No notes, books, electronic devices, or outside materials of any kind.

Read each problem carefully and simplify your answers.

Unless otherwise indicated, supporting work will be required on every problem worth more than 2 points.

Evaluate all definite integrals unless otherwise instructed.

1a(14 pts). Find the length of the curve $y = \frac{1}{8}x^4 + \frac{1}{4}x^{-2}$ from $x = 1$ to $x = 2$.

1b(6 pts). Find the area of the surface generated by rotating the curve in 1a about the x-axis. Express your answer as a definite integral, **but do not integrate**.

2(14 pts). Find the limit of the given sequence or show that it diverges.

a. $\frac{4\sqrt{n}}{\sqrt{n}-2}$ b. $\frac{(-1)^n 4\sqrt{n}}{\sqrt{n}-2}$ c. $\ln(2n+1) - \ln n$

3(20 pts). Let R be the "triangular" region in the first quadrant bounded by the curves $y = x$, $y = \dfrac{1}{x^2}$, and $x = 3$. Express the following as definite integrals, **but do not integrate**.

a. The area of R.

b. The volume of the solid obtained by rotating R about the line $y = 5$.

c. The volume of the solid obtained by rotating R about the line $x = 5$.

d. The volume of the solid whose base is R and whose cross-sections perpendicular to the x-axis are squares with one side in R.

4(10 pts). A cable weighing 3 lb/ft is used to lift 50 lb of coal up a mine shaft that is 60 ft deep. Find the work done by the lifting force. Express your answer as a definite integral, **but do not integrate**.

5. Let $f(x) = \dfrac{1}{x^2}$.

a(8 pts). Find the average value f_{ave} of $f(x)$ on the interval $[-3, -1]$.

b(4 pts). Find a number c in $[-3, -1]$ for which $f(c) = f_{\text{ave}}$, or explain why none exists.

6(24 pts). Evaluate the improper integral or show that it diverges.

a. $\int_{-2}^{1} \frac{dx}{x^2}$ b. $\int_{e}^{\infty} \frac{\ln x}{x^2}\, dx$

1a.(Source: 8.1.12) Since y is a function of x, we will integrate

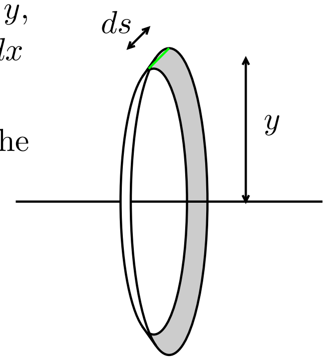
$$ds = \sqrt{(dx)^2 + (dy)^2} = \sqrt{1 + \left(\frac{dy}{dx}\right)^2} dx = \sqrt{1 + \left(\frac{1}{2}x^3 - \frac{1}{2}x^{-3}\right)^2} dx$$

$$= \sqrt{1 + \frac{1}{4}x^6 - \frac{1}{2} + \frac{1}{4}x^{-6}} dx = \sqrt{\frac{1}{4}x^6 + \frac{1}{2} + \frac{1}{4}x^{-6}} dx = \sqrt{\left(\frac{1}{2}x^3 + \frac{1}{2}x^{-3}\right)^2} dx.$$

Because $\frac{1}{2}x^3 + \frac{1}{2}x^{-3} > 0$ on the interval $[1, 2]$, $ds = \left(\frac{1}{2}x^3 + \frac{1}{2}x^{-3}\right) dx$ and so the length of the curve is

$$s = \int ds = \int_1^2 \left(\frac{1}{2}x^3 + \frac{1}{2}x^{-3}\right) dx = \left(\frac{1}{8}x^4 - \frac{1}{4}x^{-2}\right)\Big|_1^2 = 2 - \frac{1}{16} - \frac{1}{8} + \frac{1}{4} = \frac{33}{16}.$$

1b.(Source: 8.2.13) When we rotate about the x-axis, radius is y, and $A = \int dA = \int 2\pi y\, ds = 2\pi \int_1^2 \left(\frac{1}{8}x^4 + \frac{1}{4}x^{-2}\right)\left(\frac{1}{2}x^3 + \frac{1}{2}x^{-3}\right) dx$



2a. We can use l'Hospital's Rule, or simply factor out $\sqrt{n}$ from the top and bottom:

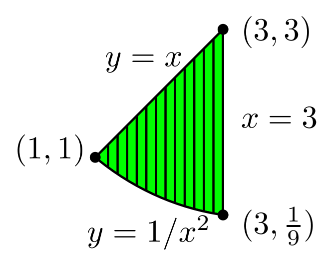
$$\frac{4\sqrt{n}}{\sqrt{n} - 2} = \frac{4\sqrt{n}}{\sqrt{n}\left(1 - \frac{2}{\sqrt{n}}\right)} = \frac{4}{1 - \frac{2}{\sqrt{n}}}$$

Since $\frac{2}{\sqrt{n}} \to 0$, the limit is 4.

2b. From part a, we know that the even-numbered terms of $\frac{(-1)^n 4\sqrt{n}}{\sqrt{n}-2}$ are going to 4, but the odd-numbered terms are going to -4. Therefore the sequence diverges.

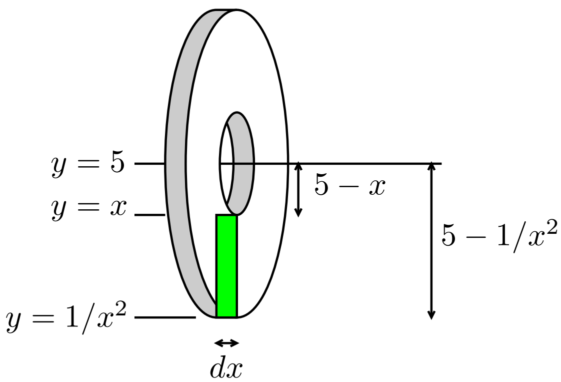
2c. $\frac{2n-1}{n} = 2 - \frac{1}{n} \to 2$, so $\ln(2n + 1) - \ln n = \ln(\frac{2n-1}{n}) \to \ln 2$.

