MATH 220–01 (Kunkle), Final Exam — Name: ______________________

200 pts, 3 hours — April 27, 2016 —

No notes, books, electronic devices, or outside materials of any kind.

Read each problem carefully and simplify your answers.

Unless otherwise indicated, supporting work will be required on every problem worth more than 2 points.

A mistake early in your solution does not rule out your receiving full credit for later steps.

You are expected to know the values of all trigonometric functions at multiples of $\pi/4$ and of $\pi/6$. You can use $\binom{k}{n}$ in your answers without having to explain what that symbol means.

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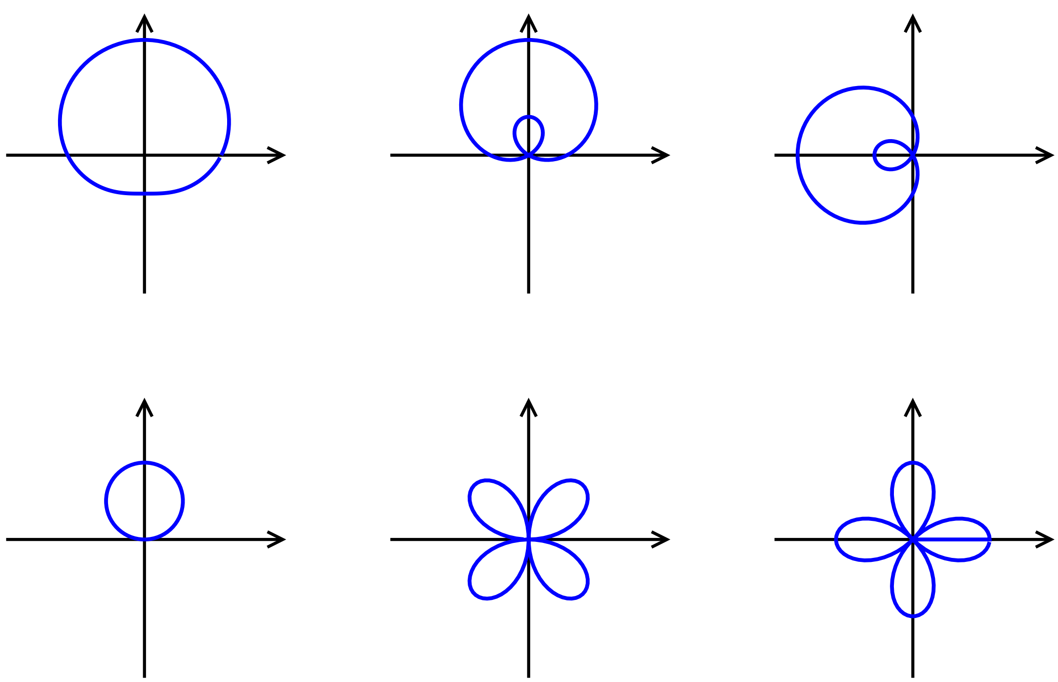
1 (6 pts). Six of the nine polar equations below are graphed in the figure. All six graphs are to the same scale. Clearly label each graph with its equation number.

i. $r = 2\cos(2\theta)$	*ii.* $r = 2\sin(2\theta)$	*iii.* $r = 2\cos(4\theta)$
iv. $r = 2\sin(4\theta)$	*v.* $r = 2\sin\theta$	*vi.* $r = 1 + 2\sin\theta$
vii. $r = 1 + \sin\theta$	*viii.* $r = 1 - 2\cos\theta$	*ix.* $r = 2 + \sin\theta$



2a (15 pts). Find the area enclosed by the polar curve $r = 1 + \sin\theta$.

2b (5 pts). Find the length of the curve in 2a. Express your answer as a definite integral, but **do not evaluate**.

3 (10 pts). Let R be the region bounded by the curves $y = 3 + \frac{1}{4}x^2$ and $y = x^2$. Find the volume of the solid obtained by rotating R about the line $x = -3$. Express your answer as a definite integral, but **do not evaluate**.

4a (10 pts). Find the general solution to the differential equation $\dfrac{dy}{dx} = \tan x \cos^4 y$. You are not required to express y as a function of x.

