MATH 220–01 (Kunkle), Exam 1 Name: ______________________

100 pts, 75 minutes Feb 7, 2017

No notes, books, electronic devices, or outside materials of any kind.

Read each problem carefully and simplify your answers.

Unless otherwise indicated, supporting work will be required on every problem worth more than 2 points.

1(16 pts). Evaluate the limit:

a. $\lim_{x\to\infty} xe^{-\sqrt{x}}$ b. $\lim_{x\to 1} x^{\frac{2}{1-x}}$

2(12 pts). Evaluate the indefinite integral: $\int x^2 \cosh x \, dx$

3(12 pts). Write as a sum of sinusoidal functions (but do not integrate): $\cos(4x)\sin(5x)$

4(9 pts). Evaluate the definite integral: $\displaystyle\int_0^{\pi/4} \sec x \, dx$

5(12 pts). Evaluate the indefinite integral: $\displaystyle\int \sin^5(\pi x)\cos^4(\pi x)\, dx$

6(12 pts). Approximate the integral $\int_0^3 e^{x^2}\, dx$ using Simpson's Rule with n = 6 subintervals. Repeat with the Trapezoid Rule. Label your answers so I can tell which is which.

7(13 pts). Evaluate the indefinite integral: $\displaystyle\int \frac{dx}{(9+x^2)^{3/2}}$

8(14 pts). Evaluate the indefinite integral: $\displaystyle\int \frac{2x^3-2x^2-x}{x^2-x-2}\, dx$

1a.(Source: 4.4.44) $\lim_{x\to\infty} xe^{-\sqrt{x}}$ has the indeterminate form $\infty \cdot 0$. Rewrite this product as the quotient $x/e^{\sqrt{x}}$. This has the form ∞/∞ as $x \to \infty$, so we can try l'Hospital's Rule:

$$\frac{x'}{(e^{\sqrt{x}})'} = \frac{1}{\frac{1}{2}x^{-1/2}e^{x^{1/2}}} = \frac{2x^{1/2}}{e^{x^{1/2}}} = \frac{\infty}{\infty}.$$

l'Hospitize again:

$$\frac{x^{-1/2}}{\frac{1}{2}x^{-1/2}e^{x^{1/2}}} = \frac{2}{e^{x^{1/2}}} = \frac{2}{\infty},$$

which means that the limit is zero. Therefore, the original limit is also zero by l'Hospital's Rule.

1b.(Source: 4.4.61) $\lim\limits_{x\to 1} x^{\frac{2}{1-x}}$ has the indeterminate form 1^∞. Let $y = x^{\frac{2}{1-x}}$. Then $\ln y = \ln x^{\frac{2}{1-x}} = \frac{2}{1-x}\ln x = \frac{2\ln x}{1-x}$. Since $\lim_{x\to 1}\ln y$ has the form $\frac{0}{0}$, try l'Hospital's Rule:

$$\frac{(2\ln x)'}{(1-x)'} = \frac{2x^{-1}}{-1} \to -2 \text{ as } x \to 1.$$

l'Hospital's Rule then implies that $\lim_{x\to 1}\ln y$ also $= -2$, and therefore $\lim_{x\to 1} y = \lim_{x\to 1} e^{\ln y} = e^{-2}$.

2.(Source: 7.1.7,14) Using integration by parts, $\begin{array}{cc} u = x^2 & dv = \cosh x\,dx \\ du = 2x\,dx & v = \sinh x \end{array}$, the indefinite integral becomes

$$\int x^2\cosh x\,dx = \int u\,dv = uv - \int v\,du = x^2\sinh x - \int 2x\sinh x\,dx.$$

Use integration by parts again: $\begin{array}{cc} u = 2x & dv = \sinh x\,dx \\ du = 2\,dx & v = \cosh x \end{array}$ and the integral equals

$$\begin{aligned} &= x^2\sinh x - \left[2x\cosh x - \int 2\cosh x\,dx\right] \\ &= x^2\sinh x - [2x\cosh x - 2\sinh x + C] \\ &= x^2\sinh x - 2x\cosh x + 2\sinh x + C \end{aligned}$$

(Since C is an arbitrary constant, so is $-C$.)

3.(Source: Euler.9a) Use Euler's formulas:

$$\begin{aligned} \cos(4x)\sin(5x) &= \left(\frac{e^{i4x}+e^{-i4x}}{2}\right)\left(\frac{e^{i5x}-e^{-i5x}}{2i}\right) = \frac{1}{4i}(e^{i9x}+e^{ix}-e^{ix}-e^{i9x}) \\ &= \frac{1}{2}\left(\frac{e^{i9x}-e^{i9x}}{2i}+\frac{e^{ix}-e^{ix}}{2i}\right) = \frac{1}{2}\big(\sin(9x)+\sin x\big). \end{aligned}$$

4.(Source: 7.2.39) $\int_0^{\pi/4}\sec x\,dx = \ln|\sec x + \tan x|\Big|_0^{\pi/4}$
$= \ln|\sec\pi/4 + \tan\pi/4| - \ln|\sec 0 + \tan 0| = \ln|\sqrt{2}+1| - \ln|1+0| = \ln(\sqrt{2}+1)$.

