No notes, books, electronic devices, or outside materials of any kind.

Read each problem carefully and simplify your answers.

Unless otherwise indicated, supporting work will be required on every problem worth more than 2 points.

1(11 pts). Let s_n be the sum of the first n terms of the series $\sum_{k=1}^{\infty}\left(\cos\left(\frac{\pi}{k}\right)-\cos\left(\frac{\pi}{k+1}\right)\right)$

a. Find a formula for s_n.

b. Find the sum of the series, if it exists.

2(5 pts). The series $\sum_{n=1}^{\infty}(-1)^n\frac{1}{n^3}$ converges. If σ is its sum and σ_{20} is its 20th partial sum, find an upper bound for the absolute error $|\sigma-\sigma_{20}|$. Express your answer in the form "$|\sigma-\sigma_{20}|\leq B$" for some number B.

3(12 pts). The series $\sum_{n=1}^{\infty}\frac{1}{n^3}$ converges. If s is its sum and s_n is its nth partial sum, how large must n be so as to ensure that $|s-s_n|\leq 10^{-9}$? Express your answer in the form "$n\geq C$" for some number C.

4(26 pts). Determine whether the series converges absolutely, converges conditionally, or diverges.

a. $\displaystyle\sum_{n=0}^{\infty}\frac{(-1)^n e^n}{1+e^n}$

b. $\displaystyle\sum_{n=0}^{\infty}(-1)^n\left(\frac{1+e^{-n}}{n}\right)$

5(28 pts). Determine whether the series converges or diverges.

a. $\displaystyle\sum_{n=1}^{\infty}\frac{e^{1/n}}{n^2}$

b. $\displaystyle\sum_{n=1}^{\infty}\frac{(n+1)^{2n}}{n^{3n}}$

6(14 pts). Find the radius of convergence of $\sum_{n=0}^{\infty}\frac{(x-1)^n}{3^n n!}$.

7(5 pts). Suppose that the power series $\sum_{n=0}^{\infty}d_n x^n$ converges at $x=-3$ and diverges at $x=5$. Determine, if possible, the behavior of the power series at the given x-values.

a.	$x=-6$	Circle one:	The series converges.	The series diverges.	Can't be determined.
b.	$x=-5$	Circle one:	The series converges.	The series diverges.	Can't be determined.
c.	$x=0$	Circle one:	The series converges.	The series diverges.	Can't be determined.
d.	$x=1$	Circle one:	The series converges.	The series diverges.	Can't be determined.
e.	$x=3$	Circle one:	The series converges.	The series diverges.	Can't be determined.

1a.(Source: 11.2.47)

$$\begin{aligned}
s_1 &= \cos(\pi) - \cos\left(\frac{\pi}{2}\right) \\
s_2 &= \cos(\pi) - \cos\left(\frac{\pi}{2}\right) + \cos\left(\frac{\pi}{2}\right) - \cos\left(\frac{\pi}{3}\right) \\
&= \cos(\pi) - \cos\left(\frac{\pi}{3}\right) \\
s_3 &= s_2 + \cos\left(\frac{\pi}{3}\right) - \cos\left(\frac{\pi}{4}\right) \\
&= \cos(\pi) - \cos\left(\frac{\pi}{4}\right)
\end{aligned}$$

so $s_n = \cos(\pi) - \cos\left(\frac{\pi}{n+1}\right)$.

1b. The sum of the series is the limit of its partial sums: $\sum_{k=1}^{\infty}\left(\cos\left(\frac{\pi}{k}\right) - \cos\left(\frac{\pi}{k+1}\right)\right) = \lim_{n\to\infty}\left(\cos(\pi) - \cos\left(\frac{\pi}{n+1}\right)\right) = \cos\pi - \cos 0 = -2$.

2.(Source: 11.5.more.1c) By the Alternating Series Test, $|\sigma - \sigma_{20}| \le b_{21} = \frac{1}{21^3}$.