4b (5 pts). Find the particular solution to the equation in 4a that passes though the point $x = \pi/3$, $y = 0$.

5 (13 pts). Find the average value of the function $f(x) = x^{3/2} \ln x$ on the interval $[1, e]$.

6 (16 pts). Integrate: $\displaystyle\int \frac{\sqrt{x^2 - 9}}{x^4}\, dx$

7 (12 pts). Integrate: $\displaystyle\int \frac{x^3 - 8x + 9}{x^2 - 3x + 2}\, dx$

8a (10 pts). Find the length of the curve $y = \frac{1}{3}\sqrt{x}(x - 3)$ for $0 \le x \le 1$.

8b (5 pts). Find the area of the surface obtained by rotating the curve in 8a about the x-axis. Express your answer as a definite integral, but **do not evaluate**.

9 (13 pts). Evaluate the limit: $\displaystyle\lim_{n\to\infty} (n + e^n)^{1/n}$

10a (6 pts). Find a formula for the nth partial sum of $\sum_{k=1}^{\infty} \left(\cos\left(\frac{\pi}{k}\right) - \cos\left(\frac{\pi}{k+1}\right)\right)$.

10b (6 pts). Find the sum of this series, if it converges.

11 (8 pts). Determine whether the series converges or diverges. $\sum_{n=1}^{\infty} \frac{n^{3/2}+1}{3n^2-2n}$

12 (16 pts). Determine whether the series converges absolutely, converges conditionally, or diverges. $\sum_{n=2}^{\infty} \frac{(-1)^n}{n(\ln n)^2}$

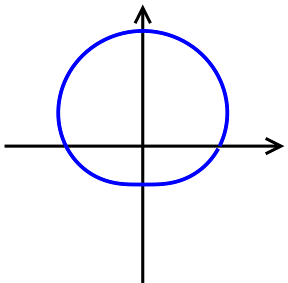
13 (19 pts). Find the interval of converge of $\sum_{n=1}^{\infty} (-1)^n \frac{(x-4)^n}{2n}$

14 (13 pts). Find the Maclaurin series for the given function.

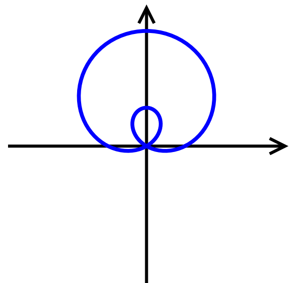
a. e^{2x} b. $\sin(x^2)$ c. $\ln(1 + x)$

15 (12 pts). Find an upper bound on the absolute error that occurs when $\frac{4}{7}x^{7/2}$ is approximated on the interval $[0.5, 1.5]$ by its degree-2 Taylor polynomial centered at $a = 1$. You are not required to find the Taylor polynomial.

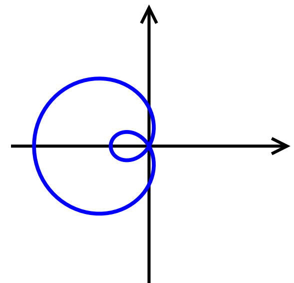
1. (Source: 10.3.31-42)



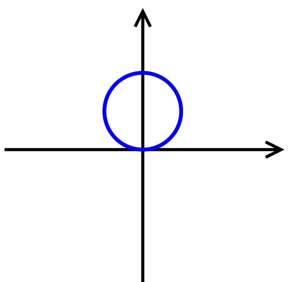
$r = 2 + \sin\theta$



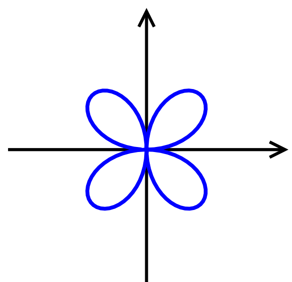
$r = 1 + 2\sin\theta$



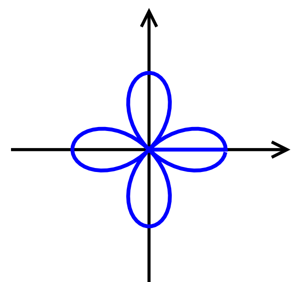
$r = 1 - 2\cos\theta$



$r = 2\sin\theta$



$r = 2\sin(2\theta)$



$r = 2\cos(2\theta)$

2a. (Source: 10.4.10) $A = \int_0^{2\pi} \frac{1}{2} r^2\, d\theta = \frac{1}{2}\int_0^{2\pi} (1+\sin\theta)^2\, d\theta = \frac{1}{2}\int_0^{2\pi} (1 + 2\sin\theta + \sin^2\theta)\, d\theta$. Now rewrite $\sin^2\theta$ using either Euler or the half-angle identity:

