1 (10 pts). Integrate: \[ \int \frac{\sqrt{9 - x^2}}{x^2} \, dx \]

Solution:

1. (Source: 7.1.23) Use trig substitution. Start with what you want for the radicand, then figure out the necessary substitution. Because we want

\[ 9 - x^2 = 9 - 9\sin^2 \theta = 9(1 - \sin^2 \theta) = 9\cos^2 \theta, \]

we let

\[ x = 3\sin \theta, \quad dx = 3\cos \theta \, d\theta. \]

Now the integral becomes

\[ \int \frac{\sqrt{9\cos^2 \theta}}{9\sin^2 \theta} \cdot 3\cos \theta \, d\theta = \int \frac{\cos^2 \theta}{\sin^2 \theta} \, d\theta = \int \cot^2 \theta \, d\theta = \int (\csc^2 \theta - 1) \, d\theta = -\cot \theta - \theta + C \]

To rewrite this answer in terms of the original variable \(x\), draw a right triangle with interior angle \(\theta\). Label two sides using \(\sin \theta = x/3\), and then find the third side by the Pythagorean theorem:

\[ \begin{array}{c}
\text{3} \\
\theta \\
\sqrt{9 - x^2} \\
x
\end{array} \quad \begin{array}{c}
\text{3} \\
\theta \\
\sqrt{9 - x^2} \\
x
\end{array} \]

So the integral equals

\[ -\cot \theta - \theta + C = -\frac{\sqrt{9 - x^2}}{x} - \sin^{-1}(x/3) + C. \]