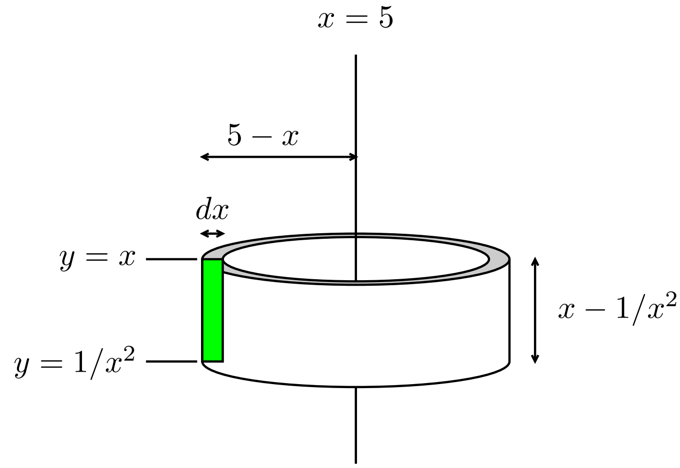
3.(Source: 6.1.9, 6.3.15-20, 6.2.11-18, 6.2.more.1) You don't need a highly accurate graph of R to answer this question. $y = x$ is the line with slope 1 passing through the origin, and $x = 3$ is a vertical line. $y = 1/x^2$ intersects $y = x$ at $(1, 1)$ and $x = 3$ at $(3, 1/9)$. R must look something like this, after slicing vertically into rectangles.



3a. The area of any one these rectangles is $(x - \frac{1}{x^2})\, dx$, so the area of R is $\int_1^3 (x - \frac{1}{x^2})\, dx$.

When we rotate each rectangle about the horizontal line $y = 5$, the result is a washer. Rotating about the vertical line $x = 5$ gives a washer.

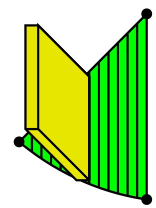




3b. $V = \int_1^3 \pi\big((5 - 1/x^2)^2 - (5-x)^2\big)\,dx$

3c. $V = \int_1^3 2\pi(5-x)(x - 1/x^2)\,dx$

3d. In this non-rotational solid, each slice perpendicular to the x-axis has a square face and an infinitesimally small thickness dx. The side-length of the square is the same as the height of the rectangle in part a. Letting dV be the volume of the slice at position x, we obtain



$$V = \int dV = \int_1^3 (x - 1/x^2)^2\,dx.$$

4.(Source: 6.4.15) Let dw be the work it takes to lift coal and remaining cable dy feet when the coal is at altitude y. At that moment, the remaining $60 - y$ feet of cable weighs $3(60-y)$ lb, so $dw = (3(60-y) + 50)\,dy$. Total work is $w = \int dw = \int_0^{50} (3(60-y)+50)\,dy$ (or $\int_0^{50}(230 - 3y)\,dy$).

5a. (Source: 6.5.10) $f_{\text{ave}} = \frac{1}{-1+3}\int_{-3}^{-1} x^{-2}\,dx = -\frac{1}{2}x^{-1}\Big|_{-3}^{-1} = \frac{1}{3}$.

5b. Since $f(x)$ is continuous on $[-3,-1]$, the existence of such a c is promised by the Mean Value Theorem for Integrals. $c^{-2} = 1/3 \implies c^2 = 3 \implies c = \pm\sqrt{3}$. Of these, only the solution $c = -\sqrt{3}$ is in $[-3,-1]$.

6a.(Source: 7.8.22,31) Integral is improper because the integrand is unbounded on the interval $[-2,1]$. Break into two: $\int_{-2}^{1}\frac{dx}{x^2} = \int_{-2}^{0}\frac{dx}{x^2} + \int_0^1 \frac{dx}{x^2}$.

The first equals $\lim_{\varepsilon\to 0^-}\int_{-2}^{\varepsilon}\frac{dx}{x^2} = \lim_{\varepsilon\to 0^-} \frac{-1}{x}\Big|_{-2}^{\varepsilon} = \lim_{\varepsilon\to 0^-}\frac{-1}{\varepsilon\frac{1}{2}} = \infty$. Since this integral diverges, the original integral from -2 to 1 also diverges.

6b. $\int_e^\infty \frac{\ln x}{x^2}\,dx = \lim_{M\to\infty}\int_e^M \frac{\ln x}{x^2}\,dx$. Integrate by parts: $u = \ln x$, $dv = x^{-}2\,dx$, $du = x^{-1}\,dx$, $v = -x^{-1}$. Then

$$\int_e^M \frac{\ln x}{x^2}\,dx = -\frac{\ln x}{x}\Big|_e^M + \int_e^M x^{-2}\,dx = \left(-\frac{\ln x}{x} - x^{-1}\right)\Big|_e^M = -\frac{\ln M}{M} - \frac{1}{M} + \frac{\ln e}{e} + \frac{1}{e}$$

Taking the limit of $\frac{\ln M}{M} \to \frac{\infty}{\infty}$ requires l'Hospital's Rule: $\frac{M^{-1}}{1} \to 0$. Now evaluate the improper integral:

$$\lim_{M\to\infty}\left(-\frac{\ln M}{M} - \frac{1}{M} + \frac{\ln e}{e} + \frac{1}{e}\right) = 0 + 0 + \frac{1}{e} + \frac{1}{e} = \frac{2}{e}.$$