$$\begin{aligned} A &= \frac{1}{2}\int_0^{2\pi} \left(1 + 2\sin\theta + \frac{1}{2}(1 - \cos(2\theta))\right) d\theta \\ &= \frac{1}{2}\int_0^{2\pi} \left(\frac{3}{2} + 2\sin\theta - \frac{1}{2}\cos(2\theta)\right) d\theta = \left(\frac{3\theta}{4} - \cos\theta - \frac{1}{8}\sin(2\theta)\right)\Bigg|^{2\pi} = \frac{3\pi}{2}. \end{aligned}$$

2b. (Source: 10.4.45-48) Arc length $= \int_0^{2\pi} \sqrt{r^2 + (\frac{dr}{d\theta})^2}\, d\theta = \int_0^{2\pi} \sqrt{(1+\sin\theta)^2 + \cos^2\theta}\, d\theta$

3. (Source: 6.2.8) These curves intersect at $x = \pm 2$. R is bounded on top by $y = 3 + \frac{1}{4}x^2$ and below by $y = x^2$. Slice R vertically. Then rotating each slice about the vertical line $x = -3$ generates a cylindrical shell of height $(3 + \frac{1}{4}x^2) - x^2 = 3 - \frac{3}{4}x^2$, radius $x - (-3) = x + 3$, and thickness dx. Volume is $\int_{-2}^{2} 2\pi(x+3)(3 - \frac{3}{4}x^2)\, dx$.

4a. (Source: 7.2.25, 9.3.1,11) Separate and integrate: $\tan x \cos^4 y = \frac{dy}{dx} \Rightarrow$ $\int \tan x\, dx = \int \sec^4 y\, dy$, or $\int \frac{\sin x}{\cos x}\, dx = \int (1 + \tan^2 y)\sec^2 y\, dy$. Substitute $u = \cos x$ in the first and $t = \tan y$ in the second, and these become $-\int u^{-1}\, du = \int (1 + t^2)\, dt$. Integrating gives the general solution:

$$-\ln|\cos x| = \tan y + \frac{1}{3}\tan^3 y + C.$$

($-\ln|\cos x|$ can be written $\ln|\sec x|$.)

4b. Plug $x = \pi/3$ and $y = 0$ into the general solution to find C.
$-\ln|\cos(\pi/3)| = \tan 0 + \frac{1}{3}\tan^3 0 + C$. Recall that $\cos(\pi/3) = 1/2$ and $\tan 0 = 0$, so the particular solution is

$$-\ln|\cos x| = \tan y + \frac{1}{3}\tan^3 y + \ln 2.$$

5. (Source: 6.5.1, 7.1.12) Average value $= \frac{1}{e-1}\int_1^e x^{3/2}\ln x\,dx$. Use integration by parts.

$$u = \ln x \qquad du = \frac{1}{x}\,dx \qquad dv = x^{3/2}\,dx \qquad v = \frac{2}{5}x^{5/2}$$

$$\int u\,dv = uv - \int v\,du$$

$$\int x^{3/2}\ln x\,dx = \frac{2}{5}x^{5/2}\ln x - \int \frac{2}{5}x^{5/2}\frac{1}{x}\,dx = \frac{2}{5}x^{5/2}\ln x - \frac{2}{5}\int x^{3/2}\,dx$$

$$= \frac{2}{5}x^{5/2}\ln x - \frac{4}{25}x^{5/2} + C.$$

$$\text{The average value} = \frac{1}{e-1}\left(\frac{2}{5}x^{5/2}\ln x - \frac{4}{25}x^{5/2}\right)\Big|_1^e = \frac{1}{e-1}\left(\frac{2}{5}e^{5/2} - \frac{4}{25}e^{5/2} + \frac{4}{25}\right)$$

