1 (10 pts). Let $R$ be the region in the $xy$-plane bounded by the curves $y = \ln x$, $y = 0$, and $x = e$.

a. Find the volume of the solid obtained by rotating $R$ about the $y$-axis. Write your answer as a definite integral, but do not integrate.

b. Find the volume of the solid whose base is $R$ and whose cross-sections perpendicular to the $x$-axis are squares with one side in $R$. Write your answer as a definite integral, but do not integrate.

Solution:

1. Here’s a sketch of $R$, sliced vertically:

   a. When one of these rectangles is rotated about $x = 0$, the result is a cylindrical shell. If we slice $R$ horizontally and rotate, the result is a washer:

   There are two correct answers:

   $$dV = 2\pi x \ln x \, dx$$
   $$V = \int_1^e 2\pi x \ln x \, dx.$$  
   $$dV = \pi (e^2 - (e^y)^2) \, dy$$
   $$V = \int_0^1 \pi (e^2 - e^{2y}) \, dy.$$  

   b. In this non-rotational solid, each slice perpendicular to the $x$-axis has a square face and an infinitesimally small thickness. Here’s one slice:

   Its sidelength equals the height of the rectangle, $\ln x$, and its thickness is $dx$, so

   $$dV = (\ln x)^2 \, dx$$ and $$V = \int_1^e (\ln x)^2 \, dx.$$