3.(Source: 11.3.more.1c) By the Integral Test, $0 \le s - s_n \le \int_n^{\infty} x^{-3}\,x$. Calculate the improper integral: $\lim_{M\to\infty}(-\frac{1}{2}x^{-2})\Big|_n^M = \lim_{M\to\infty}(-\frac{1}{2}M^{-2} + \frac{1}{2}n^{-2}) = \frac{1}{2}n^{-2}$. To ensure that $s - s_n \le 10^{-9}$, choose n so as to make $\frac{1}{2}n^{-2} \le 10^{-9}$. Solving,

$$10^9 \le 2n^2 \quad\Longrightarrow\quad 5\cdot 10^8 \le n^2 \quad\Longrightarrow\quad 10^4\sqrt{5} \le n.$$

4a.(Source: 11.7.4) Remember this useful fact from 11.1:

$$\lim_{n\to\infty} a_n = 0 \quad\Longleftrightarrow\quad \lim_{n\to\infty} |a_n| = 0.$$

In our series, $\lim_{n\to\infty}\frac{e^n}{1+e^n} = 1$, so $\lim_{n\to\infty}\frac{(-1)^n e^n}{1+e^n}$ cannot go to zero, and the series diverges by the nth Term Test.

4b.(Source: 11.5.9,12) First test for absolute convergence. $\left(\frac{1+e^{-n}}{n}\right) \ge \frac{1}{n} \ge 0$. Since $\sum_{n=1}^{\infty}\frac{1}{n}$ is the divergent Harmonic Series, $\sum_{n=1}^{\infty}\left(\frac{1+e^{-n}}{n}\right)$ diverges by the Comparison Test.

Now test for (conditional) convergence. $\lim_{n\to\infty}\left(\frac{1+e^{-n}}{n}\right) = \text{"}\frac{1+0}{\infty}\text{"} = 0$. Also, $\frac{1+e^{-n}}{n}$ is decreasing, since its numerator is positive and decreasing, and its denominator is positive and increasing. Therefore, $\sum_{n=0}^{\infty}(-1)^n\left(\frac{1+e^{-n}}{n}\right)$ converges by the Alternating Series Test. Since $\sum_{n=0}^{\infty}(-1)^n\left(\frac{1+e^{-n}}{n}\right)$ converges but not absolutely, it is conditionally convergent.

5a.(Source: 11.4.28) Since $\sum_{n=1}^{\infty} \frac{e^{1/n}}{n^2}$ is positive, we can limit-compare it with the p-series $\sum_{n=1}^{\infty} \frac{1}{n^2}$, which converges since $p = 2 > 1$.

$$\lim_{n\to\infty} \frac{e^{1/n} n^{-2}}{n^{-2}} = \lim_{n\to\infty} e^{\frac{1}{n}} = e^0 = 1.$$

Since the limit is positive and finite, the Limit Comparison Test implies that the two series either both converge or both diverge, and therefore $\sum_{n=1}^{\infty} \frac{e^{1/n}}{n^2}$ converges.

5b.(Source: 11.7.6) Apply the Root Test:

$$\lim_{n\to\infty} \left| \frac{(n+1)^{2n}}{n^{3n}} \right|^{1/n} = \lim_{n\to\infty} \frac{(n+1)^2}{n^3} = 0,$$

since the degree of the top is less than that of the bottom. Since this limit is less than 1, $\sum_{n=1}^{\infty} \frac{(n+1)^2}{n^3}$ converges.

6.(Source: 11.8.7,9) Ratio Test:

$$\lim_{n\to\infty} \frac{|x-1|^{n+1}}{3^{n+1}(n+1)!} \cdot \frac{3^n n!}{|x-1|^n} = \lim_{n\to\infty} \frac{|x-1|}{3(n+1)} = 0.$$

Since this limit is less than 1 for all x, the power series converges on the entire real number line. Therefore the radius of convergence is ∞.

7.(Source: 11.8.30) The series is centered at 0. Since it converges at $x = -3$ and diverges at $x = 5$, its radius of convergence is at least 3 and at most 5. The series must converge at points less than 3 units from zero, and must diverge at points more than 5 from zero. Other than that, we can't say more.

a. $x = -6$: diverges. b. $x = -5$, can't say. c. $x = 0$: converges.
d. $x = 1$: converges. e. $x = 3$: can't say.