No notes, books, electronic devices, or outside materials of any kind.

Read each problem carefully and simplify your answers.

Unless otherwise indicated, supporting work will be required on every problem worth more than 2 points.

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1. Let $f(x) = x \ln x$.

a(10 pts). Find the second degree Taylor polynomial $T_2(x)$ of $f(x)$ centered at $a = 1$.

b(8 pts). Find an upper bound for the absolute value of the error when $f(x)$ is approximated by $T_2(x)$ on the interval $[0.9, 1.1]$.

2(12 pts). Find the MacLaurin series for each function.

a. $\int \cos(x^3)\, dx$ b. e^{2x}

3(18 pts). Find a power series representation for each function.

a. $\tan^{-1} x$ b. $\ln(1 + x^2)$ c. $x^2 \sqrt[3]{1-x}$

4(8 pts). A particle is at position $x = 2 + 3\sin t$, $y = -2\cos t$ at time t. Sketch the path of the particle as t increases from 0 to $\frac{3\pi}{2}$. Include the particle's starting and stopping positions and the direction it travels as t increases. Help me understand your sketch by describing the particle's path in words.

5(10 pts). Sketch the curve given by the polar equation $r = 1 + 2\sin\theta$. Clearly indicate in your drawing the θ's at which r is a maximized, minimized, or zero.

6a(9 pts). Find the general solution to the differential equation $\cos x - (4y + e^{3y})\dfrac{dy}{dx} = 0$.

6b(4 pts). Find the solution of the differential equation in 6a that passes through the point $(x, y) = (\pi/2, 0)$.

7a(11 pts). Find an equation of the line tangent to the curve given by the parametric equations $x = 5 - t^2$, $y = t^4 - 32t$ at the point corresponding to $t = 1$.

7b(2 pts). Find the time(s) t, if any, when the curve in 7a has a horizontal tangent line.

7c(2 pts). Find the time(s) t, if any, when the curve in 7a has a vertical tangent line.

7d(6 pts). Find $\dfrac{d^2y}{dx^2}$ along the curve in 7a. Express your answer as a function of t.

1a.(Source: 11.11.20) For the second degree Taylor polynomial, we need the first and second derivatives of f.

n	$f^{(n)}(x)$	$f^{(n)}(1)/n!$
0	$x \ln x$	0
1	$1 + \ln x$	1
2	x^{-1}	1/2

Therefore $T_2(x) = (x-1) + \frac{1}{2}(x-1)^2$.

1b. The error depends on the size of $f'''(x) = -x^{-2}$. On the interval in question,

$$|f'''(x)| \leq \frac{1}{(0.9)^2},$$

so Taylor's inequality implies that

$$|f(x) - T_2(x)| \leq \frac{1}{(0.9)^2 3!}|x-1|^3 \leq \frac{(0.1)^3}{(0.9)^2 3!}$$

2a.(Source: 11.10.54) Starting with $\cos x = \sum_{n=0}^{\infty}(-1)^n \frac{x^{2n}}{(2n)!}$, replace x with x^3.

$$\cos(x^3) = \sum_{n=0}^{\infty}(-1)^n \frac{(x^3)^{2n}}{(2n)!} = \sum_{n=0}^{\infty}(-1)^n \frac{x^{6n}}{(2n)!}$$

Now integrate:

$$\int \cos(x^3)\,dx = \int \sum_{n=0}^{\infty}(-1)^n \frac{x^{6n}}{(2n)!}\,dx = \sum_{n=0}^{\infty}(-1)^n \frac{x^{6n+1}}{(2n)!(6n+1)} + C.$$

2b.(Source: 11.10.14) $e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$ for all real numbers x, so $e^{2x} = \sum_{n=0}^{\infty} \frac{(2x)^n}{n!} = \sum_{n=0}^{\infty} \frac{2^n x^n}{n!}$.

3. These can be obtained from series derived in class and appearing in Table 1, p. 768.

3a.(Source: 11.9.16) $\tan^{-1} x = \sum_{n=0}^{\infty}(-1)^n \frac{x^{2n+1}}{2n+1}$.

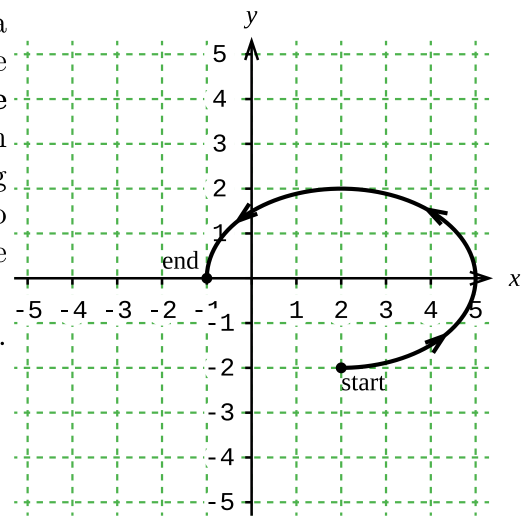
3b.(Source: 11.9.22) $\ln(1+x) = \sum_{n=1}^{\infty}(-1)^{n-1}\frac{x^n}{n}$, so $\ln(1+x^2) = \sum_{n=1}^{\infty}(-1)^{n-1}\frac{(x^2)^n}{n} = \sum_{n=1}^{\infty}(-1)^{n-1}\frac{x^{2n}}{n}$.