5.(Source: 7.2.3,5) The sine appears to an odd power, so we let $u = \cos \pi x$ and $du = -\pi \sin \pi x\, dx$.

$$\int \sin^5 \pi x \cos^4 \pi x\, dx = \frac{-1}{\pi} \int \sin^4 \pi x \cos^4 \pi x(-\pi \sin \pi x)\, dx$$
$$= \frac{-1}{\pi} \int (1 - \cos^2 \pi x)^2 \cos^4 \pi x(-\pi \sin \pi x)\, dx = \frac{-1}{\pi} \int (1 - u^2)^2 u^4\, du$$
$$= \frac{-1}{\pi} \int (1 - 2u^2 + u^4)u^4\, du = \frac{-1}{\pi} \int (u^4 - 2u^6 + u^8)\, du$$
$$= \frac{-1}{\pi} \left(\frac{1}{5}u^5 - \frac{2}{7}u^7 + \frac{1}{9}u^9 \right) + C = \frac{-1}{\pi} \left(\frac{1}{5}\cos^5 \pi x - \frac{2}{7}\cos^7 \pi x + \frac{1}{9}\cos^9 \pi x \right) + C$$

(done)

Alternately, we could make the π disappear in the first step by the simple substitution $t = \pi x$, $dt = \pi\, dx$, by which the original integral becomes $\frac{1}{\pi} \int \sin^5 t \cos^4 t\, dt$.

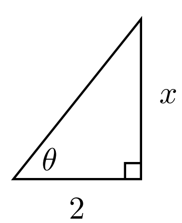
6.(Source: 7.7.20) The subintervals each have length $\Delta x = 3/6 = 1/2$, so their endpoints are 0, 1/2, 1, 3/2, 2, 5/2, and 3.

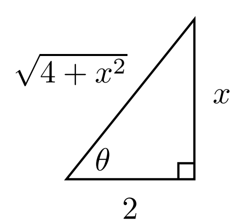
$$\texttt{TRAP} = \frac{1}{4}(e^0 + 2e^{1/4} + 2e^1 + 2e^{9/4} + 2e^4 + 2e^{25/4} + e^9)$$
$$\texttt{SIMP} = \frac{1}{4}(e^0 + 4e^{1/4} + 2e^1 + 4e^{9/4} + 2e^4 + 4e^{25/4} + e^9)$$

7.(Source: 7.3.7,9) We would like $9 + x^2 = 9 + 9\tan^2\theta$, so substitute $x = 3\tan\theta$ and $dx = 3\sec^2\theta\, d\theta$ and the integral becomes

$$\int \frac{dx}{(9 + x^2)^{3/2}} = \int \frac{3\sec^2\theta\, d\theta}{(9 + 9\tan^2\theta)^{3/2}} = \frac{3}{9^{3/2}} \int \frac{\sec^2\theta\, d\theta\, d\theta}{(1 + \tan^2\theta)^{3/2}} = \frac{1}{9} \int \frac{\sec^2\theta\, d\theta}{(\sec^2\theta)^{3/2}}$$
$$= \frac{1}{9} \int \frac{\sec^2\theta\, d\theta}{\sec^3\theta} = \frac{1}{9} \int \frac{d\theta}{\sec\theta} = \frac{1}{9} \int \cos\theta d\theta = \frac{1}{9}\sin\theta + C.$$

To rewrite this in terms of x, draw a triangle showing the tangent of θ equal $x/2$, and then find the third side by Pythagorus:





Therefore $\frac{1}{9}\sin\theta + C = \frac{x}{9\sqrt{9+x^2}} + C$.

8.(Source: 7.4.15) The degree on the top fails to be less than that of the bottom, so use long division:

$$\begin{array}{r|l} & \quad\quad\quad\quad\quad 2x \\ \hline x^2 - x - 2 & 2x^3 - 2x^2 - \ \ x \\ & -(2x^3 - 2x^2 - 4x) \\ \hline & \quad\quad\quad\quad\quad 3x \end{array}$$

So $\frac{2x^3-2x^2-x}{x^2-x-2} = 2x + \frac{3x}{x^2-x-2} = 2x + \frac{3x}{(x-2)(x+1)}$. Find the Partial Fraction Decomposition of this last fraction:

$$\frac{3x}{(x-2)(x+1)} = \frac{A}{x-2} + \frac{B}{x+1}$$

Multiply both sides by $(x-2)(x+1)$:

$$3x = A(x+1) + B(x-2)$$

By substituting $x = -1$, we learn $-3 = B(-3) \Rightarrow B = 1$. Substituting $x = 2$ gives $6 = A(3) \Rightarrow A = 2$. Therefore the integral is

$$\int \left(2x + \frac{2}{x-2} + \frac{1}{x+1}\right) dx = x^2 + 2\ln|x-2| + \ln|x+1| + C.$$