$$= \frac{1}{e-1}\left(\frac{6}{25}e^{5/2} + \frac{4}{25}\right).$$

6. (Source: 7.3.13) In order that $x^2 - 9 = 9\sec^2\theta - 9$, we let

$$x = 3\sec\theta, \qquad dx = 3\sec\theta\tan\theta\,d\theta$$

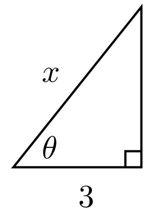
and the integral becomes

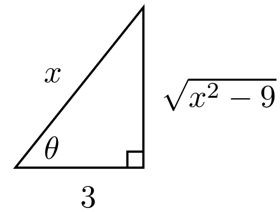
$$\int \frac{\sqrt{9\sec^2\theta - 9}}{81\sec^4\theta}3\sec\theta\tan\theta\,d\theta = \int \frac{3\sqrt{\sec^2\theta - 1}}{81\sec^3\theta}3\tan\theta\,d\theta$$

$$= \frac{1}{9}\int \frac{\sqrt{\tan^2\theta}}{\sec^3\theta}\tan\theta\,d\theta = \frac{1}{9}\int \frac{\tan^2\theta}{\sec^3\theta}, d\theta$$

$$= \frac{1}{9}\int \frac{\sin^2\theta\cos^3\theta}{\cos^2\theta}, d\theta = \frac{1}{9}\int \sin^2\theta\cos\theta, d\theta = \frac{1}{27}\sin^3\theta + C$$

To rewrite this answer in terms of the original variable x, draw a right triangle with interior angle θ. Label two sides using $\sec\theta = x/3$, and then find the third side by the Pythagorean theorem:





So $\dfrac{1}{27}\sin^3\theta + C = \dfrac{1}{27}\left(\dfrac{\sqrt{x^2-9}}{x}\right)^3 \theta + C.$

7. (Source: 7.4.16) Because the degree on top is $\geq$ the degree on bottom, we must start with long division:

$$\begin{array}{r|l} & \quad\quad\quad\quad\quad\quad x+3 \\ x^2-3x+2 & x^3 \quad\quad\quad - \; 8x+9 \\ & -(x^3-3x^2+\;2x) \\ & \quad\quad 3x^2-10x+9 \\ & \quad -(3x^2-\;9x+6) \\ & \quad\quad\quad\quad -x+3 \end{array}$$

This allows us to rewrite the integrand: $\dfrac{x^3-8x+9}{x^2-3x+2} = x+3+\dfrac{-x+3}{x^2-3x+2}$. Find find the partial fraction decomposition of

$$\frac{-x+3}{x^2-3x+2} = \frac{-x+3}{(x-1)(x-2)} = \frac{A}{x-1}+\frac{B}{x-2}$$
$$-x+3 = A(x-2)+B(x-1)$$

Letting $x=2$ gives $B=1$, and $x=1$ gives $A=-2$. So the integral becomes

$$\int\left(x+3-\frac{2}{x-1}+\frac{1}{x-2}\right)dx = \frac{1}{2}x^2+3x-2\ln|x-1|+\ln|x-2|+C.$$

8a. (Source: 8.1.11) Calculate $\frac{dy}{dx}$ *very* carefully and simplify before you square. As you've seen in the homework, a mistake under the radical will make turn this carefully designed problem into one that's impossible to finish. If that appears to happen to you, go back and look for your error.

