum_{n=1}^{\infty} (-1)^{n-1} \frac{x^{2n}}{n}$.

3c. (Source: 11.10.5, 31) Use the binomial series $(1 + x)^k = \sum_{n=0}^{\infty} \binom{k}{n} x^n$ with $k = 1/3$ and $x$ replaced by $-x$. Then multiply the result by $x^2$.

$$x^2 \sqrt{1-x} = x^2 \sum_{n=0}^{\infty} \binom{1/3}{n} (-x)^n = \sum_{n=0}^{\infty} \binom{1/3}{n} (-1)^n x^{n+2}.$$
4. (Source: 10.1.20) $x = \sin t$ and $y = -\cos t$ is a parametrization of the unit circle $x^2 + y^2 = 1$, so the graph of $x = 2 + 3\sin t$, $y = -2\cos t$ is an ellipse obtained by shifting scaling the unit circle 3 units in the $x$ direction, 2 units in the $y$ direction, and shifting the result 2 units to the right. You weren’t required to eliminate the parameter to find an $x$-$y$ equation of the ellipse, but in case you did, it’s $(x-2)^2 + \left(\frac{y}{2}\right)^2 = 1$.

The particle traces out three quarters of the ellipse. To which three, it’s easiest to plot a few points.

<table>
<thead>
<tr>
<th>$t$</th>
<th>$x$</th>
<th>$y$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>2</td>
<td>-2</td>
</tr>
<tr>
<td>$\frac{\pi}{2}$</td>
<td>5</td>
<td>0</td>
</tr>
<tr>
<td>$\pi$</td>
<td>2</td>
<td>2</td>
</tr>
<tr>
<td>$\frac{3\pi}{2}$</td>
<td>-1</td>
<td>0</td>
</tr>
</tbody>
</table>

5. (Source: 10.3.32) Since $-1 \leq 1 + 2\sin \theta \leq 3$, the curve is a limaçon passing through the origin when $\sin \theta = -1/2$. Plot the points where $r$ equals $-1$, $3$, or $0$.

<table>
<thead>
<tr>
<th>$\theta$</th>
<th>$\frac{\pi}{2}$</th>
<th>$\pi$</th>
<th>$\frac{7\pi}{6}$</th>
<th>$\frac{3\pi}{2}$</th>
<th>$2\pi$</th>
<th>$\frac{11\pi}{6}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$r$</td>
<td>1</td>
<td>3</td>
<td>1</td>
<td>0</td>
<td>-1</td>
<td>0</td>
</tr>
</tbody>
</table>

6a. (Source: 9.3.14) Separate variables and integrate:

$$\cos x = (4y + e^{3y}) \frac{dy}{dx} \implies \cos x \, dx = (4y + e^{3y}) \, dy \implies$$

$$\int \cos x \, dx = \int (4y + e^{3y}) \, dy \implies \sin x = 2y^2 + \frac{1}{3}e^{3y} + C$$

6b. Substitute $x = \pi/2$ and $y = 0$, and solve for the constant of integration.

$$\sin(\pi/2) = 1 = \frac{1}{3}e^{0} + C \implies \frac{2}{3} = C$$

The solution is $\sin x = 2y^2 + \frac{1}{3}e^{3y} + \frac{2}{3}$. 
7a. (Source: 10.2.3) The point in question is \( x = 5 - 1 = 4 \) and \( y = 1 - 32 = -31 \). At time \( t = 1 \), the slope of the tangent line is \( \frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{4t^3 - 32}{-2t} = \frac{4 - 32}{-2} = 14 \) so the tangent line is \( y + 31 = 14(x - 4) \).

7b. (Source: 10.2.17) The tangent line is horizontal when \( \frac{dy}{dt} = 0 \neq \frac{dx}{dt} \). Solving \( \frac{dy}{dt} = 4t^3 - 32 = 0 \) gives \( t^3 = 8 \), or \( t = 2 \). At the same time, \( \frac{dx}{dt} = -2t = -4 \neq 0 \), so tangent line is horizontal at \( t = 2 \).

7c. (Source: 10.2.17) The tangent line is vertical when \( \frac{dy}{dt} \neq 0 = \frac{dx}{dt} \). Solving \( \frac{dx}{dt} = -2t = 0 \) gives \( t = 0 \). At the same time, \( \frac{dy}{dt} = 4t^3 - 32 = -32 \neq 0 \), so tangent line is vertical at \( t = 0 \).

7d. (Source: 10.2.11) Find \( \frac{d^2y}{dx^2} = \frac{d}{dx} \left( \frac{dy}{dx} \right) \) using the same principle idea we use to find \( \frac{dy}{dx} \). That is, \( \frac{dU}{dx} = \frac{d}{dt} \left( \frac{dy}{dx} \right) \frac{dx}{dt} \). It will help first to simplify \( \frac{dy}{dx} = \frac{4t^3 - 32}{-2t} = -2t^2 + 16t^{-1} \). Then

\[
\frac{d\left( \frac{dy}{dx} \right)}{dx} = \frac{d\left( \frac{dy}{dx} \right)}{dt} \frac{dx}{dt} = \frac{d\left( -2t^2 + 16t^{-1} \right)}{dt} \frac{-2t}{-2t} = \frac{-4t - 16t^{-2}}{-2t} = 2 + 8t^{-3}.
\]