3c.(Source: 11.10.5, 31) Use the binomial series $(1+x)^k = \sum_{n=0}^{\infty}\binom{k}{n}x^n$ with $k = 1/3$ and x replaced by $-x$. Then multiply the result by x^2.

$$x^2\sqrt[3]{1-x} = x^2 \sum_{n=0}^{\infty}\binom{1/3}{n}(-x)^n = \sum_{n=0}^{\infty}\binom{1/3}{n}(-1)^n x^{n+2}.$$

4. (Source: 10.1.20) $x = \sin t$ and $y = -\cos t$ is a parametrization of the unit circle $x^2 + y^2 = 1$, so the graph of $x = 2 + 3\sin t$, $y = -2\cos t$ is an **ellipse** obtained by shifting scaling the unit circle 3 units in the x direction, 2 units in the y direction, and shifting the result 2 units to the right. You weren't required to eliminate the parameter to find an x-y equation of the ellipse, but in case you did, it's $\left(\frac{x-2}{3}\right)^2 + \left(\frac{y}{2}\right)^2 = 1$.

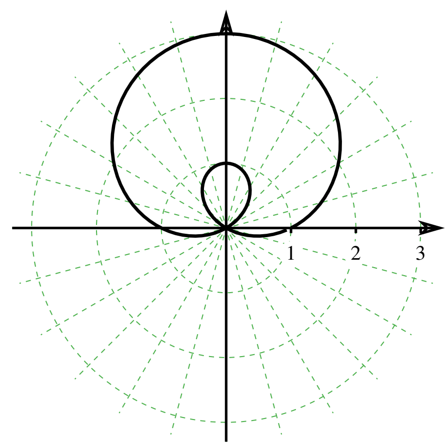
The particle traces out three quarters of the ellipse. To which three, it's easiest to plot a few points.



t	x	y
0	2	-2
$\frac{\pi}{2}$	5	0
π	2	2
$\frac{3\pi}{2}$	-1	0

5. (Source: 10.3.32) Since $-1 \le 1 + 2\sin\theta \le 3$, the curve is a limaçon passing through the origin when $\sin\theta = -1/2$. Plot the points where r equals -1, 3, or 0.

θ	0	$\frac{\pi}{2}$	π	$\frac{7\pi}{6}$	$\frac{3\pi}{2}$	2π	$\frac{11\pi}{6}$
r	1	3	1	0	-1	0	1



6a. (Source: 9.3.14) Separate variables and integrate:

$$\cos x = (4y + e^{3y})\frac{dy}{dx} \implies \cos x\,dx = (4y + e^{3y})\,dy \implies$$
$$\int \cos x\,dx = \int (4y + e^{3y})\,dy \implies \sin x = 2y^2 + \frac{1}{3}e^{3y} + C$$

6b. Substitute $x = \pi/2$ and $y = 0$, and solve for the constant of integration.

$$\sin(\pi/2) = 1 = \frac{1}{3}e^0 + C \implies \frac{2}{3} = C$$

The solution is $\sin x = 2y^2 + \frac{1}{3}e^{3y} + \frac{2}{3}$.

7a. (Source: 10.2.3) The point in question is $x = 5 - 1 = 4$ and $y = 1 - 32 = -31$. At time $t = 1$, the slope of the tangent line is $\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{4t^3-32}{-2t} = \frac{4-32}{-2} = 14$ so the tangent line is $y + 31 = 14(x - 4)$.

7b. (Source: 10.2.17) The tangent line is horizontal when $\frac{dy}{dt} = 0 \neq \frac{dx}{dt}$. Solving $\frac{dy}{dt} = 4t^3 - 32 = 0$ gives $t^3 = 8$, or $t = 2$. At the same time, $\frac{dx}{dt} = -2t = -4 \neq 0$, so tangent line is horizontal at $t = 2$.

7c. (Source: 10.2.17) The tangent line is vertical when $\frac{dy}{dt} \neq 0 = \frac{dx}{dt}$. Solving $\frac{dx}{dt} = -2t = 0$ gives $t = 0$. At the same time, $\frac{dy}{dt} = 4t^3 - 32 = -32 \neq 0$, so tangent line is vertical at $t = 0$.

7d. (Source: 10.2.11) Find $\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right)$ using the same principle idea we use to find $\frac{dy}{dx}$. That is, $\frac{dU}{dx} = \frac{\frac{dU}{dt}}{\frac{dx}{dt}}$. It will help first to simplify $\frac{dy}{dx} = \frac{4t^3-32}{-2t} = -2t^2 + 16t^{-1}$. Then

$$\frac{d(\frac{dy}{dx})}{dx} = \frac{\frac{d(\frac{dy}{dx})}{dt}}{\frac{dx}{dt}} = \frac{\frac{d(-2t^2+16t^{-1})}{dt}}{-2t} = \frac{(-4t - 16t^{-2})}{-2t} = 2 + 8t^{-3}.$$