$$y = \frac{1}{3}\sqrt{x}(x-3) = \frac{1}{3}x^{3/2}-x^{1/2} \qquad \frac{dy}{dx} = \frac{1}{2}x^{1/2}-\frac{1}{2}x^{-1/2}$$
$$s = \int ds = \int_0^1 \sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx = \int_0^1 \sqrt{1+\left(\frac{1}{2}x^{1/2}-\frac{1}{2}x^{-1/2}\right)^2}\,dx$$
$$= \int_0^1 \sqrt{1+\frac{1}{4}x-\frac{1}{2}+\frac{1}{4}x^{-1}}\,dx = \int_0^1 \sqrt{\frac{1}{4}x+\frac{1}{2}+\frac{1}{4}x^{-1}}\,dx$$
$$= \int_0^1 \sqrt{\left(\frac{1}{2}x^{1/2}+\frac{1}{2}x^{-1/2}\right)^2}\,dx = \int_0^1 \left(\frac{1}{2}x^{1/2}+\frac{1}{2}x^{-1/2}\right)dx$$
$$= \int_0^1 \left(\frac{1}{2}x^{1/2}+\frac{1}{2}x^{-1/2}\right)dx = \left(\frac{1}{3}x^{3/2}+x^{1/2}\right)\Big|_0^1 = \frac{4}{3}$$

8b. (Source: 8.2.1) $\displaystyle\int 2\pi y\,ds = \int_0^1 2\pi\left(\frac{1}{3}x^{3/2}-x^{1/2}\right)\left(\frac{1}{2}x^{1/2}+\frac{1}{2}x^{-1/2}\right)dx.$

9. (Source: 4.4.60, 11.1.39) The limit has the indeterminate form ∞^0. Let $y = (n + e^n)^{1/n}$ and take

$$\lim_{n\to\infty} \ln y = \lim_{n\to\infty} \frac{1}{n} \ln(n + e^n) = \lim_{n\to\infty} \frac{\ln(n + e^n)}{n} \to \frac{\infty}{\infty}$$

Remember that $\ln(A + B) \neq \ln A + \ln B$. l'Hôpitize:

$$\lim_{n\to\infty} \frac{\left(\frac{1}{n+e^n}\right)(1 + e^n)}{1} = \lim_{n\to\infty} \frac{1 + e^n}{n + e^n} \to \frac{\infty}{\infty}$$

l'Hôpitize again:

$$\lim_{n\to\infty} \frac{e^n}{1 + e^n} = \lim_{n\to\infty} \frac{1}{e^{-n} + 1} = \frac{1}{0 + 1} = 1.$$

By l'Hôpital's Rule, $\lim_{n\to\infty} \ln y$ also $= 1$, so $\lim_{n\to\infty} y = \lim_{n\to\infty} e^{\ln y} = e$.

10a. (Source: 11.2.40) Series is telescoping. Let's call the nth partial sum s_n and calculate a few.

$$\begin{aligned}
s_1 &= \left(\cos\left(\frac{\pi}{1}\right) - \cos\left(\frac{\pi}{2}\right)\right) \\
s_2 &= \left(\cos\left(\frac{\pi}{1}\right) - \cos\left(\frac{\pi}{2}\right)\right) + \left(\cos\left(\frac{\pi}{2}\right) - \cos\left(\frac{\pi}{3}\right)\right) \\
&= \cos\left(\frac{\pi}{1}\right) - \cos\left(\frac{\pi}{3}\right) \\
s_3 &= \cos\left(\frac{\pi}{1}\right) - \cos\left(\frac{\pi}{3}\right) + \cos\left(\frac{\pi}{3}\right) - \cos\left(\frac{\pi}{4}\right) \\
&= \cos\left(\frac{\pi}{1}\right) - \cos\left(\frac{\pi}{4}\right)
\end{aligned}$$

In general,

$$s_n = \cos\left(\frac{\pi}{1}\right) - \cos\left(\frac{\pi}{n+1}\right)$$

10b. The sum s of this series is the limit of its partial sums:

$$s = \lim_{n\to\infty} s_n = \lim_{n\to\infty} \left(\cos\left(\frac{\pi}{1}\right) - \cos\left(\frac{\pi}{n+1}\right)\right) = \cos\pi - \cos 0 = -2.$$

11. (Source: 11.4.10)

$$\frac{n^{3/2} + 1}{3n^2 - 2n} \geq \frac{n^{3/2}}{3n^2} = \frac{1}{3n^{1/2}} \geq 0$$

Since $\sum_{n=0}^{\infty} \frac{1}{n^{1/2}}$ is a divergent p-series (with $p = 1/2 < 1$), $\sum_{n=0}^{\infty} \frac{n^{3/2}+1}{3n^2-2n}$ diverges by the Comparison Test.

12. (Source: 11.3.22, 11.6.17) Check for absolute convergence first. (Note that the alternating series test can never let us conclude that a series is absolutely convergent.)

$$\sum_{n=2}^{\infty} \left| \frac{(-1)^n}{n(\ln n)^2} \right| = \sum_{n=2}^{\infty} \frac{1}{n(\ln n)^2}.$$

Let $f(x) = \frac{1}{x(\ln x)^2}$. Since $f(x)$ is positive and decreasing for all $x > 1$, we can use the Integral Test. In the improper integral $\int_2^\infty \frac{1}{x(\ln x)^2}$ substitute $u = \ln x$ and the integral becomes $\int_{\ln 2}^\infty \frac{1}{u^2}\,du$. This is a convergent p-integral (with $p = 2 > 1$), so $\sum_{n=2}^\infty \frac{(-1)^n}{n(\ln n)^2}$ is absolutely convergent by the Integral Test.

13. (Source: 11.8.16) Start with either the Root or the Ratio Test. Here's Root.

$$\lim_{n\to\infty} \sqrt[n]{\left|(-1)^n \frac{(x-4)^n}{2n}\right|} = \lim_{n\to\infty} \frac{|x-4|}{2^{1/n} n^{1/n}}.$$

As $n \to \infty$, $2^{1/n} \to 2^0 = 1$, and, by the Famous Limit Everybody Should Know, $\lim_{n\to\infty} n^{1/n} = 1$. Therefore

$$\lim_{n\to\infty} \frac{|x-4|}{2^{1/n} n^{1/n}} = |x-4|.$$

The series converges absolutely when $|x-4| < 1$, or $-1 < x-4 < 1$, or $3 < x < 5$. Check the endpoints of this interval for convergence.

At $x = 3$, the series is

$$\sum_{n=1}^\infty (-1)^n \frac{(3-4)^n}{2n} = \sum_{n=1}^\infty \frac{1}{2n} = \frac{1}{2}\sum_{n=1}^\infty \frac{1}{n}.$$

Since the Harmonic series $\sum_{n=1}^\infty \frac{1}{n}$ diverges, the power series converges at $x = 3$.

At $x = 5$, the series is

$$\sum_{n=1}^\infty (-1)^n \frac{(5-4)^n}{2n} = \sum_{n=1}^\infty (-1)^n \frac{1}{2n}.$$

$\frac{1}{2n}$ is decreasing and goes to zero as $n \to \infty$, so this series converges. Therefore, the interval of convergence of the power series is $(3, 5]$.

14. (Source: 11.10.6,9,33) Using known Maclaurin series,

a. $e^{2x} = \sum_{n=0}^\infty \frac{(2x)^n}{n!} = \sum_{n=0}^\infty \frac{2^n x^n}{n!}$,

b. $\sin(x^2) = \sum_{n=0}^\infty (-1)^n \frac{(x^2)^{2n+1}}{(2n+1)!} = \sum_{n=0}^\infty (-1)^n \frac{x^{4n+2}}{(2n+1)!}$, and

c. $\ln(1+x) \left(= \int \frac{1}{1+x}\,dx\right) = \sum_{n=0}^\infty (-1)^n \frac{x^{n+1}}{n+1}$.

15. (Source: 11.11.15) The size of the error depends on the size of the third derivative:

$$f = \frac{4}{7}x^{7/2} \quad f' = 2x^{5/2} \quad f'' = 5x^{3/2} \quad f^{(3)} = \frac{15}{2}x^{1/2}$$

Now, by Taylor's theorem,

$$f(x) - T_2(x) = \frac{f^{(3)}(c)(x-1)^3}{3!} = \frac{5c^{1/2}(x-1)^3}{4}$$

where c is a number between x and $1 (= a)$. If $0.5 \le x \le 1.5$, then $0.5 \le c \le 1.5$ also, and so $c^{1/2} \le (1.5)^{1/2}$. Therefore the absolute error is

$$|f(x) - T_2(x)| = \left|\frac{5c^{1/2}(x-1)^3}{4}\right| = \frac{5c^{1/2}|x-1|^3}{4} \le \frac{5(1.5)^{1/2}0.5^3}{4}.$$